

# Quasilinear Elliptic Systems of Second Order in Domains with Corners and Edges: Nemytskij Operator, Local Existence and Asymptotic Behaviour

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**Abstract.** We consider systems of quasilinear partial differential equations of second order in two- and three-dimensional domains with corners and edges. The analysis is performed in weighted Sobolev spaces with attached asymptotics generated by the asymptotic behaviour of the solutions of the corresponding linearized problems near boundary singularities. Applying the Local Invertibility Theorem in these spaces we find conditions which guarantee existence of small solutions of the nonlinear problem having the same asymptotic behaviour as the solutions of the linearized problem. The main tools are multiplication theorems and properties of composition (Nemytskij) operators in weighted Sobolev spaces. As application of the general results a steady-state drift-diffusion system is explained.

**Keywords:** *Quasilinear elliptic problems, weighted Sobolev spaces with attached asymptotics, asymptotic behaviour near conical points and edges, Nemytskij operators, multiplication theorems*

**AMS subject classification:** 35A07, 35B65, 35C20, 35J60, 47H30

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## 1. Introduction

Boundary value problems for systems of stationary quasilinear partial differential equations appear very often in applied sciences (drift-diffusion equations for semiconductors, two-phase flows in porous media, sedimentation processes). The mathematical study of these problems is usually restricted to problems in smooth or convex domains and continuous boundary conditions. Only a few is known about such problems in domains with corners and edges or when the boundary conditions change.

On the other hand, the theory of general linear elliptic problems in domains with a piecewise smooth boundary is well developed (see the monographs [18, 23, 38] and the references therein). In papers by Kondrat'ev, Maz'ya, Plamenevsky, Grisvard, Rossmann, Dauge and many others, the Fredholm property of linear operators in domains with conical points and edges is investigated in several scales of function spaces. Moreover, it is shown there that the solutions  $\mathbf{u}$  can be decomposed into a singular part and a more regular remainder

$$\mathbf{u} = \mathbf{u}_{sing} + \mathbf{u}_{reg}.$$

In comparison with the linear case, the theory of nonlinear problems in non-smooth domains is much less developed. Here we have to distinguish between strong nonlinear scalar problems, which are often treated by barrier methods based on maximum principles [8, 9, 15, 16, 35, 46] and problems where a linearized problem dominates the singular behaviour of bounded, small solutions [6, 17, 22, 27, 28, 35 - 37, 41, 45]. To the first class belongs for example the  $p$ -Laplacian (a quasilinear operator depending on the gradient of the solution) for which Tolksdorf [46] and Dobrowolski [15] have proved existence and asymptotic expansion of the solution near conical points. They first determine an explicit singularity  $s$  using the standard ansatz. Linearizing the operator at  $s$  and using comparison principles they obtain their results. For totally general systems we cannot hope to realize both steps.

Therefore, we investigate existence and regularity of bounded solutions of mixed boundary value problems for a class of quasilinear elliptic systems with small right-hand sides by the classical Local Invertibility Theorem. This method is not restricted to scalar operators. It requires that the operator associated with the nonlinear problem is continuously Fréchet differentiable and that the Fréchet derivative is an isomorphism between some Banach spaces. This leads to difficulties if the domain is non-smooth or if mixed boundary conditions occur. In these cases the singularities of the solutions have to be taken into account in the definition of the underlying function spaces. The proof of the Fréchet differentiability of the nonlinear elliptic operator requires differentiability results for composition operators which are well known for

standard Sobolev, Besov and Hölder spaces (see, e.g., [5, 44]) but not for weighted Sobolev spaces which we use in case of non-smooth domains.

The Local Invertibility Theorem was applied to the Dirichlet and the Neumann problem in smooth nonlinear elastic bodies [11, 48]. In [43] Recke used this approach to prove local  $W^{1,p}(\Omega)$ -solvability results with  $p > 2$  for mixed boundary value problems for a class of two-dimensional quasilinear elliptic systems. In higher-dimensional cases he investigates the local  $W^{1,2}(\Omega)$ -solvability under special growth conditions for the coefficients.

In [7] a class of semilinear elliptic boundary value problems in domains with conical boundary points was investigated by means of the Local Invertibility Theorem in usual Sobolev spaces with attached asymptotics. For quasilinear problems we use finer tools like weighted Sobolev spaces and corresponding multiplication and composition theorems [2 - 4]. In the latter papers the authors investigated the properties of multiplication and composition operators in weighted Sobolev spaces and developed the idea to use them for (Banach) iteration schemes to solve semilinear evolution equations (hyperbolic, parabolic) on domains in  $\mathbb{R}^n$  with conical singularities. In the earlier paper [1] the action of composition operators on domains of powers of certain operators (being in fact spaces with attached asymptotics) is studied. We extend these results proving a new theorem on the Fréchet differentiability of Nemytskij operators acting in weighted Sobolev spaces.

The results presented in this paper are new in several aspects:

- We study the conditions, which guarantee that the operator of the quasilinear problem acting between Sobolev spaces with attached asymptotics is Fréchet differentiable and continuous in a neighbourhood of the zero-solution and that the Fréchet derivative coincides with the formally linearized operator.
- We prove some lemmata on multiplication and composition operators in weighted Sobolev spaces of Kondrat'ev type either in domains with conical points, where the weight is the distance to the conical points or in domains of polyhedral type, where two different weights appear, generated by corner points and edges. The choice of those spaces is based on the fact that regularity results for the solutions of linearized problems are well developed in those spaces [14, 19, 31, 34, 40]. The investigation of the Nemytskij operator in such spaces has its own interest in functional analysis, independently of the special application here (for usual Sobolev, Besov and Hölder spaces, see [5, 44]).
- The method used here allows to obtain some local existence and regularity results for bounded solutions to systems of quasilinear equations in polygonal and polyhedral domains and can also be applied to interface problems and more general problems provided the singularities of the solutions of

the linearized problem are known.

The paper is organized in the following way:

After the formulation of the problem (Section 2) we describe the associated formally linearized operator and its mapping properties in weighted Sobolev spaces with attached asymptotics (Section 3). The asymptotic expansion of the solution is well known for domains with conical points in  $L_p$ -spaces, while a corresponding decomposition in polyhedral domains is only available, to our knowledge, in  $L_2$ -spaces. In Section 4 we present the main results: first we give theorems about multiplication and composition in weighted Sobolev spaces and prove the continuity and the Fréchet differentiability of nonlinear composition operators in these spaces. These theorems are the basis for the investigation of the mapping properties of the nonlinear operators presented in the second part of Section 4. There we formulate conditions, in particular on the number of asymptotic terms, which guarantee the applicability of the Local Invertibility Theorem in weighted Sobolev spaces with attached asymptotics. Here we formulate also the main results concerning the existence and the asymptotic behaviour of the solutions of the quasilinear boundary value problem near conical boundary points, vertices and edges. Since these results require quite technical proofs, we have postponed them to the last Section 5.

To facilitate the reading of our paper we present at several stages of the considerations an illustration using the special case of the steady-state drift-diffusion system.

**Acknowledgements.** The authors thank R. Farwig (Darmstadt) and A. Mielke (Stuttgart) and the referees for important and useful remarks.

## 2. Formulation of the problem

Let  $\Omega \subset \mathbb{R}^n$  ( $n = 2, 3$ ) be a bounded domain satisfying either

- (i) there exist a finite set  $\mathcal{P}$  of conical boundary points such that  $\partial\Omega \setminus \mathcal{P}$  is smooth and  $\Omega$  coincides in the vicinity of every conical boundary point  $P$  with an infinite cone  $C_P$  (whose basis  $G_P = C_P \cap S^{n-1}(P)$  is then smooth)

or

- (ii)  $\Omega \subset \mathbb{R}^3$  is a straight polyhedron.

In the case (i) let  $\partial\Omega = \bar{\Gamma}^D \cup \bar{\Gamma}^N$  be a given decomposition of the boundary with open smooth  $(n - 1)$ -dimensional manifolds  $\Gamma^D, \Gamma^N$  and  $\text{meas } \Gamma^D \neq 0$ . Moreover, we assume that  $\bar{\Gamma}^D \cap \bar{\Gamma}^N \subset \mathcal{P}$  for  $n = 2$  and  $\bar{\Gamma}^D \cap \bar{\Gamma}^N = \emptyset$  if  $n = 3$  (this last case is possible if  $\Omega$  has some holes).

In the case (ii) we suppose that  $\partial\Omega = \Gamma^D$ . Furthermore, we denote by  $\mathcal{P}$  (respectively  $\mathcal{E}$ ) the set of vertices (respectively the set of edges) of the polyhedron  $\Omega$ . The set of boundary singularities is denoted by  $\mathcal{S}$ , in other words,  $\mathcal{S} = \mathcal{P}$  in the first case and  $\mathcal{S} = \mathcal{P} \cup \mathcal{E}$  in the second case.

We consider a mixed boundary value problem for a system of  $k$  quasilinear equations of second order for the vector function  $\mathbf{u} = (u_1, \dots, u_k)$

$$\left. \begin{aligned} -\partial_j [a_{ij\sigma\tau}(\mathbf{u})\partial_i u_\sigma + b_{j\tau}(\mathbf{u})] + c_{i\sigma\tau}(\mathbf{u})\partial_i u_\sigma + d_\tau(\mathbf{u}) &= f_\tau && \text{in } \Omega \\ [a_{ij\sigma\tau}(\mathbf{u})\partial_i u_\sigma + b_{j\tau}(\mathbf{u})]n_j &= h_\tau && \text{on } \Gamma^N \\ u_\tau &= g_\tau && \text{on } \Gamma^D \end{aligned} \right\} \quad (1)$$

for  $\tau = 1, \dots, k$ . Here we denote by  $\mathbf{n} = (n_1, \dots, n_n)$  the unit outward normal vector on  $\partial\Omega$ . Moreover, we apply here and in the following the summation convention for the repeated indices  $i, j = 1, \dots, n$  and  $\sigma = 1, \dots, k$ . We assume that problem (1) with homogeneous right-hand sides has the trivial solution which means  $b_{j\tau}(\mathbf{0}) = d_\tau(\mathbf{0}) = 0$ .

