

Tensor Algebras and Displacement Structure

I. The Schur Algorithm

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Abstract. In this paper we explore the connection between tensor algebras and displacement structure. Thus, we describe a scattering experiment in this framework, we obtain a realization of the elements of the tensor algebra as transfer maps of a certain class of non-stationary linear systems, and we describe a Schur type algorithm for the Schur elements of the tensor algebra.

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1. Introduction

It was recently showed in [7] that the tensor algebra has displacement structure. The goal of this paper is to further develop this connection. The displacement structure theory, as it was initiated in the paper [13], consists in recursive factorization of matrices with additional structure encoded by a so-called displacement equation. Many applications of this theory were found in applied fields, like in the study of wave propagation in layered media, circuit synthesis, and filtering and modeling of non-stationary processes (see [14] for a survey). On the mathematical side, there are applications to bounded interpolation in upper triangular algebras (including the classical setting of bounded analytic functions on the unit disk) and to triangular factorization of operators.

On the other hand, the tensor algebra is a classical concept and in [7] it is showed that there is a connection with the displacement structure theory expressed by the fact that any element T of the unit ball of the algebra (with

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respect to the appropriate operator norm) satisfies the equation

$$A - \sum_{k=1}^N F_k A F_k^* = G J G^*$$

where $A = I - T^* T$ and the so-called generators F_k, G and J are described in the next section. This equation is of displacement type so we can develop results for the tensor algebra that parallel the ones in the displacement structure theory. It is the main goal of this paper to describe several such results.

The paper is organized as follows. In Section 2 we review the basic elements relevant to the tensor algebra. We describe the well-known isomorphic representation of this later algebra by a certain algebra $\mathcal{U}_{\mathcal{T}}^0(\mathcal{H})$ of upper triangular bounded operators since this representation allows us to rely on the results and ideas in the displacement structure theory. In this section we show that a certain class of positive definite kernels on the free semigroup with N generators has also displacement structure of the type mentioned above, and in Corollary 2.5 we explain the connection between this class of positive definite kernels and the algebra $\mathcal{U}_{\mathcal{T}}^0(\mathcal{H})$. In Section 3 we obtain results corresponding to so-called scattering experiments. Theorem 3.1 is the main result in this direction and as a by-product we can obtain Schur type parametrizations for $\mathcal{U}_{\mathcal{T}}^0(\mathcal{H})$ and the realization of the elements of the tensor algebra as transfer maps of a certain class of non-stationary linear systems. Section 4 develops a Schur type algorithm in the framework of $\mathcal{U}_{\mathcal{T}}^0(\mathcal{H})$.

We plan a more detailed study of the algorithm in a sequel of this paper.

2. Preliminaries

We briefly review the construction of the tensor algebra and we explain its displacement structure.

2.1 Tensor Algebras. We introduce some notation relevant to tensor algebras and we describe their representation as algebras of upper triangular operators (see, for instance, [15]). The associative tensor algebra $\mathcal{T}(\mathcal{H})$ generated by the complex vector space \mathcal{H} is defined by the algebraic direct sum

$$\mathcal{T}(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k}$$

where

$$\mathcal{H}^{\otimes k} = \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{k \text{ factors}}$$

is the k -fold algebraic tensor product of \mathcal{H} with itself. The elements of $\mathcal{T}(\mathcal{H})$ are terminating sequences $x = (x_0, x_1, \dots, x_k, 0, \dots)$, where $x_p \in \mathcal{H}^{\otimes p}$ is called the p th homogeneous component of x (note that $\mathcal{H}^{\otimes 0} = \mathbb{C}$ and $\mathcal{H}^{\otimes 1} = \mathcal{H}$). The addition and multiplication of elements in $\mathcal{T}(\mathcal{H})$ are defined componentwise:

$$(x + y)_n = x_n + y_n$$

and

$$(x \otimes y)_n = \sum_{k+l=n} x_k \otimes y_l \quad (x_0 \otimes y_n = y_n \otimes x_0 = x_0 y_n).$$

In order to deal with displacement structure matters it is convenient to use the fact that $\mathcal{T}(\mathcal{H})$ can be represented by upper triangular matrices of a special type. In this paper we restrict ourselves to the case $\mathcal{H} = \mathbb{C}^N$ ($N \geq 1$). Then we denote by $\mathcal{F}(\mathcal{H})$ the full Fock space associated to \mathcal{H} , that is, the Hilbert space

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k}$$

obtained by taking the Hilbert direct sum of the spaces $\mathcal{H}^{\otimes k}$ ($k \geq 0$), on which we consider the tensor Hilbert space structure induced by the Euclidean norm on $\mathcal{H} = \mathbb{C}^N$. If $\{e_1, \dots, e_N\}$ is the standard basis of \mathbb{C}^N , then

$$\left\{ e_{i_1} \otimes \cdots \otimes e_{i_k} \mid i_1, \dots, i_k \in \{1, \dots, N\} \right\}$$

is an orthonormal basis for $\mathcal{H}^{\otimes k}$. It is more convenient to write e_σ instead of $e_{i_1} \otimes \cdots \otimes e_{i_k}$, where $\sigma = i_1 \cdots i_k$ is viewed as a word in the unital free semigroup \mathbb{F}_N^+ with N generators $1, \dots, N$. The empty word is the identity element of \mathbb{F}_N^+ . The length of the word σ is denoted by $|\sigma|$. We also use to write the elements of \mathbb{F}_N^+ in lexicographic order. The space $\mathcal{H}^{\otimes k+1}$ can be identified with the direct sum of N copies of $\mathcal{H}^{\otimes k}$. More precisely, we notice that

$$\left\{ (\delta_{jl} e_\sigma)_{l=1}^N \mid j \in \{1, \dots, N\}, |\sigma| = k \right\}$$

is an orthonormal basis for

$$\underbrace{\mathcal{H}^{\otimes k} \oplus \cdots \oplus \mathcal{H}^{\otimes k}}_{N \text{ terms}} = (\mathcal{H}^{\otimes k})^{\oplus N}$$

where δ_{jl} is the Kronecker symbol. We deduce that the mapping

$$\phi_k \left(((\delta_{jl} e_\sigma)_{l=1}^N) \right) = e_{j\sigma} \quad (j = 1, \dots, N)$$

extends to a unitary operator from $(\mathcal{H}^{\otimes k})^{\oplus N}$ onto $\mathcal{H}^{\otimes k+1}$ and the formula

$$\psi_k = \phi_{k-1} \phi_{k-2}^{\oplus N} \cdots \phi_1^{\oplus N^{k-2}} \quad (k \geq 1)$$

gives a unitary operator from $\mathcal{H}_k = \mathcal{H}^{\oplus N^{k-1}}$ onto $\mathcal{H}^{\otimes k}$ (the notation $T^{\oplus l}$ means the direct sum of l copies of the operator T). We finally obtain a unitary operator $\psi : \bigoplus_{k=0}^{\infty} \mathcal{H}_k \rightarrow \mathcal{F}(\mathcal{H})$ by the formula $\psi = \bigoplus_{k=0}^{\infty} \psi_k$, where ψ_0 is the identity on $\mathbb{C} = \mathcal{H}_0$. Each \mathcal{H}_k ($k \geq 2$) is also identified with the direct sum of N copies of \mathcal{H}_{k-1} ,

$$\mathcal{H}_k = \underbrace{\mathcal{H}^{\oplus N^{k-2}} \oplus \dots \oplus \mathcal{H}^{\oplus N^{k-2}}}_{N \text{ terms}}$$

but the explicit mention of the identification can be omitted in this case. It is convenient to introduce a similar notation for an arbitrary Hilbert space \mathcal{E} . Thus, $\mathcal{E}_0 = \mathcal{E}$ and for $k \geq 1$

$$\mathcal{E}_k = \underbrace{\mathcal{E}_{k-1} \oplus \dots \oplus \mathcal{E}_{k-1}}_{N \text{ terms}} = \mathcal{E}_{k-1}^{\oplus N}.$$

Note that \mathcal{E}_k can be identified with $\mathcal{H}_k \otimes \mathcal{E}$.

