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Existence of Non-Oscillatory Solutions of Second-Order Neutral Delay Difference Equations

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Abstract. In this paper, we consider the second-order neutral delay difference equation with positive and negative coefficients

$$\Delta\ r_n\Delta(x_n + cx_{n-k})\ \ + p_{n+1}x_{n+1-m} - q_{n+1}x_{n+1-l} = 0$$

where $c \in \mathbb{R}$, $k \geq 1$ and $m, l \geq 0$ are integers, $\{r_n\}_{n=n_0}^{\infty}, \{p_n\}_{n=n_0}^{\infty}$ and $\{q_n\}_{n=n_0}^{\infty}$ are sequences of non-negative real numbers. We obtain global results (with respect to c) which are some sufficient conditions for the existences of non-oscillatory solutions.

Keywords: *Neutral difference equations, non-oscillatory solutions, existence of solutions*

AMS subject classification: 39A10

1. Introduction

Consider the second-order neutral delay difference equation with positive and negative coefficients

$$\Delta\big(r_n\Delta(x_n + cx_{n-k})\big) + p_{n+1}x_{n+1-m} - q_{n+1}x_{n+1-l} = 0 \qquad (n \geq n_0) \tag{1}$$

where $c \in \mathbb{R}$, $k \geq 1$ and $m, l \geq 0$ are integers, $\{r_n\}_{n=n_0}^{\infty}$ is a sequence of positive real numbers, $\{p_n\}_{n=n_0}^{\infty}$ and $\{q_n\}_{n=n_0}^{\infty}$ are sequences of non-negative real numbers. The forward difference Δ is defined as usual, i.e. $\Delta x_n = x_{n+1} - x_n$.

Let $\sigma = \max\{k, m, l\}$ and $N_0 \geq n_0$ be a fixed non-negative integer. By a solution of equation (1), we mean a real sequence $\{x_n\}$ which is defined for all $n \geq N_0 - \sigma$ and satisfies (1) for $n \geq N_0$. A solution $\{x_n\}$ of (1) is said to *oscillate about zero* or simply to *oscillate*, if the terms x_n of the sequence $\{x_n\}$ are neither eventually all positive nor eventually all negative. Otherwise, the solution is called *non-oscillatory*.

Recenly there have been a lot of activities concerning the oscillation and non-oscillation of delay difference equations (see, for example, [1 - 14]). Agarwal, Manuel and

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Thandapani [1- 2] establish some sufficient conditions for existence of non-oscillatory solution of second-order neutral delay difference equations

$$\Delta\big(r_n\Delta(x_n + c_n x_{n-k})\big) + p_{n+1}f(x_{n+1-m}) = 0 \qquad (n \geq n_0) \tag{2}$$

where $\{r_n\}_{n=n_0}^{\infty}, \{c_n\}_{n=n_0}^{\infty}$ and $\{p_n\}_{n=n_0}^{\infty}$ are real sequences with $r_n > 0$ and $p_n > 0$. The oscillation and non-oscillation of solutions of the first order neutral delay difference equation with positive and negative coefficients

$$\Delta(x_n + cx_{n-k}) + p_n x_{n-m} - q_n x_{n-l} = 0 \qquad (n \geq n_0)$$

have been investigated by Chen and Zhang [5], Zhang and Wang [12], and Zhou [8]. The second-order neutral difference equation with positive and negative coefficients received much less attention. In particular, there is no non-oscillation result for equation (1).

In this paper, we obtain global results (with respect to c) in the non-constant coefficient case, which are some sufficient conditions for the existence of a non-oscillatory solution of equation (1) for all values of $c \neq \pm 1$.

2. Main results

In this section, we will give four theorems for existence of non-oscillatory solution of equation (1).

