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Differential-Functional Inequalities for Bounded Vector-Valued Functions

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Abstract. For the space $\mathbb{R}^n$ ordered by a cone and some functions $f : \mathbb{R}^{n+mn} \to \mathbb{R}^n$ and $h_1, \ldots, h_m : \mathbb{R} \to \mathbb{R}$ we consider differential-functional inequalities of the type

$$v'' + cv' + f \;\; v, v(h_1), \ldots, v(h_m) \;\; \leq u'' + cu' + f \;\; u, u(h_1), \ldots, u(h_m)$$

and conclude $u \leq v$ under suitable conditions on u, v, h_k and f. The result can be applied to obtain existence and uniqueness results for differential-functional boundary value problems of the form

$$u'' + cu' + f \;\; u, u(h_1), \ldots, u(h_m) \;\; = q$$

with $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ bounded.

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1. Introduction

Let $\mathbb{R}^n$ be endowed with a norm $||\cdot||_0$, and ordered by a cone K_0, that is, K_0 is a closed, non-empty and convex subset of $\mathbb{R}^n$ with $\lambda K_0 \subseteq K_0 \;\; (\lambda \geq 0)$ and $K_0 \cap (-K_0) = \{0\}$, and let $x \leq y$ be defined by $y - x \in K_0$. Let K_0 have non-empty interior $\mathrm{Int}K_0$, and let $E = C_b(\mathbb{R}, \mathbb{R}^n)$ denote the Banach space of all bounded and continuous functions $u = (u_1, \ldots, u_n) : \mathbb{R} \to \mathbb{R}^n$, normed by $||u|| = \sup_{t \in \mathbb{R}} ||u(t)||_0$, and ordered by the cone

$$K = \big\{ u \in E : \, u(t) \in K_0 \;\, (t \in \mathbb{R}) \big\}.$$

On both spaces, $\mathbb{R}^n$ and E, the partial ordering defined by the corresponding cones K_0 and K is denoted by $\leq$, and we write $x \ll y$ or $u \ll v$ if $y - x \in \mathrm{Int}K_0$ or $v - u \in \mathrm{Int}K$, respectively. Note that K has non-empty interior since if $p \in \mathrm{Int}K_0$, then $u(t) = p \;\; (t \in \mathbb{R})$ is in $\mathrm{Int}K$.

In the sequel, let $c \in \mathbb{R}$, and let

$$f : \mathbb{R}^{n+mn} \to \mathbb{R}^n$$
$$h = (h_1, \ldots, h_m) : \mathbb{R} \to \mathbb{R}^m$$

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be continuous functions. If $u : \mathbb{R} \to \mathbb{R}^n$ is a function, we write $u(h)$ for the function

$$t \mapsto \big(u(h_1(t)), \dots, u(h_m(t))\big)$$

for short.

We consider differential-functional inequalities of the type

$$v''(t) + cv'(t) + f\big(v(t), v(h)(t)\big) \le u''(t) + cu'(t) + f\big(u(t), u(h)(t)\big)$$

for $t \in \mathbb{R}$ and want to conclude $u \le v$ under suitable conditions on u, v, h_k and f. Such a result may be used to prove uniqueness and existence of bounded solutions of first and second order systems of differential-functional equations, e.g., for boundary value problems of the type

$$u''(t) + cu'(t) + f\big(u(t), u(h)(t)\big) = q(t), \qquad u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n).$$

Note that, in particular, classical delay equations are included in our case by setting $h_k(t) = t + \tau_k \ \ (\tau_k \in \mathbb{R})$.