Let us present an example of such a system coming from semiconductor theory [25] and that we will use in the whole paper to illustrate our results:

**Example 2.1.** Consider the steady-state drift-diffusion system coming from semiconductor device modelling [25], which after scaling may be written as

$$\left. \begin{aligned} \nabla \cdot (e^\psi \nabla u) &= 0 && \text{in } \Omega \\ \nabla \cdot (e^{-\psi} \nabla v) &= 0 && \text{in } \Omega \\ -\varepsilon \Delta \psi + \delta^2 (e^\psi u - e^{-\psi} v) &= N && \text{in } \Omega \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial \psi}{\partial n} &= 0 && \text{on } \Gamma^N \\ (u, v, \psi) &= (u_D, v_D, \psi_D) && \text{on } \Gamma^D \end{aligned} \right\} \quad (2)$$

where  $\psi$  represents the electrostatic potential and  $u, v$  are the so-called Slotboom variables (associated with the concentration variables of negative and positive charges, see [25] for details). The constants  $\varepsilon > 0$  and  $\delta > 0$ , the function  $N$  (the doping profile) and  $\psi_D, u_D, v_D$  are given. In physical applications  $N$  often has jump discontinuities to create the transistor effect. Existence results for that system are well-known [25] while regularity results are less standard, especially for non-smooth domains.

In this paper we investigate under which conditions on  $a_{ij\sigma\tau}, b_{j\tau}, c_{i\sigma\tau}, d_\tau$  problem (1) is locally solvable in the neighbourhood of  $\mathbf{u} = \mathbf{0}$  in certain function spaces and how regular is this solution in the neighbourhood of corner points and edges. The key idea is to linearize the above boundary value problem, to use the mapping properties of the corresponding linearized operator in weighted Sobolev spaces and to apply the following theorem on local invertibility of nonlinear operators.

**Theorem 2.2** [49: Theorem 4.B]. *Let  $X, Y$  be two Banach spaces. Consider  $x_0 \in X$  and a neighbourhood  $\mathcal{U}(x_0)$  of  $x_0$ . Suppose  $N : \mathcal{U}(x_0) \subset X \rightarrow Y$  is a mapping from  $\mathcal{U}(x_0)$  into  $Y$  and  $y_0 = Nx_0$ . We assume the following:*

- $N$  is Fréchet differentiable in  $\mathcal{U}(x_0)$ .
- The Fréchet derivative  $N'(x_0) : X \rightarrow Y$  is bijective.
- The mapping  $x \mapsto N'(x) \in \mathcal{L}(X, Y)$  is continuous at  $x_0$ .

*Then there exists a unique mapping  $N^{-1}$  defined in a neighbourhood  $\mathcal{V}(y_0) \subset Y$ ,  $N^{-1} : \mathcal{V}(y_0) \rightarrow X$ , such that  $NN^{-1}y = y$  for all  $y \in \mathcal{V}(y_0)$ .*

### 3. The formally linearized problem and its mapping properties

We consider the formal linearization of problem (1) at  $\mathbf{u} = \mathbf{0}$ . We will show in Section 5 that the associated operator coincides indeed with the Fréchet derivative of the nonlinear operator corresponding to problem (1) acting between the weighted Sobolev spaces under consideration. The formally linearized problem reads

$$\left. \begin{aligned} -\partial_j [a_{ij\sigma\tau}(\mathbf{0})\partial_i v_\sigma + \langle \partial_{\mathbf{u}} b_{j\tau}(\mathbf{0}), \mathbf{v} \rangle] + c_{i\sigma\tau}(\mathbf{0})\partial_i v_\sigma + \langle \partial_{\mathbf{u}} d_\tau(\mathbf{0}), \mathbf{v} \rangle &= f_\tau & \text{in } \Omega \\ [a_{ij\sigma\tau}(\mathbf{0})\partial_i v_\sigma + \langle \partial_{\mathbf{u}} b_{j\tau}(\mathbf{0}), \mathbf{v} \rangle] n_j &= h_\tau & \text{on } \Gamma^N \\ v_\tau &= g_\tau & \text{on } \Gamma^D \end{aligned} \right\} \quad (3)$$

for  $\tau = 1, \dots, k$ . Here,  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $\mathbb{R}^k$  and the derivatives are to be understood in the distributional sense. For simplicity we shortly write problem (3) as

$$\left. \begin{aligned} \mathbf{A}\mathbf{v} &= \mathbf{f} & \text{in } \Omega \\ \mathbf{C}^N \mathbf{v} &= \mathbf{h} & \text{on } \Gamma^N \\ \mathbf{v} &= \mathbf{g} & \text{on } \Gamma^D \end{aligned} \right\}.$$

We assume the following:

- (E) Boundary value problem (3) is elliptic, i.e. the matrix  $\mathbf{A}$  of differential operators is properly elliptic in  $\overline{\Omega} \setminus \mathcal{S}$  and the Shapiro-Lopatinski condition is satisfied on  $\partial\Omega \setminus \mathcal{S}$ .

**Example 3.1.** For Example 2.1, the linearized operator  $\mathbf{A}$  of system (2) is the matrix

$$\mathbf{A} = \begin{pmatrix} -\Delta & 0 & 0 \\ 0 & -\Delta & 0 \\ \delta^2 I & -\delta^2 I & -\varepsilon \Delta \end{pmatrix}$$

with Dirichlet or Neumann boundary conditions. For that system assumption (E) clearly holds.

The solvability and regularity of solutions of linear elliptic problems of type (3) is thoroughly investigated for domains with conical points [19, 29], edges [12, 31, 33, 38] or for domains of polyhedral type [14, 32, 34, 42]. Here we apply these general results to problem (3). The regularity of the solutions is governed by the principal (or leading) parts  $(\mathbf{L}, \mathbf{B}^{\mathbf{N}}, \mathbf{I})$  of the operators  $(\mathbf{A}, \mathbf{C}^{\mathbf{N}}, \mathbf{I})$  of boundary value problem (3) (here and below  $\mathbf{I}$  means the identity operator on  $\Gamma^D$ ).

We formulate the solvability and regularity results in weighted Sobolev spaces for domains with conical points and domains of straight polyhedral type separately.

**3.1 Domains with conical points.** For a fixed singular point  $P \in \mathcal{P}$  we introduce spherical coordinates  $(r_P, \omega_P)$  centered at  $P$  and denote by  $C_P$  the infinite cone which coincides with  $\Omega$  in a neighbourhood of  $P$ .

We will formulate the solvability and regularity results in terms of weighted Sobolev spaces of Kondrat'ev's type that we recall here.

**Definition 3.2.** For  $d \in \mathbb{N}_0 = \{0, 1, \dots\}$ ,  $1 < p < \infty$  and  $\beta \in \mathbb{R}$  we define the weighted Sobolev space  $V_{\beta,p}^d(C_P)$  as the closure of  $C_0^\infty(\bar{C}_P \setminus P)$  with respect to the norm

$$\|u\|_{V_{\beta,p}^d(C_P)} = \sum_{|\alpha| \leq d} \|r_P^{\beta-d+|\alpha|} D^\alpha u\|_{L_p(C_P)}.$$

For  $d \in \mathbb{N}$  the space  $V_{\beta,p}^{d-1/p}(\partial C_P)$  consists of traces on  $\partial C_P$  of functions in  $V_{\beta,p}^d(C_P)$  and is equipped with the norm

$$\|u\|_{V_{\beta,p}^{d-1/p}(\partial C_P)} = \inf \|v\|_{V_{\beta,p}^d(C_P)}$$

where the infimum is taken over the set of all functions  $v \in V_{\beta,p}^d(C_P)$  such that  $v = u$  on  $\partial C_P$ .

**Definition 3.3.** For  $d \in \mathbb{N}_0$ ,  $1 < p < \infty$  and  $\vec{\beta} = (\beta_P)_{P \in \mathcal{P}}$  we define the space  $V_{\vec{\beta},p}^d(\Omega)$  as the closure of  $C_0^\infty(\bar{\Omega} \setminus \mathcal{P})$  with respect to the norm which is assembled by means of a partition of unity from the local norms  $V_{\beta_P,p}^d(C_P)$ . Let  $\zeta_P \in C_0^\infty(\mathbb{R}^n)$  with  $0 \leq \zeta_P \leq 1$  be such that  $\zeta_P = 1$  near the conical point  $P$  and  $\zeta_P = 0$  near all  $Q \in \mathcal{P}$  with  $Q \neq P$ . We set  $\zeta_0 = 1 - \sum_{P \in \mathcal{P}} \zeta_P$ . Define the norm in  $V_{\vec{\beta},p}^d(\Omega)$  by

$$\|u\|_{V_{\vec{\beta},p}^d(\Omega)} = \left( \|\zeta_0 u\|_{W_p^d(\Omega)}^p + \sum_{P \in \mathcal{P}} \|\zeta_P u\|_{V_{\beta_P,p}^d(C_P)}^p \right)^{1/p}.$$

For  $d > 0$  we denote by  $V_{\vec{\beta},p}^{d-1/p}(\partial\Omega)$  the space of traces on  $\partial\Omega \setminus \mathcal{P}$  of functions in  $V_{\vec{\beta},p}^d(\Omega)$ .

For the sake of shortness we write  $\mathbf{V}_{\vec{\beta},p}^d(\Omega)$  and  $\mathbf{V}_{\vec{\beta},p}^{d-1/p}(\partial\Omega)$  instead of  $[V_{\vec{\beta},p}^d(\Omega)]^k$  and  $[V_{\vec{\beta},p}^{d-1/p}(\partial\Omega)]^k$ , respectively.