Theorem 1. *Assume* $0 \leq c < 1$ *and*

$$\sum_{i=n_0}^{\infty} p_i \sum_{j=i}^{\infty} \frac{1}{r_j} < \infty, \qquad \sum_{i=n_0}^{\infty} q_i \sum_{j=i}^{\infty} \frac{1}{r_j} < \infty. \tag{3}$$

Further, assume there exist a constant $\alpha > \frac{1}{1-c}$ *and a sufficiently large* $N_1 \geq n_0$ *such that*

$$p_n \geq \alpha q_n \qquad (n \geq N_1). \tag{4}$$

Then equation (1) *has a non-oscillatory solution.*

Proof. By (3) and (4), there exists $n_1 \geq N_1$ sufficiently large such that

$$n_1 \geq \max\{N_1, n_0 + \sigma\}, \quad \sigma = \max\{k, m, l\} \tag{5}$$

$$\sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} < 1 - c \tag{6}$$

$$0 \leq \sum_{i=n_1}^{\infty} (\alpha M p_i - M q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \leq c - 1 + \alpha M \tag{7}$$

where $M > 0$ is a constant such that

$$\frac{1-c}{\alpha} < M \leq \frac{1-c}{1+c\alpha}. \tag{8}$$

Consider the Banach Space $l_\infty^{n_0}$ of all real sequence $x = \{x_n\}_{n=n_0}^\infty$ with the norm $||x|| = \sup_{n\geq n_0} |x_n|$. We define a closed bounded subset Ω of $l_\infty^{n_0}$ by

$$\Omega = \Big\{x = \{x_n\} \in l_\infty^{n_0} : M \leq x_n \leq \alpha M \quad (n \geq n_0)\Big\}$$

and define an operator $T : \Omega \to l_\infty^{n_0}$ by

$$Tx_n = \begin{cases} 1 - c - cx_{n-k} + \sum_{j=n}^\infty \frac{1}{r_j} \sum_{i=n_1}^{n-1} (p_i x_{i-m} - q_i x_{i-l}) & \\ + \sum_{i=n}^\infty (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^\infty \frac{1}{r_j} & \text{if } n \geq n_1 + 1 \\ Tx_{n_1} & \text{if } n_0 \leq n \leq n_1 + 1. \end{cases}$$

We shall show that $T\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $n \geq n_1$, using (7) and (8) we get

$$\begin{aligned} Tx_n &= 1 - c - cx_{n-k} + \sum_{j=n}^\infty \frac{1}{r_j} \sum_{i=n_1}^{n-1} (p_i x_{i-m} - q_i x_{i-l}) + \sum_{i=n}^\infty (p_i x_{i-m} - q_i x_{m-l}) \sum_{j=i}^\infty \frac{1}{r_j} \\ &\leq 1 - c + \sum_{j=n}^\infty \frac{1}{r_j} \sum_{i=n_1}^{n-1} (\alpha M p_i - M q_i) + \sum_{i=n_1}^{n-1} i(\alpha M p_i - M q_i) \\ &\leq 1 - c + \sum_{i=n_1}^{n-1} (\alpha M p_i - M q_i) \sum_{j=i}^\infty \frac{1}{r_j} + \sum_{i=n}^\infty (\alpha M p_i - M q_i) \sum_{j=i}^\infty \frac{1}{r_j} \\ &= 1 - c + \sum_{i=n_1}^\infty (\alpha M p_i - M q_i) \sum_{j=i}^\infty \frac{1}{r_j} \\ &\leq \alpha M. \end{aligned}$$

Furthermore, in view of (4) and (8) we have

$$\begin{aligned} Tx_n &= 1 - c - cx_{n-k} + \sum_{j=n}^\infty \frac{1}{r_j} \sum_{i=n_1}^{n-1} (p_i x_{i-m} - q_i x_{i-l}) + \sum_{i=n}^\infty (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^\infty \frac{1}{r_j} \\ &\geq 1 - c - c\alpha M + \sum_{j=n}^\infty \frac{1}{r_j} \sum_{i=n_1}^{n-1} (M p_i - \alpha M q_i) + \sum_{i=n}^\infty (M p_i - \alpha M q_i) \sum_{j=i}^\infty \frac{1}{r_j} \\ &\geq 1 - c - c\alpha M \\ &\geq M. \end{aligned}$$

Thus we proved that $T\Omega \subset \Omega$.