An operator $T \in \mathcal{L}(\bigoplus_{k=0}^{\infty} \mathcal{H}_k)$ has a matrix representation $T = [T_{ij}]_{i,j=0}^{\infty}$ with $T_{ij} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i)$. We say that T is upper triangular provided that $T_{ij} = 0$ for $i > j$. We denote by $\mathcal{U}_T^0(\mathcal{H})$ the set of upper triangular operators $T = [T_{ij}]_{i,j=0}^{\infty}$ in $\mathcal{L}(\bigoplus_{k=0}^{\infty} \mathcal{H}_k)$ with the property that for $i \leq j$ and $i, j \geq 1$

$$T_{ij} = T_{i-1, j-1}^{\oplus N} \quad (2.1)$$

and there exists k such that $T_{0j} = 0$ for $j > k$. We notice that the entries T_{0j} ($j \geq 0$) determine the matrix T . Also, a simple calculation shows that $\mathcal{U}_T^0(\mathcal{H})$ is an associative algebra.

We define an algebra isomorphism

$$\Phi : \mathcal{T}(\mathcal{H}) \rightarrow \mathcal{U}_T^0(\mathcal{H}) \quad (2.2)$$

as follows: for $x \in \mathcal{T}(\mathcal{H})$, $x = (x_0, x_1, \dots)$, there are complex numbers c_σ , $\sigma \in \mathbb{F}_N^+$ such that

$$x_j = \sum_{|\sigma|=j} c_\sigma e_\sigma. \quad (2.3)$$

For $j \geq 0$, T_{0j} is the row matrix $[c_\sigma]_{|\sigma|=j}$. Then $T_{0j} = 0$ for sufficiently large j and we can define $T \in \mathcal{L}(\bigoplus_{k=0}^{\infty} \mathcal{H}_k)$ by using (2.1). Set $\Phi(x) = T$. Clearly, Φ is a bijection. In order to see that Φ is also an algebra morphism it is convenient to introduce the following operators. First, the operator C_j^- ($j = 1, \dots, N$) is an element of $\mathcal{U}_T^0(\mathcal{H})$ defined by the formula $C_j^- = \Phi(0, e_j, 0, \dots) = [T_{lk}^j]_{l,k=0}^{\infty}$, where $T_{0k}^j = 0$ for $k \neq 1$ and $T_{01}^j = e_j$. Then $C_j^+ = (C_j^-)^*$ ($j = 1, \dots, N$). If

we use the notation C_σ^- to denote the operator $C_{i_1}^- \dots C_{i_k}^-$ where $\sigma = i_1 \dots i_k$, then we deduce that each $T \in \mathcal{U}_T^0(\mathcal{H})$ has a representation

$$T = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma C_\sigma^-. \quad (2.4)$$

Now the required properties of Φ follow from the easily verifiable fact that

$$\Phi((0, e_j, 0, \dots) \otimes (0, e_k, 0, \dots)) = C_j^- C_k^-.$$

As it is well-known, the representation of $\mathcal{T}(\mathcal{H})$ as a subalgebra of the upper triangular algebra has several advantages. We mention the introduction of natural topologies on $\mathcal{T}(\mathcal{H})$ (for some recent papers, see [11, 12]). Among these topologies, the operator norm topology was studied in papers like [2, 9, 16]. Thus, we denote by $\mathcal{U}_T(\mathcal{H})$ the algebra of all those operators in $\mathcal{L}(\oplus_{k=0}^\infty \mathcal{H}_k)$ that satisfy condition (2.1). We denote by $\mathcal{S}(\mathcal{H})$ the Schur class of all contractions in $\mathcal{U}_T(\mathcal{H})$. It is convenient to extend this setting. If \mathcal{E}_1 and \mathcal{E}_2 are Hilbert spaces, then $\mathcal{U}_T(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ denotes the set of the operators $T = [T_{ij}]_{i,j=0}^\infty$ in $\mathcal{L}(\oplus_{k=0}^\infty (\mathcal{H}_k \otimes \mathcal{E}_1), \oplus_{k=0}^\infty (\mathcal{H}_k \otimes \mathcal{E}_2))$ obeying (2.1). Instead of $\mathcal{U}_T(\mathcal{H}, \mathcal{E}, \mathcal{E})$ we write $\mathcal{U}_T(\mathcal{H}, \mathcal{E})$. Also, $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ denotes the set of the contractions in $\mathcal{U}_T(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$. We notice that $\mathcal{U}_T(\mathcal{H}) = \mathcal{U}_T(\mathcal{H}, \mathbb{C}, \mathbb{C})$.

2.2 Displacement structure. The systematic study of the displacement structure was initiated in [13], and since then manifold applications have been found in the fields of mathematics and electrical engineering (for a recent survey, see [14]). Two main themes of this theory are about scattering experiments and recursive factorizations. A matrix A has displacement structure with respect to the generators F and G provided that

$$A - FAF^* = GJ_{pq}G^* \quad (2.5)$$

where $J_{pq} = I_p \oplus -I_q$ is a signature matrix that specifies the displacement inertia of A . In many applications, $r = p + q$ is much smaller than the size of A . There is also a version of this theory allowing A to depend on a (usually integer) parameter (see [5] or [14]). A major result concerning matrices with displacement structure is that, under suitable assumptions on the generators, the successive Schur complements of A inherit a similar structure. This allows us to write the Gaussian elimination for A only in terms of generators (see [13, 14]).

We now describe a connection between displacement structure and $\mathcal{U}_T(\mathcal{H})$. Let $C_k = [T_{ij}^k]_{i,j=0}^\infty$ ($k = 1, \dots, N$) be isometries defined by the formulae $T_{ij}^k = 0$ for $i \neq j + 1$ and let $T_{i+1,i}^k$ be a block-column matrix consisting of N blocks of dimension $\dim \mathcal{H}_i$, all of them zero except for the k th block which is $I_{\mathcal{H}_i}$. These operators are closely related to the operators C_k^+ , in the sense that there is a unitary operator u such that $C_k = uC_k^+u^*$ ($k = 1, \dots, N$).

The following result was noticed in [7].

Theorem 2.1. *Let $T = [T_{ij}]_{i,j=0}^{\infty} \in \mathcal{S}(\mathcal{H})$ and $A = I - T^*T$. Then*

$$A - \sum_{k=1}^N C_k A C_k^* = G J_{11} G^* \quad (2.6)$$

where the generator G is given by the formula

$$G = \begin{bmatrix} 1 & T_{00}^* \\ 0 & T_{01}^* \\ \vdots & \vdots \end{bmatrix}.$$

Equation (2.6) can be rewritten in the form of the time variant displacement equation

$$A - F A^{\oplus N} F^* = G J_{11} G^*$$

by setting $F = [C_1, \dots, C_N]$. We now discuss some additional examples.

