Quaternionic Reformulation of Maxwell Equations for Inhomogeneous Media and New Solutions

V. V. Kravchenko

Abstract. We propose a simple quaternionic reformulation of Maxwell equations for inhomogeneous media and use it in order to obtain new solutions in a static case.

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1. Introduction

The algebra of quaternions was applied to the study of the Maxwell equations starting from the work of J. C. Maxwell himself. The standard reference for the quaternionic reformulation of the Maxwell equations is the work [4], where the Maxwell equations for vacuum were written in a simple and compact form. In this relation we should mention also the earlier article [13]. Some new integral representations for electromagnetic quantities based on the idea of quaternionic diagonalization of the Maxwell equations for homogeneous media were obtained in [6] (see also [12: Chapter 2]). A review of different applications of quaternionic analysis to the Maxwell equations can be found in [3]. In the recent work [9] with the aid of quaternionic analysis techniques the Maxwell equations for inhomogeneous but slowly changing media were diagonalized and new solutions obtained. Nevertheless, even the question how to write the Maxwell equations for arbitrary inhomogeneous media in a compact quaternionic form remained open. In the present work we propose such a reformulation and use it for obtaining new results in the static case.

V. V. Kravchenko: Escuela Sup. de Ing. y Eléc. del Inst. Polit. Nac., Dept. de Telecom., C.P. 07738, D.F., México; vkravche@maya.esimez.ipn.mx

2. Preliminaries

We denote by $\mathbb{H}(\mathbb{C})$ the algebra of complex quaternions (= biquaternions). The elements of $\mathbb{H}(\mathbb{C})$ are represented in the form $q = \sum_{k=0}^{3} q_k i_k$, where $q_k \in \mathbb{C}$, i_0 is the unit and i_k $(k = 1, 2, 3)$ are standard quaternionic imaginary units. We will use also the vector representation of complex quaternions $q = q_0 + \vec{q}$, where $\vec{q} = \sum_{k=1}^{3} q_k i_k$. The vector parts of complex quaternions we identify with vectors from $\mathbb{C}^3$. The product of two biquaternions can be written in the form

$$p \cdot q = p_0 q_0 - \langle \vec{p}, \vec{q} \rangle + [\vec{p} \times \vec{q}] + p_0 \vec{q} + q_0 \vec{p}$$

where $\langle \vec{p}, \vec{q} \rangle$ and $[\vec{p} \times \vec{q}]$ denote the usual scalar and vector products, respectively. We will use the notations $^{p}Mq = p \cdot q$ and $M^{p}q = q \cdot p$ for the operators of multiplication from the left-hand and right-hand sides, respectively.

Remark 1. The scalar product of vectors $\vec{p}$ and $\vec{q}$ can be represented as $\langle \vec{p}, \vec{q} \rangle = -\frac{1}{2}(^{\vec{p}}M + M^{\vec{p}})\vec{q}$.

On the set of differentiable $\mathbb{H}(\mathbb{C})$-valued functions the Moisil-Theodoresco operator is defined by the expression $Df = \sum_{k=1}^{3} i_k \partial_k f$. In vector form this expression can be written as

$$Df = -\operatorname{div} \vec{f} + \operatorname{grad} f_0 + \operatorname{rot} \vec{f}$$

where the first term is the scalar part of the biquaternion Df and the last two terms represent its vector part.

Let us note some simple properties of the operator D which will be used in this work. Let φ be a scalar function and f be an $\mathbb{H}(\mathbb{C})$-valued function. Then

$$D(\varphi \cdot f) = D\varphi \cdot f + \varphi \cdot Df$$

and

$$\left(D - \frac{\operatorname{grad} \varphi}{\varphi}\right) f = \varphi D\left(\frac{1}{\varphi} f\right). \tag{1}$$

3. Quaternionic reformulation of the Maxwell equations

The Maxwell equations for complex amplitudes of the time-harmonic electro-magnetic field have the form

$$\operatorname{div}(\varepsilon\vec{E}) = \rho, \qquad\qquad \operatorname{div}(\mu\vec{H}) = 0 \qquad\qquad (2)$$
$$\operatorname{rot}\vec{H} = i\omega\varepsilon\vec{E} + \vec{j} \qquad \operatorname{rot}\vec{E} = -i\omega\mu\vec{H} \qquad\qquad (3)$$

where $\vec{E}$ and $\vec{H}$ are $\mathbb{C}^3$-valued functions, ω is the frequency, ε is the permittivity and μ is the permeability of the medium. We suppose ε and μ to be two times differentiable complex-valued functions with respect to each coordinate x_k $(k = 1, 2, 3)$. Note that they are always different from zero.