Let K_0^* denote the dual cone of K_0, that is the set of all linear functionals $\varphi \in (\mathbb{R}^n)^*$ for which $\varphi(x) \ge 0 \ \ (x \ge 0)$. A function $g : \mathbb{R}^n \to \mathbb{R}^n$ is called *quasimonotone increasing*, in the sense of Volkmann [8], if for $x, y \in \mathbb{R}^n$

$$x \le y, \ \varphi \in K_0^*, \ \varphi(x) = \varphi(y) \quad \Longrightarrow \quad \varphi(g(x)) \le \varphi(g(y)).$$

2. Results

To state our results let $BUC(\mathbb{R}, \mathbb{R}^n)$ denote the space of all bounded and uniformly continuous functions on $\mathbb{R}$, and let E_1, E_2 denote the following subspaces of $E = C_b(\mathbb{R}, \mathbb{R}^n)$:

$$E_1 = \Big\{u \in E : \ u \in C^1(\mathbb{R}, \mathbb{R}^n) \ \text{ and } \ u' \in BUC(\mathbb{R}, \mathbb{R}^n)\Big\}$$
$$E_2 = \Big\{u \in E : \ u \in C^2(\mathbb{R}, \mathbb{R}^n) \ \text{ and } \ u'' \in BUC(\mathbb{R}, \mathbb{R}^n)\Big\}.$$

By $x^{[m]}$ we denote the vector $(x)_{k=1}^m \in \mathbb{R}^{mn}$, if $x \in \mathbb{R}^n$.

Theorem 1. *Let $f : \mathbb{R}^{n+mn} \to \mathbb{R}^n$ be such that for $x \in \mathbb{R}^n$ and $y = (y_1, \dots, y_m) \in \mathbb{R}^{mn}$:*

1. *$x \mapsto f(x, y)$ is quasimonotone increasing (y fixed).*

2. *$y_k \mapsto f(x, y)$ is increasing (x and $y_j \ \ (j \ne k)$ fixed), for $k = 1, \dots, m$.*

3. *There exists $p \in \operatorname{Int} K_0$ with the following property: To each compact subset $C \subseteq \mathbb{R}^{n+mn}$ and each interval $[0, \Lambda] \subseteq \mathbb{R}$ there is a constant $L > 0$ such that*

$$f(x + \lambda p, y + \lambda p^{[m]}) - f(x, y) \le -L\lambda p$$

for $(x, y) \in C$ and $\lambda \in [0, \Lambda]$.

Then, if $u, v \in E_2$ are such that

$$v''(t) + cv'(t) + f\big(v(t), v(h)(t)\big) \le u''(t) + cu'(t) + f\big(u(t), u(h)(t)\big)$$

for all $t \in \mathbb{R}$, $u \le v$ follows.

Remarks.

(i) Without condition 3, Theorem 1 fails. For example, $v'' + cv' \le u'' + cu'$ (here $f = 0$) is valid for all constant functions u and v. On the other hand, condition 3 is valid if $f \in C^1(\mathbb{R}^{n+mn}, \mathbb{R}^n)$ and if there exists $p \in \mathrm{Int} K_0$ such that

$$\Big(\frac{\partial f}{\partial x}(x,y) + \sum_{k=1}^{m} \frac{\partial f}{\partial y_k}(x,y)\Big)p \ll 0$$

for all $(x, y) \in \mathbb{R}^{n+mn}$.

(ii) As can be seen from the proof, the second order differential-functional inequality in Theorem 1 can be replaced by a first order inequality, that is, if f is as in Theorem 1 and if $u, v \in E_1$ are such that

$$v'(t) + f\big(v(t), v(h)(t)\big) \le u'(t) + f\big(u(t), u(h)(t)\big) \qquad (t \in \mathbb{R})$$

or

$$-v'(t) + f\big(v(t), v(h)(t)\big) \le -u'(t) + f\big(u(t), u(h)(t)\big) \qquad (t \in \mathbb{R}),$$

then $u \le v$.

(iii) For a one-dimensional version of Theorem 1 (for implicit inequalities, and in a different frame) see [4]. For second order inequalities on bounded intervals see [5].

The following theorems will be proved by using Theorem 1.