2.2.1 Positive definite kernels on \mathbb{F}_N^+ . A mapping $K : \mathbb{F}_N^+ \times \mathbb{F}_N^+ \rightarrow \mathbb{C}$ is called a positive definite kernel on \mathbb{F}_N^+ provided that

$$\sum_{i,j=1}^k K(\sigma_i, \sigma_j) \lambda_j \bar{\lambda}_i \geq 0$$

for each positive integer k and each choice of words $\sigma_1, \dots, \sigma_k$ in \mathbb{F}_N^+ and complex numbers $\lambda_1, \dots, \lambda_k$. Without loss of generality, we assume $K(\emptyset, \emptyset) = 1$. The semigroup \mathbb{F}_N^+ acts on itself by juxtaposition and we assume that the positive definite kernel K is invariant under this action, that is

$$K(\tau\sigma, \tau\sigma') = K(\sigma, \sigma') \quad (\tau, \sigma, \sigma' \in \mathbb{F}_N^+).$$

By the invariant Kolmogorov decomposition theorem (see, e.g., [15: Chapter II]) there exists an isometric representation u of \mathbb{F}_N^+ on a Hilbert space \mathcal{K} and a mapping $v : \mathbb{F}_N^+ \rightarrow \mathcal{K}$ such that

$$\begin{aligned} K(\sigma, \tau) &= \langle v(\tau), v(\sigma) \rangle_{\mathcal{K}} \\ u(\tau)v(\sigma) &= v(\tau\sigma) \end{aligned}$$

for all $\sigma, \tau \in \mathbb{F}_N^+$, and the set $\{v(\sigma) \mid \sigma \in \mathbb{F}_N^+\}$ is total in \mathcal{K} . The mapping v is unique up to unitary equivalence and u is uniquely determined by v . Also, the previous representation of K implies, as noticed in [17], that $u(1), \dots, u(N)$ are isometries with orthogonal ranges if and only if $K(\sigma, \tau) = 0$ for all pairs (σ, τ) with the property that there is no $\alpha \in \mathbb{F}_N^+$ such that $\sigma = \tau\alpha$ or $\tau = \sigma\alpha$.

In this case, K is called in [17] a multi-Toeplitz kernel and we will use the same terminology. We can define

$$S_{ij} = [K(\sigma, \tau)]_{|\sigma|=i, |\tau|=j} \in \mathcal{L}(\mathcal{H}_j, \mathcal{H}_i). \quad (2.7)$$

By induction on the length of words and using properties of the lexicographic order on words we check that if K is a positive definite multi-Toeplitz kernel, then for all $i, j \geq 1$

$$S_{ij} = S_{i-1, j-1}^{\oplus N}. \quad (2.8)$$

This relation is an analogue of (2.1), therefore we expect that positive definite multi-Toeplitz kernels have displacement structure.

Theorem 2.2. *Let $S = [S_{ij}]_{i, j=0}^{\infty}$ be the array of operators associated to a positive definite multi-Toeplitz kernel on \mathbb{F}_N^+ . Then*

$$S - \sum_{k=1}^N C_k S C_k^* = G J_{11} G^* \quad (2.9)$$

where the generator G is given by the formula

$$G = \begin{bmatrix} 1 & 0 \\ S_{01}^* & S_{01}^* \\ \vdots & \vdots \end{bmatrix}.$$

The equality above can be interpreted in the sense of an equality of unbounded operators with dense domain made of the terminating sequences in $\oplus_{l=0}^{\infty} \mathcal{H}_l$.

Proof of Theorem 2.2. We write $S = L + Q$, where $L = [L_{ij}]_{i, j=0}^{\infty}$ and $Q = [Q_{ij}]_{i, j=0}^{\infty}$ are the lower and, respectively, the upper parts of S , such that $L_{ij} = S_{ij}$ for $i > j$ and $L_{ij} = 0$ for $i \leq j$, while $Q_{ij} = S_{ij}$ for $i \leq j$ and $Q_{ij} = 0$ for $i > j$. Since, formally (that is, computing the entries of the arrays that are involved according to the usual rule of multiplication of operators), $LC_k = C_k L$ and $QC_k^* = C_k^* Q$ for all $k = 1, \dots, N$, we deduce that

$$\begin{aligned} S - \sum_{k=1}^N C_k S C_k^* &= L + Q - \sum_{k=1}^N C_k (L + Q) C_k^* \\ &= L + Q - \sum_{k=1}^N C_k L C_k^* - \sum_{k=1}^N C_k Q C_k^* \\ &= L + Q - \sum_{k=1}^N L C_k C_k^* - \sum_{k=1}^N C_k C_k^* Q. \end{aligned}$$

But $I - \sum_{k=1}^N C_k C_k^* = P_{\mathcal{H}_0}$, the orthogonal projection on \mathcal{H}_0 , so we can deduce that

$$\begin{aligned} S - \sum_{k=1}^N C_k S C_k^* &= L + Q - L(I - P_{\mathcal{H}_0}) - (I - P_{\mathcal{H}_0})Q \\ &= LP_{\mathcal{H}_0} + P_{\mathcal{H}_0}Q \\ &= GJ_{11}G^* \end{aligned}$$

and the assertion is proved ■

We now consider truncations of a positive definite multi-Toeplitz kernel on \mathbb{F}_N^+ . If K is such a kernel and S_{ij} are associated to K by formula (2.7), then we define $K_n = [S_{ij}]_{i,j=0}^n$. These are positive operators on $\oplus_{l=0}^n \mathcal{H}_l$ and their entries satisfy (2.8). We also introduce the operators $C_{k,n} = P_{\oplus_{l=0}^n \mathcal{H}_l} C_k |_{\oplus_{l=0}^n \mathcal{H}_l}$.

We obtain another example of a displacement equation:

Corollary 2.3. *The positive operator K_n satisfies the displacement equation*

$$K_n - \sum_{k=1}^N C_{k,n} K_n C_{k,n}^* = GJ_{11}G^* \quad (2.10)$$

where the generator G is given by the formula

$$G = \begin{bmatrix} 1 & 0 \\ S_{01}^* & S_{01}^* \\ \vdots & \vdots \\ S_{0n}^* & S_{0n}^* \end{bmatrix}.$$

Proof. This is an immediate consequence of Theorem 2.2 ■

It turns out that the operators satisfying (2.10) have a certain explicit description. For $x \in \oplus_{l=0}^n \mathcal{H}_l$ we denote by $U(x)$ the upper triangular operator on $\oplus_{l=0}^n \mathcal{H}_l$ satisfying (2.8) and whose first row is x .