Weighted Sobolev spaces have the following imbedding property, which will be used later.

**Lemma 3.4** [19]. *Let  $\vec{\beta} = (\beta_P)_{P \in \mathcal{P}}$ ,  $\vec{\gamma} = (\gamma_P)_{P \in \mathcal{P}}$  and  $d', d \in \mathbb{N}_0$  with  $d' \leq d$  and  $\beta_P - d \leq \gamma_P - d'$  for all  $P \in \mathcal{P}$ . Then  $\mathbf{V}_{\vec{\beta},p}^d(\Omega)$  is continuously imbedded into  $\mathbf{V}_{\vec{\gamma},p}^{d'}(\Omega)$ .*

Let  $G_P = C_P \cap S^{n-1}(P)$  be the intersection of  $C_P$  with the unit sphere centered at  $P$ . Since the operators  $\mathbf{L}$  and  $\mathbf{B}^N$  are homogeneous with constant coefficients, their components may be written with respect to the local spherical coordinates near  $P \in \mathcal{P}$  as follows:

$$\begin{aligned} L_{\sigma\tau}(\partial_x) &= r_P^{-2} \mathcal{L}_{P,\sigma\tau}(\omega_P, r_P \partial_{r_P}, \partial_{\omega_P}) \\ B_{\sigma\tau}^N(\partial_x) &= r_P^{-1} \mathcal{B}_{P,\sigma\tau}^N(\omega_P, r_P \partial_{r_P}, \partial_{\omega_P}). \end{aligned}$$

Applying the Mellin transform ( $r_P \partial_{r_P} \rightarrow \lambda$ ) we get a matrix operator pencil depending on the complex parameter  $\lambda$  denoted for simplicity similarly by

$$\{\mathcal{L}_P, \mathcal{B}_P^N|_{\Gamma^N \cap S^{n-1}(P)}, \mathcal{I}_P|_{\Gamma^D \cap S^{n-1}(P)}\}$$

where  $\mathcal{I}_P|_{\Gamma^D \cap S^{n-1}(P)}$  is the identity operator on  $\Gamma^D \cap S^{n-1}(P)$  whose components are  $\mathcal{L}_{P,\sigma\tau}(\omega_P, \lambda, \partial_{\omega_P})$  and  $\mathcal{B}_{P,\sigma\tau}^N(\omega_P, \lambda, \partial_{\omega_P})$ . Assumption (E) insures that the matrix operator pencil

$$\mathcal{A}_P(\lambda) = \left\{ \mathcal{L}_P(\lambda), \mathcal{B}_P^N|_{\Gamma^N \cap S^{n-1}(P)}(\lambda), \mathcal{I}_P|_{\Gamma^D \cap S^{n-1}(P)} \right\} \quad (4)$$

has countably many isolated generalized eigenvalues of finite algebraic multiplicity [23]. We associate with any eigenvalue  $\lambda$  of  $\mathcal{A}_P$  a canonical system of Jordan chains

$$\left\{ \varphi_P^{\lambda,\nu,q} : \nu = 1, \dots, M(\lambda); q = 1, \dots, \kappa(\lambda, \nu) \right\}$$

satisfying (see [10, 23] for more details)

$$\sum_{q=1}^l \mathcal{A}_P^{(l-q)}(\lambda) \frac{\varphi_P^{\lambda,\nu,q}}{(l-q)!} = 0 \quad (l = 1, \dots, \kappa(\lambda, \nu)). \quad (5)$$



We introduce singular functions corresponding to  $\lambda$  by

$$\xi_P^{\lambda, \nu, l}(r_P, \omega_P) = r_P^\lambda \sum_{q=1}^l \frac{(\log r_P)^{l-q}}{(l-q)!} \varphi_P^{\lambda, \nu, q}(\omega_P). \tag{6}$$

They satisfy

$$\begin{aligned} \mathbf{L}\xi_P^{\lambda, \nu, l} &= \mathbf{0} && \text{in } C_P \\ \mathbf{B}^N \xi_P^{\lambda, \nu, l} &= \mathbf{0} && \text{on } \Gamma_N \cap \bar{C}_P \\ \xi_P^{\lambda, \nu, l} &= \mathbf{0} && \text{on } \Gamma_D \cap \bar{C}_P. \end{aligned}$$

The signification of these singular functions is the well-known fact that the solution of the problem

$$\left. \begin{aligned} \mathbf{L}\mathbf{u} &= \mathbf{f} && \text{in } \Omega \\ \mathbf{B}^N \mathbf{u} &= \mathbf{h} && \text{on } \Gamma_N \\ \mathbf{u} &= \mathbf{g} && \text{on } \Gamma_D \end{aligned} \right\}$$

can be decomposed into a regular part and a linear combination of the above singular function (see below). In [39: Chapter 4] it is outlined that with Euler’s change of variable  $r = e^t$  and a reduction to a first order system this problem can be brought into the form of an evolution equation

$$\frac{\partial V}{\partial t} - AV = F \tag{7}$$

in a suitable Hilbert space. The solution of this equation can be reduced to the study of the resolvent of  $A$ , which is a meromorphic operator-valued function. The principle part of its Laurent series with respect to a fixed eigenvalue of  $A$  corresponds to the singular functions. They can be calculated from the principle vectors corresponding to the Jordan decomposition of the finite-dimensional eigenspace of this eigenvalue. The connection with formula (6) is explained in [39: Lemma 4.18].

In the following we denote by  $\Lambda_P(a, b)$  the set of all eigenvalues  $\lambda$  of  $\mathcal{A}_P$  with  $a < \operatorname{Re} \lambda < b$ .

**Example 3.5.** For Example 2.1 in a 2-dimensional polygonal domain  $\Omega$ , the principal part  $\mathbf{L}$  of system (2) is the diagonal matrix

$$\mathbf{L} = \begin{pmatrix} -\Delta & 0 & 0 \\ 0 & -\Delta & 0 \\ 0 & 0 & -\varepsilon\Delta \end{pmatrix} \tag{8}$$

with Dirichlet or Neumann boundary conditions. Therefore, the matrix operator  $\mathbf{L}$  is splitted into three scalar Laplace operators with Dirichlet or Neumann

boundary conditions for which the eigenvalues and associated Jordan chains are well known (see, for instance, [18, 19]). Accordingly, the eigenvalues of  $\mathcal{A}_P$  are equal to  $\frac{k\pi}{\omega_{0P}}$  with  $0 \neq k \in \mathbb{Z}$  if  $P$  belongs to two edges of  $\Gamma_D$  or two edges of  $\Gamma_N$ , otherwise they are equal to  $\frac{\pi}{2\omega_{0P}} + \frac{k\pi}{\omega_{0P}}$  with  $k \in \mathbb{Z}$  where  $\omega_{0P}$  is the interior opening angle of  $\Omega$  at  $P$ . If  $P$  belongs to two edges of  $\Gamma_D$ , for  $\lambda = \frac{k\pi}{\omega_{0P}}$  the associated Jordan chain is given by

$$\xi_P^{\lambda, i, 1}(r_P, \omega_P) = r_P^\lambda (\delta_{ij} \sin(\lambda \omega_{0P}))_{i, j=1, 2, 3}.$$

The other cases of boundary conditions can be treated similarly.

We are now able to formulate the solvability and regularity results in weighted Sobolev spaces.

**Theorem 3.6** (Solvability Theorem [19, 23, 29]). *Let  $d \in \mathbb{N}_0$ . Suppose that the ellipticity condition (E) is satisfied and that the line  $\text{Re } \lambda = d + 2 - \frac{n}{p} - \beta_P$  does not contain eigenvalues of the pencil  $\mathcal{A}_P(\lambda)$  for every  $P \in \mathcal{P}$ . Then the operator*

$$\mathcal{A}_{d, \vec{\beta}, p} = (\mathbf{A}, \mathbf{C}^{\mathbf{N}}, \mathbf{I}) : \mathbf{V}_{\vec{\beta}, p}^{d+2}(\Omega) \rightarrow \mathbf{V}_{\vec{\beta}, p}^d(\Omega) \times \mathbf{V}_{\vec{\beta}, p}^{d+1-\frac{1}{p}}(\Gamma^N) \times \mathbf{V}_{\vec{\beta}, p}^{d+2-\frac{1}{p}}(\Gamma^D)$$

is of Fredholm type.

The asymptotics (singular terms) of the linearized problem (3) with lower order terms has the same form as the asymptotics (singular terms) corresponding to the principal part of the problem if we demand that the strip of considered eigenvalues is less than 1.