Now we shall show that T is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and

$n \geq n_1$ we have

$$\begin{aligned}
&|Tx_n - Ty_n| \\
&\quad \leq c\,|x_{n-k} - y_{n-k}| + \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n-1} p_i |x_{i-m} - y_{i-m}| + \sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n-1} q_i |x_{i-l} - y_{i-l}| \\
&\qquad + \sum_{i=n}^{\infty} p_i |x_{i-m} - y_{i-m}| \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n}^{\infty} q_i |x_{i-l} - y_{i-l}| \sum_{j=i}^{\infty} \frac{1}{r_j} \\
&\quad \leq ||x-y|| \Bigg(c + \Bigg[\sum_{j=n}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n-1} (p_i + q_i) + \sum_{i=n}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \Bigg] \Bigg) \\
&\quad \leq \Bigg[c + \sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \Bigg] ||x - y|| \\
&\quad = \theta_1 ||x - y||.
\end{aligned}$$

This implies

$$||Tx - Ty|| \leq \theta_1 ||x - y||$$

where, in view of (6), $\theta_1 < 1$, which proves that T is a contraction operater on Ω. Therefore, T has a unique fixed point x in Ω, which is obviously a positive solution of equation (1). This completes the proof of Theorem 1 ∎

Theorem 2. *Assume* $1 < c < +\infty$ *and* (3) *holds. Further, assume there exist a constant* $\beta > \frac{c}{c-1}$ *and a sufficiently large* $N_1 \geq n_0$ *such that*

$$p_n \geq \beta q_n \qquad (n \geq N_1). \tag{9}$$

Then equation (1) *has a non-oscillatory solution.*

Proof. By (3) and (9), there exists $n_1 \geq N_1$ sufficiently large such that

$$n_1 + k \geq n_0 + \max\{m, l\} \tag{10}$$

$$\sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} < c - 1 \tag{11}$$

$$0 \leq \sum_{i=n_1}^{\infty} (\beta H p_i - H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \leq 1 - c + c\beta H \tag{12}$$

where $H > 0$ is a constant such that

$$\frac{c-1}{\beta c} < H \leq \frac{c-1}{c+\beta}. \tag{13}$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1, set

$$\Omega = \Big\{ x = \{x_n\} \in l_\infty^{n_0} : \, H \leq x_n \leq \beta H \;\; (n \geq n_0) \Big\}$$

and define an operator $T : \Omega \to l_\infty^{n_0}$ by

$$Tx_n = \begin{cases} 1 - \frac{1}{c}\{1 + x_{n+k} - \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ - \sum_{i=n+k}^{\infty} (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j}\} & \text{if } n \geq n_1 \\ \\ Tx_{n_1} & \text{if } n_0 \leq n \leq n_1. \end{cases}$$

We shall show that $T\Omega \subset \Omega$. In fact, for every $x \in \Omega$ and $n \geq n_1$, using (9), (12) and (13), we get

$$\begin{aligned} Tx_n &= 1 - \frac{1}{c}\Bigg(1 + x_{n+k} - \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ &\quad - \sum_{i=n+k}^{\infty} (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\leq 1 - \frac{1}{c}\Bigg(1 - \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (\beta H p_i - H q_i) - \sum_{i=n+k}^{\infty} (\beta H p_i - H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\leq 1 - \frac{1}{c}\Bigg(1 - \Bigg[\sum_{i=n_1}^{n+k-1} (\beta H p_i - H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n+k}^{\infty} (\beta H p_i - H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]\Bigg) \\ &= 1 - \frac{1}{c}\Bigg(1 - \sum_{i=n_1}^{\infty} (\beta H p_i - H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\leq \beta H. \end{aligned}$$