Theorem 2.4. *A positive operator A on $\oplus_{l=0}^n \mathcal{H}_l$ satisfies (2.10) if and only if $A = U(x)^*U(x) - U(y)^*U(y)$ for two vectors $x, y \in \oplus_{l=0}^n \mathcal{H}_l$ with the property that there is another vector $z \in \oplus_{l=0}^n \mathcal{H}_l$ such that $U(y) = U(z)U(x)$ and $U(z)$ is a contraction.*

Proof. If $A = U(x)^*U(x) - U(y)^*U(y)$, then we use the fact that, for any $x \in \oplus_{l=0}^n \mathcal{H}_l$ and $k = 1, \dots, N$, $C_{n,k}U(x)^* = U(x)^*C_{n,k}$ and deduce (2.10) as in the proof of Theorem 2.2. Conversely, if A satisfies (2.10), then we use the fact that $C_{\sigma,n} = 0$ for $|\sigma| > n$ in order to deduce that

$$A = GJ_{11}G^* + \sum_{|\sigma|=1}^n C_{\sigma,n} GJ_{11}G^* C_{\sigma,n}^*.$$

Hence, if $G = [x^*, y^*]$ for some $x, y \in \oplus_{l=0}^n \mathcal{H}_l$, then $A = U(x)^*U(x) - U(y)^*U(y)$. Since A is positive, it follows that there exists a contraction T such that $U(y) = TU(x)$. We notice that we can assume, without loss of generality, that $x_0 \neq 0$ and then we deduce that $C_{n,k}T^* = T^*C_{n,k}$ for $k = 1, \dots, N$, which implies that $T = U(z)$ for some $z \in \oplus_{l=0}^n \mathcal{H}_l$ ■

We mention a simple but useful consequence of this result. Let K be a positive definite multi-Toeplitz kernel on \mathbb{F}_N^+ with $K(\emptyset, \emptyset) = 1$ and let S_{ij} ($i, j \geq 0$) be the operators associated to K by (2.7). Let

$$\begin{aligned} x_n &= [1 \quad S_{01}^* \quad \dots \quad S_{0n}^*]^T \\ y_n &= [0 \quad S_{01}^* \quad \dots \quad S_{0n}^*]^T. \end{aligned}$$

Then $K_n = U(x_n)^*U(x_n) - U(y_n)^*U(y_n)$ and by the previous result there exists $z_n \in \oplus_{l=0}^n \mathcal{H}_l$ such that $U(y_n) = U(z_n)U(x_n)$. In fact, z_n is uniquely determined and also $P_{\oplus_{l=0}^n \mathcal{H}_l} U(z_{n+1})|_{\oplus_{l=0}^n \mathcal{H}_l} = U(z_n)$ for all $n \geq 1$. It follows that there exists a unique $T(K) \in \mathcal{S}(\mathcal{H})$ such that $P_{\oplus_{l=0}^n \mathcal{H}_l} T(K)|_{\oplus_{l=0}^n \mathcal{H}_l} = U(z_n)$ and $T(K)_{00} = 0$. Denote by $\mathcal{S}_0(\mathcal{H})$ the set of those elements $T \in \mathcal{S}(\mathcal{H})$ with $T_{00} = 0$.

Corollary 2.5. *The mapping $K \rightarrow T(K)$ gives a one-to-one correspondence between the set of positive definite multi-Toeplitz kernels on \mathbb{F}_N^+ with $K(\emptyset, \emptyset) = 1$ and $\mathcal{S}_0(\mathcal{H})$.*

2.2.2 Pick kernels. This type of kernels were recently introduced in [2, 17], and they were shown in [1] to have a certain universality property with respect to a Nevanlinna-Pick problem. Actually, these kernels belong to a class of kernels introduced in [4, 8, 19].

Let \mathcal{B}^N be the open unit ball in \mathbb{C}^N . For distinct points $\lambda_1, \dots, \lambda_L$ in \mathcal{B}^N and complex numbers b_1, \dots, b_L we define the Pick kernel by the formula

$$R = \left[\frac{1 - b_j \bar{b}_l}{1 - \langle \lambda_j, \lambda_l \rangle} \right]_{j,l=1}^L$$

where $\langle \lambda_j, \lambda_l \rangle$ denotes the Euclidean inner product in \mathbb{C}^N . We can define

$$F_j = \text{diag}(\lambda_l^{(j)})_{l=1}^L \quad (j = 1, \dots, N)$$

– the diagonal matrix with the diagonal made of the j th components of the points $\lambda_1, \dots, \lambda_L$. Also, we set

$$G = \begin{bmatrix} 1 & b_1 \\ \vdots & \vdots \\ 1 & b_L \end{bmatrix}.$$

We can check by direct computation that

$$R - \sum_{k=1}^N F_k R F_k^* = G J_{11} G^*. \quad (2.11)$$

It was shown in [7] how to use this equation in order to solve some multidimensional Nevanlinna-Pick type problems as those in [9, 17]. These problems were extended in [6] to the more natural setting of data given on the N -dimensional operator unit ball of a Hilbert space.

3. Scattering experiments

One of the main results in displacement structure theory refers to the possibility of associating a scattering operator to the data given by the generators of equation (2.5). In this section we prove a similar result for displacement equations of the type of (2.11) and we explain the connection with a special class of time variant linear systems.

Let $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{G} be Hilbert spaces. We consider a displacement equation of the form

$$A - \sum_{k=1}^N F_k A F_k^* = G J G^* \quad (3.1)$$

where $F_k \in \mathcal{L}(\mathcal{G})$ ($k = 1, \dots, N$) are given contractions on the Hilbert space \mathcal{G} . Also, $G = [U \ V] \in \mathcal{L}(\mathcal{E}_1 \oplus \mathcal{E}_2, \mathcal{G})$ and $J = I_{\mathcal{E}_1} \oplus (-I_{\mathcal{E}_2})$. The wave operators associated to (3.1) are introduced by the formulae $U_\infty^* = [U_k]_{k=0}^\infty$ and $V_\infty^* = [V_k]_{k=0}^\infty$ where $U_k = [F_\sigma U]_{|\sigma|=k} : \mathcal{E}_{1,k} \rightarrow \mathcal{G}$ and $V_k = [F_\sigma V]_{|\sigma|=k} : \mathcal{E}_{2,k} \rightarrow \mathcal{G}$. We will assume that both U_∞ and V_∞ are bounded and also that $\lim_{k \rightarrow \infty} \sum_{|\sigma|=k} \|F_\sigma^* g\| = 0$ for all $g \in \mathcal{G}$. Under these assumptions we deduce that (3.1) has a unique solution given by

$$A = U_\infty^* U_\infty - V_\infty^* V_\infty. \quad (3.2)$$

The next result was already noticed in [7], but here we present the details of a different approach based on system theoretic ideas.

Theorem 3.1. *Solution (3.2) of the displacement equation (3.1) is positive if and only if there exists $T \in \mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ such that $V_\infty = T U_\infty$.*

Proof. Assume $A = U_\infty^* U_\infty - V_\infty^* V_\infty \geq 0$ and let $A = LL^*$ be a factorization of A with $L \in \mathcal{L}(\mathcal{F}, \mathcal{G})$ for some Hilbert space \mathcal{F} . From (3.1) we deduce that

$$LL^* + VV^* = \sum_{k=1}^N F_k LL^* F_k^* + UU^*.$$

In matrix form,

$$[L \ V] \begin{bmatrix} L^* \\ V^* \end{bmatrix} = [F_1 L \ \dots \ F_N L \ U] \begin{bmatrix} L^* F_1^* \\ \vdots \\ L^* F_N^* \\ U^* \end{bmatrix}. \quad (3.3)$$

Defining $A^* = [L \ V]$ and $B^* = [F_1 L \ \dots \ F_N L \ U]$, we deduce from (3.3) that there exists a unitary operator $\theta_0 \in \mathcal{L}(\overline{\mathcal{R}(B)}, \overline{\mathcal{R}(A)})$ such that $A = \theta_0 B$. It follows that there exist Hilbert spaces \mathcal{R}_1 and \mathcal{R}_2 , and a unitary extension $\theta \in \mathcal{L}(\mathcal{F}^{\oplus N} \oplus \mathcal{E}_1 \oplus \mathcal{R}_1, \mathcal{F} \oplus \mathcal{E}_2 \oplus \mathcal{R}_2)$ of θ_0 , hence this extension satisfies the relation

$$\begin{bmatrix} A \\ 0_{\mathcal{R}_2} \end{bmatrix} = \theta \begin{bmatrix} B \\ 0_{\mathcal{R}_1} \end{bmatrix}. \quad (3.4)$$