Theorem 2. *Let $f : \mathbb{R}^{n+mn} \to \mathbb{R}^n$ be as in Theorem 1, let h be uniformly continuous on $\mathbb{R}$, and let $q \in BUC(\mathbb{R}, \mathbb{R}^n)$. Then the boundary value problems*

$$u''(t) + c\,u'(t) + f\big(u(t), u(h)(t)\big) = q(t), \qquad u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n)$$

and

$$\pm u'(t) + f\big(u(t), u(h)(t)\big) = q(t), \qquad u \in E \cap C^1(\mathbb{R}, \mathbb{R}^n)$$

have at most one solution.

Remarks.

(i) For example, the equation $-u'(t) + u(2t) = 0$ has non-trivial bounded solutions $u \in C^\infty(\mathbb{R}, \mathbb{R})$, even satisfying $u(0) = 0$ (see [6]). Here condition 3 from Theorem 1 is not satisfied.

(ii) Bounded solutions of differential-functional equations in the case $n = 1$ were studied by Staňek [7]. For ordinary differential equations, results related to Theorem 2 were first obtained by Bebernes and Jackson [2] and Belova [3] in the case $n = 1$. For a survey on boundary value problems on infinite intervals we refer to [1] and the references given therein.

Theorem 3. *Let $f : \mathbb{R}^{n+mn} \to \mathbb{R}^n$ be as in Theorem 1, and in addition let there exist $a > 0$ such that $x \mapsto f(x,y) + a^2 x$ is increasing ($y \in \mathbb{R}^{mn}$ fixed). Let h be uniformly continuous on $\mathbb{R}$, and let $q \in BUC(\mathbb{R}, \mathbb{R}^n)$. Moreover, let there exist $u_0, v_0 \in E_2$ such that*

$$v_0''(t) + f\big(v_0(t), v_0(h)(t)\big) \le q(t) \le u_0''(t) + f\big(u_0(t), u_0(h)(t)\big)$$

for $t \in \mathbb{R}$. Then, the boundary value problem

$$u''(t) + f\big(u(t), u(h)(t)\big) = q(t), \qquad u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n)$$

is uniquely solvable, and the solution u satisfies $u_0 \le u \le v_0$.

Remark. Related existence results for $c \neq 0$ can be obtained by the same method and more technical effort.

As an example we consider the linear case: Let $\mathbb{R}^n$ be ordered by the natural cone

$$K = \Big\{ x \in \mathbb{R}^n : x_k \ge 0 \ (k = 1, \ldots, n) \Big\}.$$

It is well known [8] that $g = (g_1, \ldots, g_n) : \mathbb{R}^n \to \mathbb{R}^n$ is quasimonotone increasing if and only if $x_k \mapsto g_j(x_1, \ldots, x_n)$ is increasing ($j \neq k$). Now consider real $(n \times n)$-matrices $A, B_1, \ldots, B_m$ with $x \mapsto Ax$ quasimonotone increasing (that is, all off diagonal entries of A are ≥ 0), and $x \mapsto B_k x$ increasing ($k = 1, \ldots, m$). Let there exist $p \gg 0$ (that is $p_1, \ldots, p_n > 0$ in this case) such that

$$Ap + \sum_{k=1}^{m} B_k p \le -Lp$$

for some $L > 0$. Then Theorem 3 applies to $f(x,y) = Ax + \sum_{k=1}^{m} B_k y_k$, that is, we have unique solvability of the problem

$$u''(t) + Au(t) + \sum_{k=1}^{m} B_k u(h_k(t)) = q(t), \qquad u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n)$$

if h is uniformly continuous and $q \in BUC(\mathbb{R}, \mathbb{R}^n)$ (choose $\mu > 0$ such that $-\mu Lp \le q(t) \le \mu Lp$, and set $u_0(t) = -\mu p$ and $v_0(t) = \mu p$ for $t \in \mathbb{R}$). Moreover, according to Theorem 1 the solution depends monotone decreasing on q.

For example, the system

$$\left.\begin{array}{l} u_1''(t) - 4u_1(t) + u_2(t) + u_1(2t) + u_2(t+1) = \sin^2(t) \\ u_2''(t) - 4u_2(t) + u_1(t) + u_1(3t) + u_2(t-1) = \exp(-t^2) \end{array}\right\},$$

$u = (u_1, u_2) \in E \cap C^2(\mathbb{R}, \mathbb{R}^2)$, has a unique solution (choose $p = (1,1)$), and $u \le 0$.