Furthermore, in view of (9) and (13) we get

$$\begin{aligned} Tx_n &= 1 - \frac{1}{c}\Bigg(1 + x_{n+k} - \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ &\quad - \sum_{i=n+k}^{\infty} (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\leq 1 - \frac{1}{c}\Bigg(1 + \beta H - \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (H p_i - \beta H q_i) - \sum_{i=n+k}^{\infty} (H p_i - \beta H q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\geq 1 - \frac{1}{c}(1 + \beta H) \\ &\geq H. \end{aligned}$$

Thus we proved that $T\Omega \subset \Omega$.

Now we shall show that T is a contraction operator on Ω. In fact, for $x, y \in \Omega$ and

$n \geq n_1$ we have

$$\begin{aligned}
&|Tx_n - Ty_n| \\
&\leq \frac{1}{c}\Bigg(|x_{n+k} - y_{n+k}| \\
&\quad + \Bigg[\sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} p_i|x_{i-m} - y_{i-m}| + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} q_i|x_{i-l} - y_{i-l}|\Bigg] \\
&\quad + \Bigg[\sum_{i=n+k}^{\infty} p_i|x_{i-m} - y_{i-m}| \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n+k}^{\infty} q_i|x_{i-l} - y_{i-l}| \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]\Bigg) \\
&\leq \frac{1}{c}||x-y||\Bigg(1 + \Bigg[\sum_{i=n_1}^{n+k-1} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n+k}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]\Bigg) \\
&= \frac{1}{c}\Bigg[1 + \sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]||x-y|| \\
&= \theta_2||x-y||.
\end{aligned}$$

This implies

$$||Tx - Ty|| \leq \theta_2||x-y||$$

where, in view of (11), $\theta_2 < 1$, which prove that T is a contraction operator. Consequently, T has the unique fixed point x, which is obviously a positive solution of equation (1). This completes the proof of Theorem 2 ∎

Theorem 3. *Assume $-1 < c < 0$ and (3) holds. Further, assume there exist a constant $\gamma > 1$ and a sufficiently large $N_1 \geq n_0$ such that*

$$p_n \geq \gamma q_n \qquad (n \geq N_1). \tag{14}$$

Then equation (1) has a non-oscillatory solution.

Proof. By (3) and (14), there exists $n_1 \geq N_1$ sufficiently large such that (5) and the inequalities

$$\sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} < c + 1 \tag{15}$$

$$0 \leq \sum_{i=n_1}^{\infty} (\gamma M_1 p_i - M_1 q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \leq (c+1)(\gamma M_1 - 1) \tag{16}$$

where the constant M_1 satisfies

$$\frac{1}{\gamma} < M_1 \leq 1. \tag{17}$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1, set

$$\Omega = \Big\{x = \{x_n\} \in l_\infty^{n_0} : M_1 \leq x_n \leq \gamma M_1 \quad (n \geq n_0)\Big\}$$

and define an operator $T:\,\Omega\to l_\infty^{n_0}$ by

$$Tx_n=\begin{cases}1+c-cx_{n-k}+\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}(p_ix_{i-m}-q_ix_{i-l})\\ +\sum_{i=n}^{\infty}(p_ix_{i-m}-q_ix_{i-l})\sum_{j=i}^{\infty}\frac{1}{r_j} & \text{if } n\geq n_1+1\\ Tx_{n_1} & \text{if } n_0\leq n\leq n_1+1.\end{cases}$$