Equations (2) can be rewritten as

$$\left.\begin{aligned}
\operatorname{div}\vec{E} + \left\langle \frac{\operatorname{grad}\varepsilon}{\varepsilon}, \vec{E}\right\rangle &= \frac{\rho}{\varepsilon}\\
\operatorname{div}\vec{H} + \left\langle \frac{\operatorname{grad}\mu}{\mu}, \vec{H}\right\rangle &= 0
\end{aligned}\right\}.$$

Combining these equations with (3) we have the Maxwell equations in the form

$$\left.\begin{aligned}
D\vec{E} &= \left\langle \frac{\operatorname{grad}\varepsilon}{\varepsilon}, \vec{E}\right\rangle - i\omega\mu\vec{H} - \frac{\rho}{\varepsilon}\\
D\vec{H} &= \left\langle \frac{\operatorname{grad}\mu}{\mu}, \vec{H}\right\rangle + i\omega\varepsilon\vec{E} + \vec{j}
\end{aligned}\right\}.$$

Taking into account Remark 1 we rewrite them as

$$\left.\begin{aligned}
\left(D + \frac{1}{2}\frac{\operatorname{grad}\varepsilon}{\varepsilon}\right)\vec{E} &= -\frac{1}{2}M^{\frac{\operatorname{grad}\varepsilon}{\varepsilon}}\vec{E} - i\omega\mu\vec{H} - \frac{\rho}{\varepsilon}\\
\left(D + \frac{1}{2}\frac{\operatorname{grad}\mu}{\mu}\right)\vec{H} &= -\frac{1}{2}M^{\frac{\operatorname{grad}\mu}{\mu}}\vec{H} + i\omega\varepsilon\vec{E} + \vec{j}
\end{aligned}\right\}.$$

Due to (1) we obtain

$$\left.\begin{aligned}
\frac{1}{\sqrt{\varepsilon}}D(\sqrt{\varepsilon}\,\vec{E}) + \vec{E}\cdot\vec{\varepsilon} &= -i\omega\mu\vec{H} - \frac{\rho}{\varepsilon}\\
\frac{1}{\sqrt{\mu}}D(\sqrt{\mu}\,\vec{H}) + \vec{H}\cdot\vec{\mu} &= i\omega\varepsilon\vec{E} + \vec{j}
\end{aligned}\right\}$$

where

$$\vec{\varepsilon} = \frac{\operatorname{grad}\sqrt{\varepsilon}}{\sqrt{\varepsilon}} \qquad \text{and} \qquad \vec{\mu} = \frac{\operatorname{grad}\sqrt{\mu}}{\sqrt{\mu}}.$$

Introducing the notations

$$\vec{\mathcal{E}} = \sqrt{\varepsilon}\vec{E}, \qquad \vec{\mathcal{H}} = \sqrt{\mu}\vec{H}, \qquad k = \omega\sqrt{\varepsilon\mu}$$

we finally arrive at the system

$$(D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} = -ik\vec{\mathcal{H}} - \frac{\rho}{\sqrt{\varepsilon}} \tag{4}$$

$$(D + M^{\vec{\mu}})\vec{\mathcal{H}} = ik\vec{\mathcal{E}} + \sqrt{\mu}\vec{j} \tag{5}$$

which is equivalent to (2) - (3). This pair of equations is the quaternionic reformulation of the Maxwell equations for inhomogeneous media.

The operator $D + M^{\alpha}$ with α being a constant complex quaternion was studied in detail in [12]. Note that $\vec{\varepsilon}$ and $\vec{\mu}$ are constants if ε and μ are functions of the form $\exp(ax_1 + bx_2 + cx_3 + d)$ with constant a, b, c. For the case when α is not a constant there were proposed some classes of exact solutions in [8, 10, 11]. In the same articles the reader can see that the classical Dirac operator with different potentials is closely related to the operator $D + M^{\alpha}$. A simple matrix transform proposed in [7] turns the classical Dirac operator into the operator $D + M^{\alpha}$, where α contains the mass and the energy of the particle as well as the terms corresponding to potentials.