Let θ_{ij} ($i \in \{1, 2, 3\}, j \in \{1, 2, \dots, N+2\}$) be the matrix coefficients of θ . It is convenient to rename some of these coefficients. Thus, we set

$$\begin{aligned} X_k &= \theta_{1k} \quad (k = 1, \dots, N), & Z &= \theta_{1, N+1} \\ Y_k &= \theta_{2k} \quad (k = 1, \dots, N), & W &= \theta_{2, N+1}. \end{aligned}$$

From (3.4) we deduce that

$$\begin{aligned} L^* &= \sum_{k=1}^N X_k L^* F_k^* + Z U^* \\ V^* &= \sum_{k=1}^N Y_k L^* F_k^* + W V^*. \end{aligned}$$

By induction we deduce that

$$V^* = W U^* + \sum_{k=1}^N \sum_{|\sigma|=0}^n Y_k X_\sigma Z U^* F_{k\sigma}^* + \sum_{|\tau|=n+1} Q_\tau L^* F_\tau^* \quad (3.5)$$

where Q_τ are monomials of length $|\tau|$ in the variables $X_1, \dots, X_N, Y_1, \dots, Y_N$. Since θ is unitary it follows that all Q_τ are contractions.

We define $T_{00} = W$ and, for $j > 0$,

$$T_{0j} = [Y_k X_\sigma Z]_{|\sigma|=j-1; k=1, \dots, N}.$$

Then we define T_{ij} ($i > 0, j \geq i$) by formula (2.1) and $T_{ij} = 0$ for $i > j$. We show that $T = [T_{ij}]_{i,j=0}^\infty$ belongs to $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$. To that end, we introduce the following notation for $n \geq 0$:

$$\begin{aligned} X_1(n) &= [X_1 \ \dots \ X_N]^{\oplus N^n} & Y_1(n) &= [Y_1 \ \dots \ Y_N]^{\oplus N^n} \\ X_2(n) &= Z^{\oplus N^n} & Y_2(n) &= W^{\oplus N^n}. \end{aligned}$$

We check by induction that

$$\begin{aligned} T_{00} &= Y_2(0) \\ T_{01} &= Y_1(0)X_2(1) \\ T_{0j} &= Y_1(0)X_1(1) \cdots X_1(j-1)X_2(j) \quad (j \geq 2) \end{aligned}$$

Consequently, T^* is the transfer map of the linear time variant system

$$\begin{cases} x(n+1) = X_1(n)^*x(n) + Y_1(n)^*u(n) \\ y(n) = X_2(n)^*x(n) + Y_2(n)^*u(n) \end{cases} \quad (n \geq 0).$$

Since each matrix $\begin{bmatrix} X_1(n) & X_2(n) \\ Y_1(n) & Y_2(n) \end{bmatrix}$ ($n \geq 0$) is a contraction, it follows that T^* is a contraction, hence T is a contraction and $T \in \mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$. Since $\lim_{k \rightarrow \infty} \sum_{|\sigma|=k} \|F_\sigma^* g\| = 0$ for all $g \in \mathcal{G}$, we deduce from (3.5) that $V_\infty = TU_\infty$ ■

This result is a generalization of Theorem 3.1. It can be used to solve interpolation and moment problems in several non-commuting variables as suggested in [7]. Details can be found in [6]. The proof given here emphasizes the fact that this type of problems appears as an interesting particular case of similar questions in time variant setting (as described, for instance, in [5]). Next we develop this idea by giving a parametrization of the elements in $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$.

3.1 Schur Parameters. We introduce a Schur type parametrization for $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ and then give a number of applications. A first instance of such a result might be considered Euler's description of $SO(3)$. Schur's classical result in [18] gives a parametrization of the contractive holomorphic functions on the unit disk and Szegő's theory of orthogonal polynomials provides an alternative to this result for probability measures on the unit circle (see [3, 10] for more details).

Here we provide a parametrization of $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ in terms of so-called Schur parameters. For a contraction C we denote by \mathcal{D}_C its defect space, that is the closure of the range of the defect operator $D_C = (I - C^*C)^{1/2}$. Let $\Pi(\mathcal{E}_1, \mathcal{E}_2)$ be the family of sequences of contractions $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ with the properties $\gamma_\emptyset \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ and for $|\sigma| > 0$

$$\gamma_{k\sigma} \in \mathcal{L}(\mathcal{D}_{\gamma_\sigma}, \mathcal{D}_{\gamma_{k\sigma-1}^*}) \quad (3.6)$$

where $k = 1, \dots, N$ and $k\sigma - 1$ denotes the predecessor of $k\sigma$ with respect to the lexicographic order.

Theorem 3.2. *There exists a one-to-one correspondence between $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ and $\Pi(\mathcal{E}_1, \mathcal{E}_2)$.* ■

Proof. A simple proof can be obtained as an application of the Schur parametrization of upper triangular contractions in [5]. Let $T \in \mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ and, for each $j \geq 0$, T_{0j} is the row matrix $[c_\sigma]_{|\sigma|=j}$. We use [5: Theorem 2.2.1] in order to associate to T a family Γ of Schur parameters. The Schur parameters of the contraction

$$\begin{bmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{bmatrix} = \begin{bmatrix} c_\emptyset & c_1 & \dots & & c_N \\ 0 & c_\emptyset & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & & c_\emptyset & 0 \\ 0 & & \dots & & c_\emptyset \end{bmatrix}$$

are given by $\Gamma_{00} = \gamma_\emptyset = c_\emptyset$, $\Gamma_{11} = \Gamma_{00}^{\oplus N}$ and $\Gamma_{01} \in \mathcal{L}(\mathcal{D}_{\gamma_\emptyset}^{\oplus N}, \mathcal{D}_{\gamma_\emptyset})$.

For the next step we associate Schur parameters to the contraction

$$\begin{bmatrix} T_{00} & T_{01} & T_{02} \\ 0 & T_{11} & T_{12} \\ 0 & 0 & T_{22} \end{bmatrix}.$$

We notice that

$$\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} = \begin{bmatrix} T_{00}^{\oplus N} & T_{01}^{\oplus N} \\ 0 & T_{11}^{\oplus N} \end{bmatrix}.$$

By [5: Formula (2.2.3)], the entries of $\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$ are polynomials in the Schur parameters, their adjoints and their defect operators. Consequently, the contraction associated to the Schur parameters $\Gamma_{00}^{\oplus N}, \Gamma_{01}^{\oplus N}, \Gamma_{11}^{\oplus N}$ is precisely

$$\begin{bmatrix} T_{00}^{\oplus N} & T_{01}^{\oplus N} \\ 0 & T_{11}^{\oplus N} \end{bmatrix}.$$