**Theorem 3.7** (Regularity Theorem [19, 23, 29]). *Let  $d \in \mathbb{N}_0, \mathbf{u} \in \mathbf{V}_{\vec{\beta}, p}^{d+2}(\Omega)$  be a solution of the linearized boundary value problem (3), where  $\mathbf{f} \in \mathbf{V}_{\vec{\mu}, p}^d(\Omega), \mathbf{h} \in \mathbf{V}_{\vec{\mu}, p}^{d+1-1/p}(\Gamma^N), \mathbf{g} \in \mathbf{V}_{\vec{\mu}, p}^{d+2-1/p}(\Gamma^D)$ , and let the vectors  $\vec{\beta}$  and  $\vec{\mu}$  have the components  $\beta_P$  and  $\mu_P$ , respectively. Suppose that ellipticity condition (E) is satisfied and that the lines*

$$\begin{aligned} \text{Re } \lambda = d + 2 - \frac{n}{p} - \beta_P &= H_{1, P} \\ \text{Re } \lambda = d + 2 - \frac{n}{p} - \mu_P &= H_{2, P} \end{aligned}$$

do not contain eigenvalues of the pencil  $\mathcal{A}_P(\lambda)$  for every  $P \in \mathcal{P}$ . Suppose further that  $0 < \beta_P - \mu_P < 1$  for all  $P \in \mathcal{P}$ . Then the solution  $\mathbf{u}$  admits the decomposition

$$\mathbf{u} = \mathbf{u}_{reg} + \sum_{P \in \mathcal{P}} \sum_{\lambda \in \Lambda_P(H_{1, P}, H_{2, P})} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda, \nu)} \chi_P c_P^{\lambda, \nu, q} \xi_P^{\lambda, \nu, q} \quad (9)$$

where the singular vector functions  $\xi_P^{\lambda,\nu,q}$  are defined by (6) and  $\mathbf{u}_{reg} \in \mathbf{V}_{\vec{\mu},p}^{d+2}(\Omega)$ ,  $\chi_P$  are cut-off functions equal to 1 near 0 and  $c_P^{\lambda,\nu,q}$  are constants. ■

**3.2 Straight polyhedral domains.** Let us consider the case that  $\Omega$  is a three-dimensional straight polyhedron. We treat here only the Dirichlet problem with  $\mathbf{g} = \mathbf{0}$ . It is well known [14, 18, 19, 31, 32, 40] that a solution  $\mathbf{u}$  of problem (3) has edge and/or vertex singularities if the right-hand sides are smooth enough. We underline that the decomposition in singular and regular terms is known in weighted  $L_2$ -Sobolev spaces [14, 38, 40]; the  $L_p$  theory is developed only for edge singularities [31, 32]. To describe this decomposition we introduce some further notations.

Firstly, we fix  $P$  in the set  $\mathcal{P}$  of vertices of  $\Omega$ . Let  $C_P$  be the infinite polyhedral cone of  $\mathbb{R}^3$  which coincides with  $\Omega$  in a neighbourhood of  $P$ ; we set  $G_P = C_P \cap S^2(P)$ , the intersection of  $C_P$  with the unit sphere centered at  $P$ . We now proceed as for conical points and introduce the operator pencil  $\mathcal{A}_P$  and the corresponding singular vector functions  $\xi_P^{\lambda,\nu,q}$  analogously to (4) and (6). The spectrum of  $\mathcal{A}_P$  will be denoted by  $\Lambda_P$ . Note that  $G_P$  is a curvilinear polygon on the sphere and the singular functions depend also on the shape of its corners (edge singularities).

Secondly, we consider the edge singularities. Let  $E$  be an edge of  $\Omega$  and let  $\omega_{0E}$  be the opening angle of the edge. We write the operator  $\mathbf{L}(\partial_{\mathbf{x}})$  in local Cartesian coordinates  $(y_{E1}, y_{E2}, z_E)$ , where the  $z_E$ -axis coincides with the edge  $E$  (for shortness we drop the subscript  $E$  for a moment)

$$\mathbf{L}(\partial_{\mathbf{x}}) = \mathbf{L}(\partial_{y_1}, \partial_{y_2}, \partial_z).$$

After the application of the Fourier transform with respect to  $z \rightarrow \eta$  we obtain the operator  $\mathbf{L}(\partial_{y_1}, \partial_{y_2}, \eta)$ , where  $\mathbf{L}(\partial_{y_1}, \partial_{y_2}, 0)$  has the decomposition

$$\mathbf{L}(\partial_{y_1}, \partial_{y_2}, 0) = r^{-2} \mathcal{L}(\omega, \partial_\omega, r \partial_r).$$

Here  $(r, \omega)$  denote the polar coordinates in the  $(y_1, y_2)$ -plane. Then we introduce the operator pencil  $\mathcal{A}_E(\lambda)$  corresponding to the parameter depending boundary value problem

$$\left. \begin{aligned} \mathcal{L}(\omega, \partial_\omega, \lambda)u(\omega) &= 0 & (\omega \in (0, \omega_0)) \\ u(\omega) &= 0 & (\omega \in \{0, \omega_0\}) \end{aligned} \right\}.$$

The spectrum of  $\mathcal{A}_E$  will be denoted by  $\Lambda_E$ , and for any  $\lambda \in \Lambda_E$  we write the associated singular function  $\xi_E^{\lambda,\nu,q}$  ( $q = 1, \dots, \kappa(\lambda, \nu)$ ;  $\nu = 1, \dots, M(\lambda)$ ).

**Example 3.8.** For Example 2.1, a principal part  $\mathbf{L}$  of system (2) is the diagonal matrix (8) with Dirichlet boundary conditions. Therefore,  $\mathbf{L}$  can be decomposed into three scalar Laplace operators with Dirichlet boundary conditions for which the eigenvalues and Jordan chains at corners and along edges are wellknown (see, for instance, [14, 18]). Accordingly, the edge eigenvalues of  $\mathcal{A}_E$  are equal to  $\frac{k\pi}{\omega_{0E}}$  with  $0 \neq k \in \mathbb{Z}$  and the associated Jordan chain is given in Example 3.5 replacing  $P$  by  $E$ . Any corner eigenvalue  $\lambda$  associated with a corner  $P$  is given by  $\lambda = -\frac{1}{2} \pm \sqrt{\nu + \frac{1}{4}}$  where  $\nu$  is an eigenvalue of the Laplace-Beltrami operator  $L_P^{Dir}$  on  $G_P$  with Dirichlet boundary conditions (see [14, 18] for details). Further, the associated Jordan chain is given by

$$\xi_P^{\lambda, i, 1}(r_P, \omega_P) = r_P^\lambda (\delta_{ij} \psi_\nu(\omega_P))_{j=1,2,3} \quad (i = 1, 2, 3)$$

where  $\psi_\nu$  is the eigenvector of  $L_P^{Dir}$  associated with the eigenvalue  $\nu$ .

Finally, we shall use the ‘‘angular’’ distance to the edges  $\theta = \min_{E \in \mathcal{E}} \theta_E$ , where for  $E \in \mathcal{E}$  we define  $\theta_E = r_E/d_E$ , when  $r_E$  is the distance to  $E$  and  $d_E$  is a function in  $\bar{\Omega}$ , smooth on  $\bar{E}$  and equivalent to the distance to the endpoints of  $E$ . Note that  $\theta(x)$  corresponds to the distance between  $x$  and the edges far from the corners, while near a corner  $P$  and an edge  $E$  it is equivalent to the angle between the line  $Px$  and  $E$ .

We now recall some weighted Sobolev spaces of Kondrat’ev’s type with double weight (for edge and vertex singularities) already used in [13, 32, 34].

**Definition 3.9.** For two real numbers  $\alpha, \beta$  and a positive integer  $d$  we set

$$H_{\alpha, \beta}^d(\Omega) = \left\{ v : r^{\alpha+|\gamma|-d} \theta^{\beta+|\gamma|-d} D^\gamma v \in L^2(\Omega) \text{ for all } \gamma \in \mathbb{N}_0^3 \text{ with } |\gamma| \leq d \right\}$$

where  $r = r(x)$  is the distance between  $x$  and the set of vertices of  $\Omega$  and  $\theta$  as defined above. It is a Hilbert space with the norm

$$\|v\|_{H_{\alpha, \beta}^d(\Omega)} = \left\{ \sum_{|\gamma| \leq d} \|r^{\alpha+|\gamma|-d} \theta^{\beta+|\gamma|-d} D^\gamma v\|_{L^2(\Omega)}^2 \right\}^{1/2}.$$

Furthermore, we set  $\mathbf{H}_{\alpha, \beta}^d(\Omega) = (H_{\alpha, \beta}^d(\Omega))^k$ .

Let us note that in [13, 32, 34] the above space is denoted by  $V_{\alpha, \beta}^{d, 2}(\Omega)$ . We have used the notation  $H_{\alpha, \beta}^d(\Omega)$  in order to avoid confusion with the space  $V_{\beta, 2}^d(\Omega)$  introduced before.

We now give the regularity results in weighted Sobolev spaces. We first state a decomposition result extending [14: Theorem 17.13]. Then we deduce an isomorphism property.

**Theorem 3.10.** *Assume that the line  $\operatorname{Re} \lambda = \frac{1}{2}$  contains no eigenvalues of  $\mathcal{A}_P(\lambda)$ , for all vertices  $P \in \mathcal{P}$ , and that the line  $\operatorname{Re} \lambda = 1$  contains no eigenvalues of  $\mathcal{A}_E(\lambda)$ , for all edges  $E \in \mathcal{E}$ . Then the solution  $\mathbf{u} \in \mathring{H}^1(\Omega)$  of problem (3) with  $\mathbf{f} \in \mathbf{L}_2(\Omega)$  admits the decomposition*

$$\begin{aligned} \mathbf{u} = & \mathbf{u}_{reg} + \sum_{P \in \mathcal{P}} \sum_{\lambda \in \Lambda_P(-\frac{1}{2}, \frac{1}{2})} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda, \nu)} \chi_P c_P^{\lambda, \nu, q} \boldsymbol{\xi}_P^{\lambda, \nu, q} \\ & + \sum_{E \in \mathcal{E}} \sum_{\lambda \in \Lambda_E(0, 1)} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda, \nu)} \mathbf{Z}_E^{\lambda, \nu, q}(c_E^{\lambda, \nu, q}). \end{aligned} \quad (10)$$