In a static case ($\omega = 0$) we arrive at the equations

$$\left.\begin{array}{l} (D + M^{\vec{\varepsilon}})\vec{\mathcal{E}} = -\dfrac{\rho}{\sqrt{\varepsilon}} \\[2mm] (D + M^{\vec{\mu}})\vec{\mathcal{H}} = \sqrt{\mu}\vec{j} \end{array}\right\}.$$

Thus we are interested in the solutions for the operator $D + M^{\vec{\alpha}}$, where the complex quaternion $\vec{\alpha}$ has the form $\vec{\alpha} = \frac{\operatorname{grad}\varphi}{\varphi}$ and the function φ is different from zero. Note that due to property (1) the operator $D +^{\vec{\alpha}} M$ permits a complete study which can be found in [14] because it practically reduces to the operator D. In the case of the operator $D + M^{\vec{\alpha}}$ the situation as we will see later on is quite different.

Consider the equation

$$(D + M^{\vec{\alpha}})\vec{f} = 0. \tag{6}$$

Denote $v = \frac{\Delta\varphi}{\varphi}$. In other words, φ is a solution of the Schrödinger equation

$$-\Delta\varphi + v\varphi = 0. \tag{7}$$

Proposition 2. *Let ψ be another solution of equation (7). Then the function*

$$\vec{f} = (D - \vec{\alpha})\psi \tag{8}$$

is a solution of equation (6).

Proof. The proof consists of a simple calculation. Consider

$$D\vec{f} = -\Delta\psi - D\psi \cdot \frac{D\varphi}{\varphi} - \psi \cdot D\left(\frac{D\varphi}{\varphi}\right)$$

$$= -v\psi - D\psi \cdot \frac{D\varphi}{\varphi} + \psi \cdot \left(\frac{D\varphi}{\varphi}\right)^2 + \psi\frac{\Delta\varphi}{\varphi}$$

$$= -\left(D\psi - \frac{D\varphi}{\varphi} \cdot \psi\right)\frac{D\varphi}{\varphi}$$

$$= -\vec{f} \cdot \vec{\alpha}.$$

This way the assertion is proved ■

This proposition gives us the possibility to reduce the solution of equation (6) to that of the Schrödinger equation (7). Moreover, if ψ is a fundamental solution of the Schrödinger operator $(-\Delta + v)\psi = \delta$, then the function $\vec{f}$ defined by (8) is a fundamental solution of the operator $D + M^{\vec{\alpha}}$ that can be seen following the proof of Proposition 2.

Remark 3. Proposition 2 is closely related to the factorization of the Schrödinger operator proposed in [1, 2]. Namely, for a scalar function u we have the equality

$$(D + M^\alpha)(D - M^\alpha)u = (-\Delta + v)u$$

if the complex quaternionic function α satisfies the equation

$$D\alpha + \alpha^2 = -v. \tag{9}$$

It is easy to check that for $\alpha = \frac{\text{grad}\varphi}{\varphi}$ equation (9) is equivalent to equation (7). Equation (9) can be considered as a natural generalization of the ordinary differential Riccati equation. In [5, 15] the corresponding generalizations of the well known Euler theorems for the Riccati equation were obtained.

Let us consider the following simple example of application of Proposition 2.

Example 4. Consider equation (6) in some domain $\Omega \subset \mathbb{R}^3$ and let $\frac{\Delta\varphi}{\varphi} = -c^2$ in Ω, where c is a complex constant. In this case we are able to construct a fundamental solution for the operator $D + M^{\vec{\alpha}}$. Denote

$$\psi(x) = \frac{e^{ic|x|}}{4\pi|x|}.$$

This is a fundamental solution of the operator $-\Delta - c^2$. Then the fundamental solution of $D + M^{\vec{\alpha}}$ is constructed as

$$\vec{f}(x) = \left(D - \frac{\mathrm{grad}\varphi(x)}{\varphi(x)}\right)\frac{e^{ic|x|}}{4\pi|x|} = \left(-\frac{x}{|x|^2} + ic\frac{x}{|x|} - \frac{\mathrm{grad}\varphi(x)}{\varphi(x)}\right)\frac{e^{ic|x|}}{4\pi|x|}$$

where $x = \sum_{k=1}^{3} x_k i_k$.

Note that it is not clear how to obtain this result for the Maxwell operators $D + M^{\vec{\varepsilon}}$ and $D + M^{\vec{\mu}}$ using other known methods.

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