From the uniqueness of the Schur parameters, we deduce that the Schur parameters of $\begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}$ are precisely $\Gamma_{00}^{\oplus N}, \Gamma_{01}^{\oplus N}, \Gamma_{11}^{\oplus N}$. By induction, we deduce that we can associate to T the family of Schur parameters $\Gamma = \{\Gamma_{ij}\}_{i \leq j}$ such that each Γ_{ij} satisfies (2.1). We notice that $\Gamma_{0j} : \mathcal{D}_{\Gamma_{0,j-1}}^{\oplus N} \rightarrow \mathcal{D}_{\Gamma_{0,j-1}^*}$ for $j \geq 1$. Also, each Γ_{0j} is a row contraction, $\Gamma_{0j} = [\gamma_\sigma^0]_{|\sigma|=j}$. By [5: Proposition 1.4.2],

$$\Gamma_{01} = \begin{bmatrix} \gamma_1 & D_{\gamma_1^*} \gamma_2 & \dots & D_{\gamma_1^*} \dots D_{\gamma_{N-1}^*} \gamma_N \end{bmatrix}$$

where $\gamma_k \in \mathcal{L}(\mathcal{D}_{\gamma_\emptyset}, \mathcal{D}_{\gamma_{k-1}^*})$ ($k = 1, \dots, N$). Also, there exist unitary operators $u_1 : \mathcal{D}_{\Gamma_{01}} \rightarrow \bigoplus_{k=1}^N \mathcal{D}_{\gamma_k}$ and $v_1 : \mathcal{D}_{\Gamma_{01}^*} \rightarrow \mathcal{D}_{\gamma_N^*}$. Next, we have that $\Gamma_{02} :$

$\mathcal{D}_{\Gamma_{01}}^{\oplus N} \rightarrow \mathcal{D}_{\Gamma_{01}^*}$ and we can define the row contraction $\tilde{\Gamma}_{02} = [\gamma_\sigma^0]_{|\sigma|=2}$ from $(\bigoplus_{k=1}^N \mathcal{D}_{\gamma_k})^{\oplus N}$ into $\mathcal{D}_{\gamma_N^*}$ by the formula $\tilde{\Gamma}_{02} = v_1 \Gamma_{02} (u_1^{\oplus N})^*$. As above,

$$\tilde{\Gamma}_{02} = \left[\gamma_{11} \ D_{\gamma_{11}^*} \ \gamma_{12} \ \dots \ D_{\gamma_{11}^*} \ \dots \ D_{\gamma_{N,N-1}^*} \ \gamma_{NN} \right]$$

where $\gamma_{k\tau} \in \mathcal{L}(\mathcal{D}_{\gamma_\tau}, \mathcal{D}_{\gamma_{k\tau-1}^*})$ ($k = 1, \dots, N; |\tau| = 1$). We proceed by induction in order to construct the whole family of Schur parameters $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ with $\gamma_\emptyset \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ and obeying (3.6) ■

The parameters $\{\Gamma_{0j}\}_{j \geq 0}$ are the Schur parameters associated to T in [16]. The above proof explains that they are just a particular case of more general constructions as those in [5]. The new parameters γ_σ are more like "true" Schur parameters in the sense that they are contractions subject to the only restrictions given by compatibility relations (3.6). Still, the parameters $\{\Gamma_{0j}\}_{j \geq 0}$ are useful since they preserve the natural filtration of $\mathcal{U}_T(\mathcal{H})$.

A similar parametrization can be obtained for the set of positive multi-Toeplitz kernels. Denote by $\Pi_0(\mathcal{E}_1, \mathcal{E}_2)$ the set of families $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ in $\Pi(\mathcal{E}_1, \mathcal{E}_2)$ ■ with $\gamma_\emptyset = 0$.

Theorem 3.3. *There exists a one-to-one correspondence between the set of \mathcal{E} -valued positive multi-Toeplitz kernels with $K(\emptyset, \emptyset) = 1$ and $\Pi_0(\mathcal{E})$.*

Proof. This is a consequence of Theorem 3.2 and Corollary 2.5. A direct proof can be obtained as an application of [5: Theorem 1.5.3] ■

This shows that the main result of [17] is just a particular case of [5: Theorem 1.5.3], consequently all the other results in [17] follow from corresponding results in [5]. An application of the Schur type parametrization gives another representation of the class $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$. Motivated by the proof of Theorem 3.1 we consider the class $\mathcal{BS}_N(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ of systems

$$\begin{cases} x(n+1) = A(n)x(n) + B(n)u(n) \\ y(n) = C(n)x(n) + D(n)u(n) \end{cases} \quad (n \geq 0) \quad (\Sigma)$$

where $\mathcal{E}_1, \mathcal{E}_2$ and \mathcal{F} are Hilbert spaces, $A(0) \in \mathcal{L}(\mathcal{F}, \mathcal{F}^{\oplus N})$, $B(0) \in \mathcal{L}(\mathcal{E}_2, \mathcal{F}^{\oplus N})$, $C(0) \in \mathcal{L}(\mathcal{F}, \mathcal{E}_1)$, $D \in \mathcal{L}(\mathcal{E}_2, \mathcal{E}_1)$, and $A(n) = A(n-1)^{\oplus N}$, $B(n) = B(n-1)^{\oplus N}$, $C(n) = C(n-1)^{\oplus N}$, $D(n) = D(n-1)^{\oplus N}$ (these relations are all of the type of (2.1)). Also, we assume that for each $n \geq 0$ the operator $\begin{bmatrix} A(n) & B(n) \\ C(n) & D(n) \end{bmatrix}$ is a coisometry. The evolution of the system Σ can be encoded by the transfer map T_Σ that shows how the infinite input sequence $(u(0), u(1), \dots)$ is transformed by the system into an output sequence $(y(0), y(1), \dots)$ provided that

$x(0) = 0$. It is easy to see that the transfer map T_Σ is a lower triangular contraction with the matrix coefficients

$$\begin{aligned} T_\Sigma(n, n) &= D(n) \\ T_\Sigma(n+1, n) &= C(n+1)B(n) \\ T_\Sigma(n+j, n) &= C(n+j)A(n+j-1)\cdots A(n+1)B(n) \quad (j > 1). \end{aligned}$$

Theorem 3.4. *T belongs to $\mathcal{S}(\mathcal{H}, \mathcal{E}_1, \mathcal{E}_2)$ if and only if T^* is the transfer map of a system in $\mathcal{BS}_N(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ for some Hilbert space \mathcal{F} .*

Proof. We use the construction from the proof of [5: Theorem 2.3.3]. Thus, let $\{\Gamma_{ij}\}_{i \leq j}$ be the Schur parameters associated to T in the proof of Theorem 3.2. For each $i \geq 0$ we consider the row contraction

$$L_i = \left[\Gamma_{ii} \ D_{\Gamma_{ii}^*} \Gamma_{i,i+1} \ D_{\Gamma_{ii}^*} D_{\Gamma_{i,i+1}^*} \Gamma_{i,i+2} \ \cdots \right].$$

Let V_i be the isometry associated to L_i by [5: Formula 1.6.3], that is,

$$V_i = \begin{bmatrix} \Gamma_{ii} & D_{\Gamma_{ii}^*} \Gamma_{i,i+1} & D_{\Gamma_{ii}^*} D_{\Gamma_{i,i+1}^*} \Gamma_{i,i+2} & \cdots \\ D_{\Gamma_{ii}^*} & -\Gamma_{ii}^* \Gamma_{i,i+1} & -\Gamma_{ii}^* D_{\Gamma_{i,i+1}^*} \Gamma_{i,i+2} & \cdots \\ 0 & D_{\Gamma_{i,i+1}^*} & -\Gamma_{i,i+1}^* \Gamma_{i,i+2} & \cdots \\ 0 & 0 & D_{\Gamma_{i,i+2}^*} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} \Gamma_{ii} & B(i)^* \\ C(i)^* & A(i)^* \end{bmatrix}. \quad (3.7)$$