For every $x\in\Omega$ and $n\geq n_1$, using (14) and (16), we get

$$\begin{aligned}Tx_n&=1+c-cx_{n-k}+\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}(p_ix_{i-m}-q_ix_{i-l})+\sum_{i=n}^{\infty}(p_ix_{i-m}-q_ix_{i-l})\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\leq 1+c-c\gamma M_1+\sum_{i=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}(\gamma M_1p_i-M_1q_i)+\sum_{i=n}^{\infty}(\gamma M_1p_i-M_1q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\leq 1+c-c\gamma M_1+\sum_{i=n_1}^{\infty}(\gamma M_1p_i-M_1q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\leq 1+c-c\gamma M_1+(c+1)(\gamma M_1-1)\\ &=\gamma M_1.\end{aligned}$$

Further, in view of (14) and (17) we have

$$\begin{aligned}Tx_n&=1+c-cx_{n-k}+\sum_{i=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}(p_ix_{i-m}-q_ix_{i-l})+\sum_{i=n}^{\infty}(p_ix_{i-m}-q_ix_{i-l})\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\geq 1+c-cM_1+\sum_{i=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}(M_1p_i-\gamma M_1q_i)+\sum_{i=n}^{\infty}(M_1p_i-\gamma M_1q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\geq 1+c-cM_1\\ &\geq M_1.\end{aligned}$$

Thus, we proved that $T\Omega\subset\Omega$.

For $x,y\in\Omega$ and $n\geq n_1$ we have

$$\begin{aligned}&|Tx_n-Ty_n|\\ &\quad\leq -c|x_{n-k}-y_{n-k}|+\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}p_i|x_{i-m}-y_{i-m}|+\sum_{j=n}^{\infty}\frac{1}{r_j}\sum_{i=n_1}^{n-1}q_i|x_{i-l}-y_{i-l}|\\ &\qquad+\sum_{i=n}^{\infty}p_i|x_{i-m}-y_{i-m}|\sum_{j=i}^{\infty}\frac{1}{r_j}+\sum_{i=n}^{\infty}q_i|x_{i-l}-y_{i-l}|\sum_{j=i}^{\infty}\frac{1}{r_j}\\ &\quad\leq ||x-y||\Bigg(-c+\Bigg[\sum_{i=n_1}^{\infty}(p_i+q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}+\sum_{i=n}^{\infty}(p_i+q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}\Bigg]\Bigg)\\ &\quad=\Bigg[-c+\sum_{i=n_1}^{\infty}(p_i+q_i)\sum_{j=i}^{\infty}\frac{1}{r_j}\Bigg]||x-y||\\ &\quad=\theta_3||x-y||.\end{aligned}$$

This implies

$$||Tx - Ty|| \le \theta_3 ||x - y||$$

where, in view of (15), $\theta_3 < 1$. This proves that T is a contraction operator. Consequently, T has a unique fixed point x, which is obviously a positive solution of equation (1). This completes the proof of Theorem 3 ∎

Theorem 4. *Assume $-\infty < c < -1$ and (3) holds. Further, assume there exists a constant $\delta > 1$ and a sufficiently large $N_1 \ge n_0$ such that*

$$p_n \ge \delta q_n \qquad (n \ge N_1). \tag{18}$$

Then equation (1) has a non-oscillatory solution.

Proof. By (3) and (18), there exists a $n_1 \ge n_0$ sufficiently large such that (10) and the inequalities

$$\sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} < -c - 1 \tag{19}$$

$$0 \le \sum_{i=n_1}^{\infty} (\delta H_1 p_i - H_1 q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} \le (c+1)(H_1 - 1) \tag{20}$$

hold where the constant $H_1 > 0$ satisfies

$$\frac{1}{\delta} \le H_1 < 1. \tag{21}$$

Let $l_\infty^{n_0}$ be the set as in the proof of Theorem 1, set

$$\Omega = \Big\{ x = \{x_n\} \in l_\infty^{n_0} : \ H_1 \le x_n \le \delta H_1 \ \ (n \ge n_0) \Big\}$$

and define an operator $T : \Omega \to l_\infty^{n_0}$ by

$$Tx_n = \begin{cases} 1 + \frac{1}{c}\{1 - x_{n+k} + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ + \sum_{i=n+k}^{\infty} i(p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j} \} & \text{if } n \ge n_1 \\ Tx_{n_1} & \text{if } n_0 \le n \le n_1. \end{cases}$$