3. Preliminaries

For the proofs of our results we will use the following propositions.

As we noted in the introduction, K has non-empty interior. The functions from $\mathrm{Int}K$ are characterized by

Proposition 1. *Let $u \in K$. Then $u \gg 0$ if and only if to each $0 \neq \varphi \in K_0^*$ there exists $\varepsilon_\varphi > 0$ such that $\varphi(u(t)) \geq \varepsilon_\varphi$ for all $t \in \mathbb{R}$.*

Proof. Let $p \in \mathrm{Int}K_0$ and let $||\cdot||_1$ denote the norm on $(\mathbb{R}^n)^*$ induced by $||\cdot||_0$. If $0 \neq \varphi \in K_0^*$ is such that $\inf_{t\in\mathbb{R}} \varphi(u(t)) = 0$, then $u - \delta p \notin K$ for each $\delta > 0$, since $\varphi(p) > 0$. Hence $u \notin \mathrm{Int}K$.

On the other hand, if $\varphi(u(t)) \geq \varepsilon_\varphi > 0 \ \ (t \in \mathbb{R})$ for all $0 \neq \varphi \in K_0^*$, then there exists $\varepsilon_0 > 0$ such that $\varphi(u(t)) \geq \varepsilon_0 \ \ (t \in \mathbb{R})$ for all $\varphi \in K_0^*$ with $||\varphi||_1 = 1$. If $v \in E$ is such that $||v|| < \frac{\varepsilon_0}{2}$, then $\varphi(u(t) + v(t)) \geq \frac{\varepsilon_0}{2} \ \ (t \in \mathbb{R})$ for all $\varphi \in K_0^*$ with $||\varphi||_1 = 1$, and therefore $u + v \geq 0$. Thus $u \gg 0$ ∎

For the first and second derivative of functions in K we have

Proposition 2. *Let $u \in K \cap E_2$ and let $(t_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}$ such that the limits $u_0 = \lim_{k\to\infty} u(t_k)$, $u_1 = \lim_{k\to\infty} u'(t_k)$ and $u_2 = \lim_{k\to\infty} u''(t_k)$ exist. Then $u_0 \geq 0$, and if $\varphi(u_0) = 0$ for some $\varphi \in K^*$, then $\varphi(u_1) = 0$ and $\varphi(u_2) \geq 0$.*

Proof. It is obvious that $u_0 \geq 0$. Now let $\varphi \in K_0^*$ be such that $\varphi(u_0) = 0$. Consider a sequence $(\tau_j)_{j=1}^\infty$ in $\mathbb{R}\backslash\{0\}$ with limit 0, and set $\psi(t) = \varphi(u(t)) \ \ (t \in \mathbb{R})$. Since ψ, ψ', ψ'' are bounded on $\mathbb{R}$ and since $\psi'' \in BUC(\mathbb{R}, \mathbb{R})$, we have

$$
\begin{aligned}
\frac{\psi(t+\tau_j) - \psi(t)}{\tau_j} - \psi'(t) &\to 0 \\
\frac{\psi(t-\tau_j) - \psi(t)}{\tau_j} + \psi'(t) &\to 0 \\
\frac{\psi(t+\tau_j) - 2\psi(t) + \psi(t-\tau_j)}{\tau_j^2} - \psi''(t) &\to 0
\end{aligned}
$$

uniformly on $\mathbb{R}$ as $j \to \infty$. We can find a subsequence (t_{k_j}) such that $\frac{\psi(t_{k_j})}{\tau_j} \to 0$ as $j \to \infty$, and therefore

$$
0 \leq \frac{\psi(t_{k_j} + \tau_j)}{\tau_j} \to \varphi(u_1) \qquad (j \to \infty).
$$

Analogously we get $\varphi(u_1) \leq 0$ and $\varphi(u_2) \geq 0$ ∎

4. Proofs

We will now give the proofs of our main results.