In particular, the operator V_0 is defined on the space $\mathcal{E}_1 \oplus \bigoplus_{k=0}^{\infty} \mathcal{D}_{\Gamma_{0k}^{\oplus N}}$ with values in $\mathcal{E}_2 \oplus \bigoplus_{k=0}^{\infty} \mathcal{D}_{\Gamma_{0k}}$. We set $\mathcal{F} = \bigoplus_{k=0}^{\infty} \mathcal{D}_{\Gamma_{0k}}$ and let $\phi : (\bigoplus_{k=0}^{\infty} \mathcal{D}_{\Gamma_{0k}})^{\oplus N} \rightarrow \bigoplus_{k=0}^{\infty} \mathcal{D}_{\Gamma_{0k}^{\oplus N}}$ be the unitary operator defined by the formula $\phi(\bigoplus_{l=1}^N \bigoplus_{k=0}^{\infty} u_{kl}) = \bigoplus_{k=0}^{\infty} \bigoplus_{l=1}^N u_{kl}$. Then we define

$$\begin{bmatrix} \bar{D}^*(i) & \bar{B}^*(i) \\ \bar{C}^*(i) & \bar{A}^*(i) \end{bmatrix} = \begin{bmatrix} \Gamma_{ii} & B^*(i)\phi^{\oplus N^i} \\ (\phi^{\oplus N^{i-1}})^* C^*(i) & (\phi^{\oplus N^{i-1}})^* A^*(i)\phi^{\oplus N^i} \end{bmatrix}.$$

By [5: Theorem 2.3.3], $T_{i,i+1}^* = C(i+1)B(i) = \bar{C}(i+1)\bar{B}(i)$ and for $j > i$

$$T_{ij}^* = C(j)A(j-1)\cdots A(i+1)B(i) = \bar{C}(j)\bar{A}(j-1)\cdots \bar{A}(i+1)\bar{B}(i). \quad (3.8)$$

Since the elements of the family Γ satisfy (2.1) and the matrix coefficients of V_i are monomials in Γ_{ij} , their adjoints and defect operators, it follows that the system associated to the families $\{\bar{A}(n)\}_{n \geq 0}$, $\{\bar{B}(n)\}_{n \geq 0}$, $\{\bar{C}(n)\}_{n \geq 0}$, $\{\bar{D}(n)\}_{n \geq 0}$ belongs to $\mathcal{BS}_N(\mathcal{E}_1, \mathcal{E}_2, \mathcal{F})$ and T^* is the transfer map of this system ■

4. Recursive constructions

As noticed in the proof of Theorem 3.4, the Schur parameters introduced in Theorem 3.2 are basically given by the parametrization of the unitary group. A convenient way to encode the dependence of the matrix coefficients of T on the Schur parameters is given by a continued fraction algorithm introduced for $N = 1$ by Schur in [18]. Here we describe such an algorithm for $N > 1$. Of course, the main difficulty consists in keeping all the constructions within the tensor algebra. Due to the fact that there is a standard filtration on the tensor algebra, the constructions do not go through as in the one-dimensional case, still we can obtain a sort of graded Schur-type algorithm. For the algebra of upper triangular matrices, a Schur-type algorithm was known (as a consequence of a similar algorithm for the spectral factorization of a family of matrices with displacement structure (see [14] or [5] for a detailed discussion). If we use it directly to our situation, it will produce the parameters $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ of Theorem 3.2, but the objects produced at each step are no longer in the tensor algebra. In this section we show how to remedy this situation.

We first notice another useful representation of a transfer map (we will assume in this section that $\mathcal{E}_1 = \mathcal{E}_1 = \mathbb{C}$). It is convenient to extend the symbol map (2.2). Thus, for $T \in \mathcal{L}(\oplus_{k=0}^{\infty} \mathcal{E}_{1,k}, \oplus_{k=0}^{\infty} \mathcal{E}_{2,k})$ an upper triangular operator satisfying (2.1), we write

$$\Phi([T_{0j}]_{j=0}^{\infty}) = T.$$

Now, if T_Σ is the transfer map of a system Σ in $\mathcal{BS}_N(\mathcal{F}) = \mathcal{BS}_N(\mathbb{C}, \mathbb{C}, \mathcal{F})$, then

$$\begin{aligned} T_\Sigma^* &= \Phi(D(0)^*, 0, \dots) + \Phi(0, B(0)^*, 0, \dots) \\ &\quad \times (I - \Phi(0, A(0)^*, 0, \dots))^{-1} \Phi(C(0)^*, 0, \dots) \end{aligned}$$

provided that $\|\Phi(0, A(0)^*, 0, \dots)\| < 1$. We notice that this is the case if and only if $\|A(0)\| < 1$. Let $A(0)^* = [A_j(0)^*]_{j=1}^N$ and $B(0)^* = [B_j(0)^*]_{j=1}^N$. Then

$$\begin{aligned} \Phi(0, A(0)^*, 0, \dots) &= \sum_{j=1}^N \Phi(A_j(0)^*, 0, \dots) C_j^- \\ \Phi(0, B(0)^*, 0, \dots) &= \sum_{j=1}^N \Phi(B_j(0)^*, 0, \dots) C_j^- \end{aligned}$$

so that

$$\begin{aligned} T_\Sigma^* &= \Phi(D(0)^*, 0, \dots) + \left(\sum_{j=1}^N \Phi(B_j(0)^*, 0, \dots) C_j^- \right) \\ &\quad \times \left(I - \sum_{j=1}^N \Phi(A_j(0)^*, 0, \dots) C_j^- \right)^{-1} \Phi(C(0)^*, 0, \dots) \end{aligned} \tag{4.1}$$

provided that $\|A(0)\| < 1$.

Next, we prove a refinement of (4.1) which is the key step of a Schur-type algorithm. We introduce the operator given by the matrix

$$S = \begin{bmatrix} 0 & 1 & 0 & & \dots \\ 0 & 0 & 1 & & \dots \\ & 0 & 0 & 1 & \dots \\ \vdots & & & \ddots & \ddots \end{bmatrix}.$$

Theorem 4.1. *If T belongs to $\mathcal{S}(\mathcal{H})$, then there exists a unique $T_1 \in \mathcal{S}(\mathcal{H}, \mathcal{D}_{T_{00}}^{\oplus N}, \mathcal{D}_{T_{00}^*})$ such that*

$$T = \Phi(T_{00}, 0, \dots) + \Phi(D_{T_{00}^*}, 0, \dots) T_1 S (I - \Phi(-T_{00}^*, 0, \dots) T_1 S)^{-1} \Phi(D_{T_{00}}, 0, \dots). \blacksquare$$