Here the regular part  $\mathbf{u}_{reg}$  belongs to  $\mathbf{H}_{0,0}^2(\Omega)$ , the coefficients  $c_P^{\lambda, \nu, q}$  of the vertex singularities are real numbers. The coefficient functions  $c_E^{\lambda, \nu, q}$  of the edge singularities belong to  $H_{-\operatorname{Re} \lambda}^{1-\operatorname{Re} \lambda}(E)$  (usual weighted Sobolev space of Kondrat'ev's type defined on the edge  $E$ , where the weight is the distance to the endpoint of  $E$ ; see, for instance, [14: Appendix A]). Further,  $\mathbf{Z}_E^{\lambda, \nu, q}$  is a pseudo-differential operator which maps continuously  $H_{-\operatorname{Re} \lambda}^{1-\operatorname{Re} \lambda}(E)$  into  $\mathring{\mathbf{H}}^1(\Omega)$  which may be written as

$$\mathbf{Z}_E^{\lambda, \nu, q}(c) = \mathcal{K}(c) \chi_E(\theta_E) \boldsymbol{\xi}_E^{\lambda, \nu, q}(\theta_E, \omega_E),$$

where  $\chi_E$  is a cut-off function equal to 1 near 0 and  $\mathcal{K}(c)$  is a convolution operator defined by

$$\mathcal{K}(c)(\theta_E, \omega_E, z_E) = \int_{\mathbb{R}} \varphi\left(\frac{t}{\theta_E}\right) \tilde{c}(t - \tilde{z}_E) \frac{dt}{\theta_E},$$

$\varphi$  being a smooth function in  $\mathcal{S}(\mathbb{R})$  of mean 1,  $\tilde{c}(\tilde{z}_E) = c(z_E)$  and finally  $\tilde{z}_E$  is the stretched variable  $\tilde{z}_E = \int_0^{z_E} \frac{dz}{d_E(z)}$  (so that the mapping  $z_E \rightarrow \tilde{z}_E$  is one-to-one from  $E$  to  $\mathbb{R}$ ). Moreover, there exists a constant  $C > 0$  such that

$$\begin{aligned} \|\mathbf{u}_{reg}\|_{\mathbf{H}_{0,0}^2(\Omega)} + \sum_{P \in \mathcal{P}} \sum_{\lambda \in \Lambda_P(-\frac{1}{2}, \frac{1}{2})} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda, \nu)} |c_P^{\lambda, \nu, q}| \\ + \sum_{E \in \mathcal{E}} \sum_{\lambda \in \Lambda_E(0, 1)} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda, \nu)} \|c_E^{\lambda, \nu, q}\|_{H_{-\operatorname{Re} \lambda}^{1-\operatorname{Re} \lambda}(E)} \leq C \|\mathbf{f}\|_{\mathbf{L}_2(\Omega)}. \end{aligned} \quad (11)$$

**Proof.** The proof follows the lines of [14: Theorem 17.13] (a localized version, see [14: Remark 17.18]) where the usual Sobolev spaces are replaced by weighted Sobolev spaces of Kondrat'ev's type. The main ingredient is the edge decomposition in the spaces  $V_{\beta, 2}^d(D)$ , where  $D$  is a dihedral cone, which is available due to the results in 2-dimensional cones from [23: Chapter 8] ■

**Corollary 3.11.** *Let  $\alpha_0$  and  $\beta_0$  be defined by*

$$\alpha_0 = \min_{\lambda \in \Lambda_P, P \in \mathcal{P}, \operatorname{Re} \lambda > -\frac{1}{2}} \operatorname{Re} \lambda \quad \text{and} \quad \beta_0 = \min_{\lambda \in \Lambda_E, E \in \mathcal{E}, \operatorname{Re} \lambda > 0} \operatorname{Re} \lambda$$

and let  $\alpha, \beta \in \mathbb{R}$  with  $\alpha > \min\{0, \frac{1}{2} - \alpha_0\}$  and  $\beta > \min\{0, 1 - \beta_0\}$ . Then the solution  $\mathbf{u} \in \mathring{\mathbf{H}}^1(\Omega)$  of problem (3) with  $\mathbf{f} \in \mathbf{L}_2(\Omega)$  belongs to  $\mathbf{H}_{\alpha, \beta}^2(\Omega)$ . Moreover, there exists a constant  $C > 0$  such that

$$\|\mathbf{u}\|_{\mathbf{H}_{\alpha, \beta}^2(\Omega)} \leq C \|\mathbf{f}\|_{\mathbf{L}_2(\Omega)}. \quad (12)$$

**3.3 Weighted Sobolev spaces with attached asymptotics.** Theorems 3.7 and 3.10 motivate the introduction of Sobolev spaces with attached asymptotics which will be identified later with the spaces  $X$  and  $Y$  occurring in the Local Invertibility Theorem 2.2. In order to apply this theorem we have to ensure that the linearized problem is uniquely solvable in appropriate spaces. Therefore we make the following assumption:

(U) If  $\Omega$  has only conical singularities, then the operator  $\mathcal{A}_{d, \vec{\beta}, p}$  in Theorem 3.6 is an isomorphism for  $d + 2 - \beta_P - \frac{n}{p} = 1 - \frac{n}{2}$  for all  $P \in \mathcal{P}$ . If  $\Omega$  is a polyhedron, we assume that the operator  $\mathbf{A}$  is an isomorphism from  $\mathring{\mathbf{H}}^1(\Omega)$  into  $\mathbf{H}^{-1}(\Omega)$ .

**Remark 3.12.** For domains with conical points, the choice of the values of  $d$  and  $\vec{\beta}$  is motivated by existence and uniqueness theorems for weak solutions in  $\mathbf{H}^1(\Omega)$ . If problem (3) is not uniquely solvable, then it is necessary to use appropriate factor spaces using the fact that  $\mathcal{A}_{d, \vec{\beta}, p}$  is a Fredholm operator.

**3.3 Domains with conical points.** For any fixed  $\mu_P < d + 1 - \frac{n}{p} + \frac{n}{2}$  ( $P \in \mathcal{P}$ ) let  $\mathbf{v}_1, \dots, \mathbf{v}_z$  be a basis of the spaces generated by all singular vector functions  $\chi_P \xi_P^{\lambda, \nu, q}$  from Theorem 3.7 corresponding to the eigenvalues in  $\bigcup_{P \in \mathcal{P}} \Lambda_P(1 - \frac{n}{2}, d + 2 - \frac{n}{p} - \mu_P)$ . We define

$$\mathbf{D}_{\vec{\mu}, p}^{d+2}(\Omega) = \operatorname{span}\{\mathbf{v}_1, \dots, \mathbf{v}_z\} \oplus \mathbf{V}_{\vec{\mu}, p}^{d+2}(\Omega).$$

The norm of an element  $\mathbf{u} \in \mathbf{D}_{\vec{\mu}, p}^{d+2}(\Omega)$  with the decomposition  $\mathbf{u} = \sum_{j=1}^z c_j \mathbf{v}_j + \tilde{\mathbf{u}}$  is defined by

$$\|\mathbf{u}\|_{\mathbf{D}_{\vec{\mu}, p}^{d+2}(\Omega)} = \sum_{j=1}^z |c_j| + \|\tilde{\mathbf{u}}\|_{\mathbf{V}_{\vec{\mu}, p}^{d+2}(\Omega)}.$$

For every  $P \in \mathcal{P}$  let

$$\alpha_P = \min \left\{ \operatorname{Re} \lambda : \lambda \in \Lambda_P \left( 1 - \frac{n}{2}, d + 2 - \frac{n}{p} - \mu_P \right) \right\}$$

$$\nu_P = d + 2 - \frac{n}{p} - \alpha_P + \varepsilon$$

where  $\varepsilon$  is a small positive number. Furthermore, let  $\vec{\nu}$  be the vector with the components  $\nu_P$  ( $P \in \mathcal{P}$ ).

**Lemma 3.13.** *The imbedding  $\mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega) \rightarrow \mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega)$  is continuous.*

**Proof.** This is a direct consequence of the fact that  $\mathbf{v}_j \in \mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega)$  and of the continuous imbedding of  $\mathbf{V}_{\vec{\mu},p}^{d+2}(\Omega)$  into  $\mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega)$  ■

Theorem 3.7 and uniqueness assumption (U) allow to formulate the mapping properties of the operator  $(\mathbf{A}, \mathbf{C}^{\mathbf{N}}, \mathbf{I})$  of the linearized problem (3) in the space  $\mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega)$ .

**Theorem 3.14.** *Suppose that the line  $\operatorname{Re} \lambda = d + 2 - \frac{n}{p} - \mu_P$  does not contain eigenvalues of the pencil  $\mathcal{A}_P(\lambda)$ , that  $0 < d + 1 - \frac{n}{p} + \frac{n}{2} - \mu_P < 1$  for every  $P \in \mathcal{P}$  and that condition (U) is satisfied. Then the operator*

$$(\mathbf{A}, \mathbf{C}^{\mathbf{N}}, \mathbf{I}) : \mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega) \rightarrow \mathbf{V}_{\vec{\mu},p}^d(\Omega) \times \mathbf{V}_{\vec{\mu},p}^{d+1-1/p}(\Gamma^N) \times \mathbf{V}_{\vec{\mu},p}^{d+2-1/p}(\Gamma^D)$$

is an isomorphism.