Proof of Theorem 1. The idea of the proof is as follows: We write u, v as limits of functions $U, V : [0,\infty) \to E$, where U, V satisfy a differential inequality, and adapt the method in [8] to prove $U(s) \ll V(s)$.

We write Dw and D^2w for the first and second derivative of a function $w : \mathbb{R} \to \mathbb{R}^n$, and $D^0w := w$. In a first step we prove that there are constants $L, \mu > 0$ such that the functions $U, V : [0,\infty) \to E$ defined by

$$\left.\begin{aligned} U(s) &= u \\ V(s) &= \mu(||v|| + 1)\exp(-Ls)p + v \end{aligned}\right\}$$

satisfy $U(0) \ll V(0)$, and

$$\begin{aligned} H_u(s) &:= U'(s) - D^2(U(s)) - c\,D(U(s)) - f\big(U(s), (U(s))(h)\big) \\ &\ll V'(s) - D^2(V(s)) - c\,D(V(s)) - f\big(V(s), (V(s))(h)\big) \\ &=: H_v(s) \quad (s \geq 0). \end{aligned}$$

Since $p \gg 0$, we can manage

$$U(0) = u \ll \mu(||v|| + 1)p + v = V(0)$$

by choosing $\mu > 0$ sufficiently large. According to condition 3 in Theorem 1, there is a constant $L > 0$ such that

$$f\big(x + \lambda p, y + \lambda p^{[m]})\big) - f(x, y) \leq -2L\lambda p$$

for $||x||_0 \leq ||v||$, $||y_k||_0 \leq ||v||$ $(k = 1, \ldots, m)$ and $0 \leq \lambda \leq \mu(||v|| + 1)$. Then

$$\begin{aligned} &H_v(s) - H_u(s) \\ &\quad = -L\mu(||v|| + 1)\exp(-Ls)p - D^2(v - u) - c\,D(v - u) \\ &\qquad - \Big(f\big(\mu(||v|| + 1)\exp(-Ls)p + v, \mu(||v|| + 1)\exp(-Ls)p^{[m]} + v(h)\big) - f(v, v(h))\Big) \\ &\qquad - \big(f(v, v(h)) - f(u, u(h))\big) \\ &\quad \geq -L\mu(||v|| + 1)\exp(-Ls)p \\ &\qquad - \Big(f\big(\mu(||v|| + 1)\exp(-Ls)p + v, \mu(||v|| + 1)\exp(-Ls)p^{[m]} + v(h)\big) - f(v, v(h))\Big) \\ &\quad \geq -L\mu(||v|| + 1)\exp(-Ls)p + 2L\mu(||v|| + 1)\exp(-Ls)p \\ &\quad \gg 0 \end{aligned}$$

for $s \geq 0$. Note that $U, V : [0,\infty) \to E$ are continuously differentiable, and that $V(s) \in E_2$ $(s \geq 0)$, since each function $V(s)$ is the sum of $v \in E_2$ and a constant function.

Now, we set $d(s) = V(s) - U(s)$ $(s \geq 0)$. We prove $d(s) \gg 0$ $(s \geq 0)$. Assume that this is not true. Since $d(0) \gg 0$ and since $d : [0, \infty) \to E$ is continuous, there exists $s_0 > 0$ such that $d(s) \gg 0$ $(s \in [0, s_0))$, and $d(s_0) \geq 0$ but $d(s_0) \notin \mathrm{Int}K$. According to Proposition 1 there is a functional $0 \neq \varphi \in K_0^*$ such that $\inf_{t\in\mathbb{R}} \varphi\big((d(s_0))(t)\big) = 0$. Let $(t_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ with $\lim_{k\to\infty} \varphi\big((d(s_0))(t_k)\big) = 0$, and without loss of generality assume that (t_k) is chosen such that

$$\left.\begin{aligned} &D^j(U(s_0))(t_k) \to U_j \\ &D^j(V(s_0))(t_k) \to V_j \\ &(U(s_0)(h))(t_k) \to U_h \\ &(V(s_0)(h))(t_k) \to V_h \end{aligned}\right. \qquad (j = 0, 1, 2)$$

as $k \to \infty$. We have

$$\left.\begin{aligned} &(U'(s_0))(t_k) = 0 \\ &(V'(s_0))(t_k) = -L\mu(||v|| + 1)\exp(-Ls_0)p \end{aligned}\right\} \qquad (k \in \mathbb{N}).$$