Proof. Let $\{\Gamma_{ij}\}_{i \leq j}$ be the Schur parameters associated to T as in the proof of Theorem 3.2. Then $|\Gamma_{00}| \leq 1$. If $|\Gamma_{00}| = 1$, then we must have $\Gamma_{ij} = 0$ for $j > i$, so that $T = \Phi(\Gamma_{00}, 0, \dots)$ and $T_1 = 0$. Assume $|\Gamma_{00}| < 1$. Then T^* is the transfer map of the system associated to the operators V_i defined by (3.7). Thus, we have

$$\begin{aligned} V_0 &= \begin{bmatrix} \Gamma_{00} & B(0)^* \\ C(0)^* & A(0)^* \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{00} & D_{\Gamma_{00}^*} \Gamma_{01} & D_{\Gamma_{00}^*} D_{\Gamma_{01}^*} \Gamma_{02} & \dots \\ D_{\Gamma_{00}} & -\Gamma_{00}^* \Gamma_{01} & -\Gamma_{00}^* D_{\Gamma_{01}^*} \Gamma_{02} & \dots \\ 0 & D_{\Gamma_{01}} & -\Gamma_{01}^* \Gamma_{02} & \dots \\ 0 & 0 & D_{\Gamma_{02}} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{00} & D_{\Gamma_{00}^*} & 0 & \dots \\ D_{\Gamma_{00}} & -\Gamma_{00}^* & 0 & \dots \\ 0 & 0 & I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} I & 0 & 0 & \dots \\ 0 & \Gamma_{01} & D_{\Gamma_{01}^*} \Gamma_{02} & \dots \\ 0 & D_{\Gamma_{01}} & -\Gamma_{01}^* \Gamma_{02} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{00} & D_{\Gamma_{00}^*} & 0 \\ D_{\Gamma_{00}} & -\Gamma_{00}^* & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & \Gamma_{01} & B'(0)^* \\ 0 & C'(0)^* & A'(0)^* \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_{00} & D_{\Gamma_{00}^*} \Gamma_{01} & D_{\Gamma_{00}^*} B'(0)^* \\ D_{\Gamma_{00}} & -\Gamma_{00}^* \Gamma_{01} & -\Gamma_{00}^* B'(0)^* \\ 0 & C'(0)^* & A'(0)^* \end{bmatrix}. \end{aligned}$$

We assume first that $\|A(0)\| < 1$. Then formula (4.1) gives

$$\begin{aligned} T &= \Phi(\Gamma_{00}, 0, \dots) + \Phi(0, B(0)^*, 0, \dots) (I - \Phi(0, A(0)^*, 0, \dots))^{-1} \Phi(C(0)^*, 0, \dots) \\ &= \Phi(\Gamma_{00}, 0, \dots) + \left[\Phi(0, D_{\Gamma_{00}^*} \Gamma_{01}, 0, \dots) \quad \Phi(0, D_{\Gamma_{00}^*} B'(0)^*, 0, \dots) \right] \\ &\quad \times \begin{bmatrix} I - \Phi(0, -\Gamma_{00}^* \Gamma_{01}, 0, \dots) & -\Phi(0, -\Gamma_{00}^* B'(0)^*, 0, \dots) \\ -\Phi(0, C'(0)^*, 0, \dots) & I - \Phi(0, A'(0)^*, 0, \dots) \end{bmatrix}^{-1} \begin{bmatrix} \Phi(D_{\Gamma_{00}}, 0, \dots) \\ 0 \end{bmatrix}. \end{aligned}$$

Since $\|A(0)\| < 1$, we also have $\|A'(0)\| < 1$ so that we can use a well-known formula for the inversion of a 2×2 matrix in order to deduce that

$$\begin{aligned} T &= \Phi(\Gamma_{00}, 0, \dots) + \Phi(D_{\Gamma_{00}^*}, 0, \dots) \left(\Phi(0, \Gamma_{00}, 0, \dots) \Delta^{-1} \right. \\ &\quad \left. + \Phi(0, B'(0)^*, 0, \dots) (I - \Phi(0, A'(0)^*, 0, \dots))^{-1} \right. \\ &\quad \left. \times \Phi(0, C'(0)^*, 0, \dots) \Delta^{-1} \right) \Phi(D_{\Gamma_{00}}, 0, \dots) \end{aligned}$$

where

$$\begin{aligned} \Delta &= I - \Phi(0, -\Gamma_{00}^* \Gamma_{01}, 0, \dots) - \Phi(0, -\Gamma_{00}^* B'(0)^*, 0, \dots) \\ &\quad \times (I - \Phi(0, A'(0)^*, 0, \dots))^{-1} \Phi(0, C'(0)^*, 0, \dots) \\ &= I - \Phi(0, -\Gamma_{00}^*, 0, \dots) \left(\Phi(0, -\Gamma_{00}, 0, \dots) \right. \\ &\quad \left. + \Phi(0, B'(0)^*, 0, \dots) (I - \Phi(0, A'(0)^*, 0, \dots))^{-1} \Phi(0, C'(0)^*, 0, \dots) \right). \end{aligned}$$

Since $\|\Phi(0, -\Gamma_{00}^*, 0, \dots)\| < 1$, Δ is invertible and the previous calculation makes sense. Also, we define

$$\begin{aligned} R &= \Phi(0, \Gamma_{01}, 0, \dots) + \Phi(0, B'(0)^*, 0, \dots) \\ &\quad \times (I - \Phi(0, A'(0)^*, 0, \dots))^{-1} \Phi(0, C'(0)^*, 0, \dots). \end{aligned}$$

Since $\begin{bmatrix} \Gamma_{01} & B'(0)^* \\ C'(0)^* & A'(0)^* \end{bmatrix}$ is a contraction (in fact, it is an isometry), we can easily check that $R \in \mathcal{S}_0(\mathcal{H})$. Actually, we can notice that R is obtained from the Schur parameters $\{\tilde{\Gamma}_{ij}\}_{i \geq j}$, where $\tilde{\Gamma}_{ii} = 0$ and $\tilde{\Gamma}_{ij} = \Gamma_{ij}$ for $j > i$. We obtain that

$$T = \Phi(\Gamma_{00}, 0, \dots) + \Phi(D_{\Gamma_{00}^*}, 0, \dots) R (I - \Phi(-\Gamma_{00}^*, 0, \dots) R)^{-1} \Phi(D_{\Gamma_{00}}, 0, \dots). \quad (4.2)$$

Since $\|\Phi(-\Gamma_{00}^*, 0, \dots)\| < 1$, the previous formula makes sense without our assumption that $\|A(0)\| < 1$. In fact, an approximation argument shows that (4.2) holds without that assumption. Thus, let $\rho \in (0, 1)$ and consider T_ρ^* – the transfer map of the system $\{\rho A(n), B(n), C(n), D(n)\}$ (not necessarily in $\mathcal{BS}_N(\mathcal{H})$). All the previous calculations go through and we obtain a contraction R_ρ such that (4.2) holds for T_ρ . Letting $\rho \rightarrow 1$ we obtain (4.2) in any case ■

This result leads to the following Schur-type algorithm for a $T \in \mathcal{S}(\mathcal{H})$:

$$\begin{cases} T_0 = T, \gamma_n = T_{n,00} \quad (n \geq 0) \\ T_{n+1} = \text{the unique solution of the equation} \\ T_n = \Phi(\gamma_n, 0, \dots) + \Phi(D_{\gamma_n^*}, 0, \dots)T_{n+1}S^{\oplus \sum_{k=0}^n N^k} \\ (I - \Phi(-\gamma_n^*, 0, \dots)T_{n+1}S^{\oplus \sum_{k=0}^n N^k})^{-1}\Phi(D_{\gamma_n}, 0, \dots). \end{cases}$$

The algorithm generates a sequence of elements $T_n \in \mathcal{S}(\mathcal{H}, \mathcal{D}_{T_{n-1,00}^{\oplus N}}, \mathcal{D}_{T_{n-1,00}^*})$ and it is easily seen that $\gamma_n = \Gamma_{0n}$ ($n \geq 0$). Therefore $\{\gamma_n\}_{n \geq 0}$ uniquely determine T and the algorithm encodes the dependence of T on these parameters.

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