**Proof.** The mapping is bijective due to expansion (9). Furthermore, the continuity of the inverse mapping follows from the estimate (see [20: Theorem 4])

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega)} &= \sum_{j=1}^z |c_j| + \|\mathbf{u}_{reg}\|_{\mathbf{V}_{\vec{\mu},p}^{d+2}(\Omega)} \\ &\leq C \left( \|\mathbf{A}\mathbf{u}\|_{\mathbf{V}_{\vec{\mu},p}^d(\Omega)} + \|\mathbf{C}^{\mathbf{N}}\mathbf{u}\|_{\mathbf{V}_{\vec{\mu},p}^{d+1-1/p}(\Gamma^N)} + \|\mathbf{u}\|_{\mathbf{V}_{\vec{\mu},p}^{d+2-1/p}(\Gamma^D)} \right) \end{aligned}$$

and the assertion is proved ■

**3.2.2 Straight polyhedral domains.** Similarly, if  $\Omega$  is a straight polyhedron, we define  $\mathbf{D}_{0,0}(\Omega)$  as the space consisting of all functions of form (10) with  $c_P^{\lambda,\nu,q} \in \mathbb{R}$  and  $c_E^{\lambda,\nu,q} \in H_{-\operatorname{Re} \lambda}^{1-\operatorname{Re} \lambda}(E)$  equipped with the norm

$$\begin{aligned} \|\mathbf{u}\|_{\mathbf{D}_{0,0}(\Omega)} &= \|\mathbf{u}_{reg}\|_{\mathbf{H}_{0,0}^2(\Omega)} + \sum_{P \in \mathcal{P}} \sum_{\lambda \in \Lambda_P(-\frac{1}{2}, \frac{1}{2})} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda,\nu)} |c_P^{\lambda,\nu,q}| \\ &\quad + \sum_{E \in \mathcal{E}} \sum_{\lambda \in \Lambda_E(0,1)} \sum_{\nu=1}^{M(\lambda)} \sum_{q=1}^{\kappa(\lambda,\nu)} \|c_E^{\lambda,\nu,q}\|_{H_{-\operatorname{Re} \lambda}^{1-\operatorname{Re} \lambda}(E)}. \end{aligned}$$

**Theorem 3.15.** *Let the assumptions of Theorem 3.10 and condition (U) be satisfied. Then the operator  $\mathbf{A}$  is an isomorphism between the spaces  $\mathbf{D}_{0,0}(\Omega)$  and  $\mathbf{L}_2(\Omega)$ . Moreover, the space  $\mathbf{D}_{0,0}(\Omega)$  is continuously embedded into  $\mathbf{H}_{\alpha,\beta}^2(\Omega) \cap \dot{\mathbf{H}}^1(\Omega)$  with  $\alpha$  and  $\beta$  from Theorem 3.11.*

**Proof.** The first assertion follows from estimate (11) and the closed graph theorem. The second assertion follows from estimate (12) and the inverse estimate (11) ■

## 4. Main results

Theorems 3.14 and 3.15 suggest the choices of the spaces  $X$  and  $Y$  in the Local Invertibility Theorem 2.2. Namely, if  $\Omega$  has only conical singularities, then we will take

$$\begin{aligned} X &= \mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega) \\ Y &= Y_{\vec{\mu},p}^d = \mathbf{V}_{\vec{\mu},p}^d(\Omega) \times \mathbf{V}_{\vec{\mu},p}^{d+1-\frac{1}{p}}(\Gamma^N) \times \mathbf{V}_{\vec{\mu},p}^{d+2-\frac{1}{p}}(\Gamma^D). \end{aligned}$$

On the other hand, if  $\Omega$  is a polyhedron, then we will clearly take  $X = \mathbf{D}_{0,0}(\Omega)$  and  $Y = \mathbf{L}_2(\Omega)$ . For these choices of spaces we have to show that the operator  $N$  corresponding to nonlinear boundary value problem (1) is continuously Fréchet differentiable. For that purpose we prove multiplication and composition theorems in weighted Sobolev spaces extending some results from [1 - 3]. In Subsection 4.1 we state the multiplication and composition theorems, in Subsection 4.2 we present the results of the application of the Local Invertibility Theorem, the main results of this paper. The proofs are postponed to Section 5.

**4.1 Multiplication and composition theorems in weighted Sobolev spaces.** In this subsection we state some multiplication and composition theorems.

**4.1.1 Domains with conical points.** Let us start with a multiplication theorem for scalar functions in domains with only one conical point. Replacing in Definition 3.2 the infinite cone  $C_P$  by the bounded domain  $\Omega$  with the conical boundary point  $P$ , we get the space  $V_{\beta,p}^d(\Omega)$ .

**Theorem 4.1** [1, 3]. *Let  $2 \leq q \in \mathbb{N}_0$ ,  $1 < p < \infty$ , and  $d_i \in \mathbb{N}_0$ ,  $\gamma_i \in \mathbb{R}$  for all  $i = 1, \dots, q$ . Further, let  $d \in \mathbb{N}_0$  be such that  $d \leq d_i$  for all  $i = 1, \dots, q$  and  $\sum_{i \in I_j} d_i > (j-1)\frac{n}{p} + d$  for all  $j = 2, \dots, q$  and all  $I_j \subset \{1, \dots, q\}$  such that  $\#I_j = j$ . Then for all  $u_i \in V_{\gamma_i,p}^{d_i}(\Omega)$  ( $i = 1, \dots, q$ )*

$$\prod_{i=1}^q u_i \in V_{\gamma,p}^d(\Omega)$$

where

$$\gamma \geq \sum_{i=1}^q \gamma_i + d - \sum_{i=1}^q d_i + \frac{n}{p}(q-1).$$

Moreover, there exists a constant  $C > 0$  such that

$$\left\| \prod_{i=1}^q u_i \right\|_{V_{\gamma,p}^d(\Omega)} \leq C \prod_{i=1}^q \|u_i\|_{V_{\gamma_i,p}^{d_i}(\Omega)}.$$



We proceed with the mapping properties of the composition operator

$$G : \mathbf{u} \rightarrow g \circ \mathbf{u}$$

in weighted Sobolev spaces, defined for simplicity for domains with only one conical point ( $r = r_P$ ), and formulate a result which differs slightly from [1, 2].

**Theorem 4.2** [1, 2]. *Let  $d, d' \in \mathbb{N}_0$  with  $d > \frac{n}{p}$  and  $d \geq d'$ , and let  $\gamma, \gamma', p \in \mathbb{R}$  with  $1 < p$ .*

(i) *Assume that  $d - \gamma - \frac{n}{p} > 0$ . If  $g \in C^{d'}(\mathbb{R}^k)$  satisfies the flatness condition*

$$|D^\alpha g(\mathbf{x})| \leq C|\mathbf{x}|^{s-|\alpha|} \tag{13}$$

*for all  $|\mathbf{x}| \leq 1$  and  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $|\alpha| \leq d'$ , for some  $C > 0$  (independent of  $\mathbf{x}$ ) and some real number  $s \geq d'$  such that*

$$(d' - \frac{n}{p} - \gamma') < s(d - \frac{n}{p} - \gamma), \tag{14}$$

*then the scalar-valued composition operator  $G : \mathbf{u} \rightarrow g \circ \mathbf{u}$  maps  $\mathbf{V}_{\gamma,p}^d(\Omega) \rightarrow V_{\gamma',p}^{d'}(\Omega)$  and*

$$\|g \circ \mathbf{u}\|_{V_{\gamma',p}^{d'}(\Omega)} \leq C \left\{ \|\mathbf{u}\|_{\mathbf{V}_{\gamma,p}^d(\Omega)}^s + \chi_{[1,+\infty)}(\|\mathbf{u}\|_{\mathbf{V}_{\gamma,p}^d(\Omega)}) \|\mathbf{u}\|_{\mathbf{V}_{\gamma,p}^d(\Omega)} \right\}$$

*where  $\chi_{[1,+\infty)}$  is the characteristic function of  $[1, +\infty)$ .*

(ii) *If  $d - \gamma - \frac{n}{p} < 0$ , then the same assertion holds, if we add to condition (13) the growth condition*

$$|D^\alpha g(\mathbf{x})| \leq C|\mathbf{x}|^{s-|\alpha|} \tag{15}$$

*for all  $|\mathbf{x}| > 1$  and  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $|\alpha| \leq d'$ .*

Finally, we discuss the continuity and the differentiability of the Nemytskij operator  $G$  in the neighbourhood of  $\mathbf{u} = \mathbf{0}$ . Note that the local Lipschitz continuity of the Nemytskij operator has been proved in [2: Theorem 4.11].

**Theorem 4.3.** *Let  $d, d' \in \mathbb{N}_0$  with  $d > \frac{n}{p}$  and  $d \geq d'$ , and let  $\gamma, \gamma', p \in \mathbb{R}$  with  $1 < p$ . Assume that  $d - \gamma - \frac{n}{p} > 0$  and  $d - d' - \gamma + \gamma' > 0$ . If  $g \in C^{d'+1}(\mathbb{R}^k)$  satisfies the flatness condition*

$$|D^\alpha g(\mathbf{x})| \leq C|\mathbf{x}|^{s-|\alpha|} \tag{16}$$

*for all  $|\mathbf{x}| \leq 1$  and  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $|\alpha| \leq d' + 1$ , for some  $C > 0$  (independent of  $\mathbf{x}$ ) and some real number  $s \geq d' + 1$  such that (14) holds,*

then the scalar-valued composition operator  $G : \mathbf{u} \rightarrow g \circ \mathbf{u}$ , which maps  $\mathbf{V}_{\gamma,p}^d(\Omega) \rightarrow V_{\gamma',p}^{d'}(\Omega)$ , is differentiable in a neighbourhood  $\mathcal{U}(\mathbf{0}) \subset \mathbf{V}_{\gamma,p}^d(\Omega)$  of  $\mathbf{u} = \mathbf{0}$ . The Fréchet derivative  $G'(\mathbf{v})$  is given by

$$G'(\mathbf{v})\mathbf{u} = \langle \partial_{\mathbf{u}}g(\mathbf{v}), \mathbf{u} \rangle. \quad (17)$$

Furthermore, the composition operator  $H_j : \mathbf{V}_{\gamma,p}^d(\Omega) \rightarrow V_{\eta,p}^{d'}$  defined by  $H_j : \mathbf{u} \mapsto \partial_{u_j}g \circ \mathbf{u}$  is continuous for  $\eta = \gamma' - \gamma + d - \frac{n}{p}$ .

**4.1.2 Straight polyhedral domains.** We start with a multiplicative result in weighted Sobolev spaces  $H_{\alpha,\beta}^l(\Omega)$  that will be useful later on (compare with Theorem 4.1).