Note that $U_0 \leq V_0$ and $\varphi(V_0 - U_0) = 0$, and according to Proposition 2 we have $\varphi(V_1 - U_1) = 0$ and $\varphi(V_2 - U_2) \geq 0$. Moreover, $V_h - U_h \in (K_0)^m$. Since

$$\varepsilon < \varphi\big((H_v(s_0))(t_k) - (H_u(s_0))(t_k)\big) \qquad (k \in \mathbb{N})$$

for some $\varepsilon > 0$, we get (as $k \to \infty$)

$$\begin{aligned} 0 < \varphi\Big(&- L\mu(||v|| + 1)\exp(-Ls_0)p - (V_2 - U_2) - c\,(V_1 - U_1) \\ &- \big(f(V_0, V_h) - f(V_0, U_h)\big) - \big(f(V_0, U_h) - f(U_0, U_h)\big)\Big) \\ \leq{}& -L\mu(||v|| + 1)\exp(-Ls_0)\varphi(p) \\ <{}& 0 \end{aligned}$$

according to the properties of f, which is a contradiction. Hence we have $U(s) = u \ll \mu(||v|| + 1)\exp(-Ls) + v = V(s)$ $(s \geq 0)$ and therefore, as $s \to \infty$, $u \leq v$ ∎

Proof of Theorem 2. We consider a solution $u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n)$ of the second order problem in Theorem 2. Then $f(u, u(h)) \in E$, hence $u'' + cu' \in E$. We first prove $u' \in E$. In the case $c = 0$ this follows from coordinatewise application of Taylor's formula. Let $c \neq 0$. Then, the two point boundary value problem

$$\left.\begin{aligned} z''(t) + cz'(t) = 0 \quad (t \in [0, 1]) \\ z(0) = z(1) = 0 \end{aligned}\right\}$$

has only the trivial solution $z : [0, 1] \to \mathbb{R}$, $z(t) = 0$. Hence there exists Green's function $G : [0, 1]^2 \to \mathbb{R}$ for this problem, and if $r \in C([0, 1], \mathbb{R}^n)$ and $w \in C^2([0, 1], \mathbb{R}^n)$ solves $w'' + cw' = r$, then

$$w(t) = \frac{\exp(-ct) - 1}{\exp(-c) - 1}\big(w(1) - w(0)\big) + w(0) + \int_0^1 G(t, \tau)r(\tau)\,d\tau$$

for $t \in [0,1]$. Hence

$$w'(t) = \frac{-c\exp(-ct)}{\exp(-c)-1}\big(w(1)-w(0)\big) + \int_0^1 G_t(t,\tau)r(\tau)\,d\tau$$

for $t \in [0,1]$, and since G_t is bounded, there is a constant $\alpha \geq 0$ such that

$$||w'(t)||_0 \leq \alpha\Big(||w(1)||_0 + ||w(0)||_0 + \max_{\tau\in[0,1]} ||r(\tau)||_0\Big)$$

for $t \in [0,1]$. Fix $t_0 \in \mathbb{R}$ and consider $w(t) = u(t+t_0)$ $(t \in [0,1])$. We obtain