For every $x \in \Omega$ and $n \ge n_1$, using (18) and (21) we get

$$\begin{aligned} Tx_n &= 1 + \frac{1}{c}\Bigg(1 - x_{n+k} + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ &\quad + \sum_{i=n+k}^{\infty} (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\le 1 + \frac{1}{c}\Bigg(1 - \delta H_1 + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (H_1 p_i - \delta H_1 q_i) \\ &\quad + \sum_{i=n+k}^{\infty} (H_1 p_i - \delta H_1 q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\le 1 + \frac{1}{c}(1 - \delta H_1) \\ &\le \delta H_1. \end{aligned}$$

Furthermore, in view of (20) and (21) we have

$$\begin{aligned} Tx_n &= 1 + \frac{1}{c}\Bigg(1 - x_{n+k} + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (p_i x_{i-m} - q_i x_{i-l}) \\ &\quad + \sum_{i=n+k}^{\infty} (p_i x_{i-m} - q_i x_{i-l}) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\geq 1 + \frac{1}{c}\Bigg(1 - H_1 + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} (\delta H_1 p_i - H_1 q_i) \\ &\quad + \sum_{i=n+k}^{\infty} (\delta H_1 p_i - H_1 q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\geq 1 + \frac{1}{c}\Bigg(1 - H_1 + \sum_{i=n_1}^{\infty} (\delta H_1 p_i - H_1 q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\geq 1 + \frac{1}{c}\big(1 - H_1 + (c+1)(H_1 - 1)\big) \\ &= H_1. \end{aligned}$$

Thus, we proved $T\Omega \subset \Omega$.

For $x, y \in \Omega$ and $n \geq n_1$ we have

$$\begin{aligned} &|Tx_n - Ty_n| \\ &\quad \leq -\frac{1}{c}\Bigg(|x_{n+k} - y_{n+k}| \\ &\qquad + \Bigg[\sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} p_i |x_{i-m} - y_{i-m}| + \sum_{j=n+k}^{\infty} \frac{1}{r_j} \sum_{i=n_1}^{n+k-1} q_i |x_{i-l} - y_{i-l}|\Bigg] \\ &\qquad + \Bigg[\sum_{i=n+k}^{\infty} p_i |x_{i-m} - y_{i-m}| \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n+k}^{\infty} q_i |x_{i-l} - y_{i-l}| \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]\Bigg) \\ &\quad \leq -\frac{1}{c}||x - y||\Bigg(1 + \sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j} + \sum_{i=n+k}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg) \\ &\quad = -\frac{1}{c}\Bigg[1 + \sum_{i=n_1}^{\infty} (p_i + q_i) \sum_{j=i}^{\infty} \frac{1}{r_j}\Bigg]||x - y|| \\ &\quad = \theta_4 ||x - y||. \end{aligned}$$

This immediately implies

$$||Tx - Ty|| \leq \theta_4 ||x - y||.$$

In view of (19), $\theta_4 < 1$. This proves that T is a contraction operator. Consequently, T has a unique fixed point x, which is obviously a positive solution of equation (1). This completes the proof of Theorem 4 ∎

Finally, in the special case where $q_n = 0$, conditions (4), (9), (14) and (18) are redundant. By Theorems 1 - 4, we have the following result.

Corollary 1. *Assume* $-\infty < c < +\infty$ *and* $\sum_{i=n_0}^{\infty} p_i \sum_{j=i}^{\infty} \frac{1}{r_j} < \infty$. *Then the neutral difference equation*

$$\Delta(r_n \Delta(x_n + cx_{n-k}) + p_{n+1}x_{n+1-m} = 0$$

has a non-oscillatory solution.

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