**Theorem 4.4.** *Let  $2 \leq q \in \mathbb{N}_0$ , and  $d_i \in \mathbb{N}_0, \alpha_i, \beta_i \in \mathbb{R}$  for all  $i = 1, \dots, q$ . Further, let  $d \in \mathbb{N}_0$  be such that  $d \leq d_i$  for all  $i = 1, \dots, q$  and  $\sum_{i \in I_j} d_i > (j-1)\frac{3}{2} + d$  for all  $j = 2, \dots, q$  and all  $I_j \subset \{1, \dots, q\}$  such that  $\#I_j = j$ . Then, for all  $u_i \in H_{\alpha_i, \beta_i}^{d_i}(\Omega)$  ( $i = 1, \dots, q$ ),*

$$\prod_{i=1}^m u_i \in H_{\alpha,\beta}^d(\Omega)$$

where

$$\alpha \geq \sum_{i=1}^q \alpha_i + d - \sum_{i=1}^q d_i + \frac{3}{2}(q-1)$$

$$\beta \geq \sum_{i=1}^q \beta_i + d - \sum_{i=1}^q d_i + (q-1).$$

Moreover, there exists a constant  $C > 0$  such that

$$\left\| \prod_{i=1}^q u_i \right\|_{H_{\alpha,\beta}^d(\Omega)} \leq C \prod_{i=1}^q \|u_i\|_{H_{\alpha_i, \beta_i}^{d_i}(\Omega)}. \quad (18)$$

We now pass to the composition result. Our goal is to give sufficient conditions on  $g$  which insure that  $G$  becomes differentiable from  $\mathbf{H}_{\alpha,\beta}^d(\Omega)$  into  $H_{\alpha',\beta'}^{d'}(\Omega)$ , for any  $d, d' \in \mathbb{N}_0$  with  $d' \leq d$ .

**Theorem 4.5.** *Let  $d, d' \in \mathbb{N}_0$  with  $d \geq 2$  and  $d \geq d'$ , and let  $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$ . Further, let  $g \in C^{d'}(\mathbb{R}^k)$  satisfy (13) for some real number  $s \geq d'$  such that*

$$(d' - \frac{3}{2} - \alpha') < s(d - \frac{3}{2} - \alpha) \quad (19)$$

$$(d' - 1 - \beta') < s(d - 1 - \beta). \quad (20)$$

If  $d - \frac{3}{2} - \alpha$  or  $d - 1 - \beta$  is negative, we further require that (15) holds. Then, for all  $\mathbf{u} \in \mathbf{H}_{\alpha,\beta}^d(\Omega)$ ,

$$g \circ \mathbf{u} \in H_{\alpha',\beta'}^{d'}(\Omega).$$

Moreover, there exists a continuous function  $C$  such that

$$\|g \circ \mathbf{u}\|_{H_{\alpha',\beta'}^{d'}(\Omega)} \leq C(\|\mathbf{u}\|_{\mathbf{H}_{\alpha,\beta}^d(\Omega)})\|\mathbf{u}\|_{\mathbf{H}_{\alpha,\beta}^d(\Omega)}^s.$$

We finish with the differentiability properties of  $G$ .

**Theorem 4.6.** *Let  $d, d' \in \mathbb{N}_0$  with  $d \geq 2$  and  $d \geq d'$ , and let  $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$  such that*

$$\begin{aligned} d - \frac{3}{2} - \alpha &> 0 \\ d - 1 - \beta &> 0 \\ d - \alpha - d' + \alpha' &> 0 \\ d - \beta - d' + \beta' &> 0. \end{aligned}$$

Further, let  $g \in C^{d'+1}(\mathbb{R}^k)$  satisfy growth condition (16) for some real number  $s \geq d' + 1$  such that (19) – (20) hold. Then the operator

$$G : \mathbf{H}_{\alpha,\beta}^d(\Omega) \ni \mathbf{u} \rightarrow g \circ \mathbf{u} \in H_{\alpha',\beta'}^{d'}(\Omega)$$

is differentiable in a neighbourhood  $\mathcal{U}(\mathbf{0}) \subset \mathbf{H}_{\alpha,\beta}^d(\Omega)$  of  $\mathbf{u}$ .

**4.2 Existence, uniqueness and asymptotic behaviour of the solution of the quasilinear problem.** Now we use our results on the mapping properties of the composition operator on domains with conical points or corners and edges to study the Fréchet-differentiability near the zero solution of the quasilinear operator associated with problem (1). Then we apply the Local Invertibility Theorem to deduce existence, uniqueness and asymptotic behaviour of the solution of (1). In this section we present only the main results, the proofs are postponed to Section 5.

**4.2.1 Domains with conical points.** For all  $P \in \mathcal{P}$ , introduce the following notations:

$$\alpha_P = \min \left\{ \mathbf{R} \lambda : \lambda \in \Lambda_P \left( 1 - \frac{n}{2}, \vartheta_P \right) \right\} \quad (21)$$

$$\delta_P = \max \left\{ \mathbf{R} \lambda : \lambda \in \Lambda_P \left( 1 - \frac{n}{2}, \vartheta_P \right) \right\} \quad (22)$$

$$\nu_P = d + 2 - \frac{n}{p} - \alpha_P + \varepsilon \quad (23)$$

$$\mu_P = d + 2 - \frac{n}{p} - \delta_P - \varepsilon \quad (24)$$

$$Y_{\bar{\mu},p}^d = \mathbf{V}_{\bar{\mu},p}^d(\Omega) \times \mathbf{V}_{\bar{\mu},p}^{d+1-\frac{1}{p}}(\Gamma^N) \times \mathbf{V}_{\bar{\mu},p}^{d+2-\frac{1}{p}}(\Gamma^D) \quad (25)$$

where  $\vartheta_P > 1 - \frac{n}{2}$  are real numbers given a-priori for all  $P \in \mathcal{P}$  such that the set  $\Lambda_P(1 - \frac{n}{2}, \vartheta_P)$  is not empty and  $\varepsilon > 0$  is a fixed sufficiently small number.

**Theorem 4.7.** *Let us assume the following:*

- (A1)  $a_{ij\sigma\tau} \in \mathbb{C}^2(\mathbb{R}^k), b_{j\tau} \in \mathbb{C}^2(\mathbb{R}^k), c_{i\sigma\tau} \in \mathbb{C}^1(\mathbb{R}^k), d_\tau \in \mathbb{C}^2(\mathbb{R}^k).$
- (A2)  $b_{j\tau}(\mathbf{0}) = d_\tau(\mathbf{0}) = \mathbf{0}.$
- (A3)  $\alpha_P > 0, 2\alpha_P > \delta_P, \alpha_P + 1 > \delta_P, 2 - \frac{n}{2} > \delta_P$  for all  $P \in \mathcal{P}.$
- (A4)  $(\mathbf{f}, \mathbf{h}, \mathbf{g}) \in Y_{\bar{\mu}, p}^d$  with sufficiently small norms and  $d \in \{0, 1\}, p > n.$
- (A5) Assumptions (E) and (U) are satisfied.

Then nonlinear problem (1) has a unique solution  $\mathbf{u} \in \mathbf{D}_{\bar{\mu}, p}^{d+2}(\Omega).$

**Remark 4.8.** The assumption  $2\alpha_P > \delta_P$  is a restriction on the length of the asymptotics, which is generated by the non-linearity of the coefficients  $a_{ij\sigma\tau}.$  In the semilinear case we need only the condition  $1 + 2\alpha_P > \delta_P$  instead of  $2\alpha_P > \delta_P.$

We now formulate an existence result for nonlinear problem (1) under weaker assumptions on the given right-hand sides as in Theorem 4.7, however, the solution admits no singular decomposition of form (9).

**Theorem 4.9.** *Let  $\mu'_P = d + 2 - \frac{n}{p} - \beta_P$  with  $0 \leq \beta_P < \min\{\alpha_P, 2 - \frac{n}{2}\}$  for all  $P \in \mathcal{P}.$  We assume that assumptions (A1), (A2), (A5) and the assumption*

- (A4')  $(\mathbf{f}, \mathbf{h}, \mathbf{g}) \in Y_{\bar{\mu}', p}^d$  with sufficiently small norms and  $d \in \{0, 1\}, p > n$

are satisfied. Then nonlinear problem (1) has a unique solution  $\mathbf{u} \in \mathbf{V}_{\bar{\mu}', p}^{d+2}(\Omega).$

Let us illustrate the above results for our Example 2.1:

**Example 4.10.** Let  $\Omega$  be a 2-dimensional polygonal domain and  $\omega_{0P}$  the interior opening angle and  $\omega_P$  the running polar angle to the corner point  $P.$  For Example 2.1 with Dirichlet boundary conditions on  $\partial\Omega$  (only for the sake of simplicity), assumptions (A1) and (A2) clearly hold. To check assumption (A3) we remark that  $\alpha_P = \frac{\pi}{\omega_{0P}}$  and that we can chose  $\vartheta_P < \min\{\frac{2\pi}{\omega_{0P}}, \frac{\pi}{\omega_{0P}} + 1, 2\}.$  Thus we have  $\alpha_P = \delta_P$  for reentrant corners. Finally, assumption (A5) holds if  $\delta$  is sufficiently small with respect to  $\varepsilon$  (due to Poincaré's inequality). Under these assumptions, if  $N \in V_{\bar{\mu}, p}^d(\Omega)$  and  $(\psi_D, u_D, v_D) \in [V_{\bar{\mu}, p}^{d+2-1/p}(\partial\Omega)]^3$  with a sufficiently small norm (cf. assumption (A4)), problem (2) admits a unique solution  $(\psi, u, v)$  which admits the decomposition

$$\psi = \psi_{reg} + \sum_{P \in \mathcal{P}} \chi_P(r_P) c_P^\psi r_P^{\frac{\pi}{\omega_{0P}}} \sin\left(\frac{\pi\omega_P}{\omega_{0P}}\right)$$

where  $\psi_{reg}$  belongs to  $V_{\bar{\mu}, p}^{d+2}(\Omega)$  and  $c_P^\psi$  is some constant. A similar expansion for  $u$  and  $v$  holds replacing  $\psi$  by  $u$  and  $v$ , respectively.