$$||u'(t+t_0)||_0 \leq \alpha\big(2||u|| + ||f(u,u(h))|| + ||q||\big)$$

for $t \in [0,1]$. Since t_0 was arbitrary, we have $u' \in E$ and therefore $u'' \in E$. In particular, $u, u' \in BUC(\mathbb{R},\mathbb{R}^n)$, and since h is uniformly continuous, also $u(h) \in BUC(\mathbb{R},\mathbb{R}^{mn})$. Since f is uniformly continuous on compact subsets of $\mathbb{R}^{n+mn}$, we get $f(u,u(h)) \in BUC(\mathbb{R},\mathbb{R}^n)$ and, finally, $u'' \in BUC(\mathbb{R},\mathbb{R}^n)$. Altogether $u \in E_2$. Now, if v is another solution of our problem, then $v \in E_2$ and Theorem 1 gives $u \leq v$ and $v \leq u$, hence $u = v$.

Analogously we get uniqueness of the solution of the first order problem (compare Remark ii following Theorem 1) ∎

Proof of Theorem 3. First note that the problem in Theorem 3 has at most one solution according to Theorem 2, and if u is a solution, then $u \in E_2$ (compare the proof of Theorem 2). According to Theorem 1 we have $u_0 \leq u \leq v_0$. It remains to prove the existence of a solution.

Consider the operator T defined by

$$(Tw)(t) = \int_{\mathbb{R}} \frac{1}{2a}\exp(-a|t-\tau|)\Big(f(w(\tau),w(h)(\tau)) + a^2w(\tau) - q(\tau)\Big)d\tau.$$

For $w \in E$ it is easy to check that $Tw \in E \cap C^2(\mathbb{R},\mathbb{R}^n)$, $(Tw)' \in E$, and

$$(Tw)'' - a^2Tw + f(w,w(h)) + a^2w = q.$$

In particular, $(Tw)'' \in E$, and therefore $Tw \in E_2$ if $w \in E_2$. Moreover, $T : E \to E$ is increasing. Let the sequence (w_n) be defined recursively by

$$\left.\begin{aligned} w_0 &= u_0 \\ w_{n+1} &= Tw_n \quad (n \in \mathbb{N}_0) \end{aligned}\right\}.$$

Then $Tw_0 = Tu_0 \geq u_0$. Indeed, for $t \in \mathbb{R}$ we have

$$\begin{aligned}(Tu_0)(t) &= \int_{\mathbb{R}} \frac{1}{2a}\exp(-a|t-\tau|)\Big(f\big(u_0(\tau),u_0(h)(\tau)\big) + a^2u_0(\tau) - q(\tau)\Big)d\tau \\ &\geq \int_{\mathbb{R}} \frac{1}{2a}\exp(-a|t-\tau|)\Big(a^2u_0(\tau) - u_0''(\tau)\Big)d\tau \\ &=: g(t).\end{aligned}$$

Now $g'' - a^2 g = u_0'' - a^2 u_0$ and $g \in E_2$. Theorem 1 (applied to the function $x \mapsto -a^2 x$ and $c = 0$) proves $g = u_0$. Therefore, (w_n) is increasing. Theorem 1 gives $w_n \leq v_0 \ \ (n \in \mathbb{N}_0)$ since

$$\begin{aligned} w''_{n+1} &+ f\big(w_{n+1}, w_{n+1}(h)\big) \\ &= f\big(w_{n+1}, w_{n+1}(h)\big) + a^2 w_{n+1} - \big(f(w_n, w_n(h)) + a^2 w_n\big) + q \\ &\geq q. \end{aligned}$$

Hence the sequence (w_n) converges pointwise to a measurable and bounded function $u : \mathbb{R} \to \mathbb{R}^n$ with $u_0(t) \leq u(t) \leq v_0(t) \ \ (t \in \mathbb{R})$. From Lebesgue's theorem of dominated convergence we get $\varphi \circ Tw_n \to \varphi \circ Tu \ \ (\varphi \in K_0^*)$ pointwise on $\mathbb{R}$ as $k \to \infty$. Therefore, $Tu = u$, hence $u \in E$. Then $u \in E \cap C^2(\mathbb{R}, \mathbb{R}^n)$ and $u'' + f(u, u(h)) = q$ ∎

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