Alternatively, Theorem 4.9 gives a solution  $(\psi, u, v)$  in  $[V_{\vec{\mu}', p}^{d+2}(\Omega)]^3$  provided  $\beta_P < \min\{1, \frac{\pi}{\omega_{0P}}\}$  for appropriated data.

**4.2.2 Straight polyhedral domains.** We modify Theorem 4.7 for polyhedral domains.

**Theorem 4.11.** *We suppose that the conditions of Theorem 3.10, assumptions (A2) and (A5) as well as the assumptions*

(A1'')  $a_{ij\sigma\tau} \in \mathbb{C}^3(\mathbb{R}^k), b_{j\tau} \in \mathbb{C}^2(\mathbb{R}^k), c_{i\sigma\tau} \in \mathbb{C}^1(\mathbb{R}^k), d_\tau \in \mathbb{C}^2(\mathbb{R}^k)$

(A3'')  $\alpha_0 > \frac{1}{4}$  and  $\beta_0 > \frac{1}{2}$  with  $\alpha_0$  and  $\beta_0$  defined in Corollary 3.11

(A4'')  $\mathbf{f} \in L_2(\Omega)$  with a sufficiently small norm

hold. Then nonlinear problem (1) with  $\mathbf{g} = 0$  has a unique solution  $\mathbf{u} \in \mathbf{D}_{0,0}(\Omega)$ .

**Remark 4.13.** Note that [21: Theorem 1] gives sufficient conditions which guarantee that  $\beta_0 > \frac{1}{2}$ . These conditions always hold for scalar operators ( $k = 1$ ). The condition  $\alpha_0 > \frac{1}{4}$  is relatively strong but cannot be removed in our level of generality, e.g., if  $k = 1$  and  $\mathbf{A}$  is the Laplace operator, then  $\alpha_0 > \frac{1}{4}$  if, for all  $P \in \mathcal{P}$ ,  $C_P$  is included in a revolution cone of opening  $\xi < \xi_0$ , with  $\xi_0 \approx 160^\circ$  (see [14: Section 18.D]). If Theorem 3.10 would have been established in  $L_p$ -spaces, then we could get similar results as in Theorem 4.7 without the assumption  $\alpha_0 > \frac{1}{4}$ .

**Example 4.13.** For Example 2.1 assumptions (A1'') and (A2) directly hold while assumption (A5) holds for appropriated  $\varepsilon$  and  $\delta$  (see above). For assumption (A3'') the condition  $\beta_0 > \frac{1}{2}$  is equivalent to  $\omega_{0E} < 2\pi$ , while the condition  $\alpha_0 > \frac{1}{4}$  is simply equivalent to  $\nu_{1P} > \frac{5}{16}$  for all  $P \in \mathcal{P}$ , where  $\nu_{1P}$  is the first eigenvalue of  $L_P^{Dir}$  (which could be explicitly checked either numerically or analytically, cf. Remark 4.12). Under these assumptions, if  $N \in L_2(\Omega)$  with a sufficiently small norm and  $(\psi_D, u_D, v_D) = (0, 0, 0)$ , problem (2) admits a unique solution  $(\psi, u, v)$  which admits decomposition (10) with singularities as described in Example 3.8.

## 5. Proofs of the main results

**5.1 Multiplication and composition theorems in weighted Sobolev spaces.** We prove the results of Subsection 4.1 in the following subsections.

**5.1.1 Domains with conical points.** We start with the proof of the multiplication Theorem 4.1.

**Proof of Theorem 4.1.** We remark that the weighted Sobolev spaces  $V_{\gamma,p}^d(\Omega)$  coincide with the standard Sobolev spaces  $W_p^d(\Omega)$  outside a small neighbourhood of the conical point  $P$ . Since the multiplication theorem is valid for the spaces  $W_p^d(\Omega)$  (see, e.g., [18, 48]), it is enough here to prove the assertion for the weighted spaces  $V_{\gamma,p}^d(C_P)$ .

Let  $G_P := C_P \cap S^{n-1}$  and  $Z_P := G_P \times \mathbb{R}$ , and let  $(r, \omega)$  be the spherical coordinates in  $\mathbb{R}^n$  with origin in  $P$ . For arbitrary functions  $u(r, \omega)$  on  $\mathbb{R}_+ \times G_P$  let  $\mathcal{T}u$  be the Euler transformation of  $u$ , i.e.

$$(\mathcal{T}u)(t, \omega) = u(e^t, \omega). \tag{26}$$

The assertion of the theorem follows from the equivalence of the norms  $\|u\|_{V_{\gamma,p}^d(C_P)}$  and  $\|e^{(\gamma-d+\frac{n}{p})t}\mathcal{T}u\|_{W_p^d(Z_P)}$  (see [23: p. 193]) and the multiplication properties of (non-weighted) Sobolev spaces (see, e.g., [18, 48]) ■

**Proof of Theorem 4.2.** We have to show that  $g \circ \mathbf{u} \in V_{\gamma',p}^{d'}(\Omega)$ , i.e.

$$r^{|\alpha|-d'+\gamma'} D^\alpha(g \circ \mathbf{u}) \in L_p(\Omega) \quad (|\alpha| \leq d'). \tag{27}$$

In order to verify this we use Theorem 4.1, flatness condition (13), the estimate

$$|\mathbf{u}(\mathbf{x})| \leq C|\mathbf{x}|^{d-\gamma-\frac{n}{p}} \|\mathbf{u}\|_{\mathbf{V}_{\gamma,p}^d(C_P)} \tag{28}$$

valid for all  $\mathbf{u} \in \mathbf{V}_{\gamma,p}^d(C_P)$  with  $d > \frac{n}{p}$  [30: Lemma 1.1] (see also [38: p. 88]) and Faa di Bruno's formula (see, e.g., [26])

$$D^\alpha(g \circ \mathbf{u}) = \sum_{1 \leq |\beta| \leq |\alpha|} (D^\beta g) \circ \mathbf{u} \cdot \left( \sum_q c_q \prod_{i=1}^n \prod_j D^{q_{ij}} u_i \right) \quad (|\alpha| \geq 1) \tag{29}$$

where the summation is taken over all multi-indices  $q = (q_{ij})$  such that

$$\sum_{i,j} q_{ij} = \alpha, \quad |q_{ij}| \geq 1, \quad \sum_{i,j} (|q_{ij}| - 1) = |\alpha| - |\beta|$$

and  $c_q > 0$  are real constants (see [2: Theorem 4.5] for the details).

The case (i) describes the case when  $\mathbf{u}$  vanishes near the corner point  $P$ , whereas the case (ii) allows a moderate unbounded behaviour of  $\mathbf{u}$  near  $P$ . Note that in the case (ii) estimate (14) implies an upper bound for  $s$ , namely

$$d' \leq s < \frac{-d'+\frac{n}{p}+\gamma'}{-d+\frac{n}{p}+\gamma} \quad \blacksquare$$

**Proof of Theorem 4.3.** Clearly, Theorem 4.2 implies that  $G$  maps  $\mathbf{V}_{\gamma,p}^d(\Omega)$  to  $V_{\gamma',p}^{d'}(\Omega)$ . Let  $\mathbf{u}, \mathbf{v} \in \mathcal{U}(\mathbf{0})$ . Applying the multiplication Theorem 4.1 we obtain

$$\begin{aligned} & \|G(\mathbf{u}) - G(\mathbf{v}) - \langle \partial_{\mathbf{u}}g(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle\|_{V_{\gamma',p}^{d'}(\Omega)} \\ &= \left\| \left\langle \mathbf{u} - \mathbf{v}, \int_0^1 \partial_{\mathbf{u}}g(\mathbf{v} + t(\mathbf{u} - \mathbf{v})) - \partial_{\mathbf{u}}g(\mathbf{v}) dt \right\rangle \right\|_{V_{\gamma',p}^{d'}(\Omega)} \quad (30) \\ &\leq \|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}_{\gamma,p}^d(\Omega)} \int_0^1 \|\partial_{\mathbf{u}}g(\mathbf{v} + t(\mathbf{u} - \mathbf{v})) - \partial_{\mathbf{u}}g(\mathbf{v})\|_{\mathbf{V}_{\eta,p}^{d'}} dt. \end{aligned}$$

The flatness condition guarantees that the right-hand sides of the above inequality are well defined (compare Theorem 4.2).

It remains to show that the composition operator  $H_j : \mathbf{V}_{\gamma,p}^d(\Omega) \rightarrow V_{\eta,p}^{d'}(\Omega)$  is continuous. We start with the case  $d' = 0$ . Let  $(\mathbf{u}_n)$  be a sequence of elements in  $\mathbf{V}_{\gamma,p}^d(\Omega)$  which converges to an element  $\mathbf{u}$  in  $\mathbf{V}_{\gamma,p}^d(\Omega)$ . Due to estimate (28) and the assumption  $d - \gamma - \frac{n}{p} > 0$  we have  $\mathbf{u}_n \rightarrow \mathbf{u}$  in  $\mathbf{C}(\bar{\Omega})$ . Since  $g \in \mathbf{C}^1(\mathbb{R}^k)$ , we have  $H_j(\mathbf{u}_n)(\mathbf{x}) \rightarrow H_j(\mathbf{u})(\mathbf{x})$  in  $\mathbf{C}(\bar{\Omega})$  which implies that  $H_j(\mathbf{u}) \in \mathbf{C}(\bar{\Omega}) \cap V_{\eta,p}^0(\Omega)$ . Now we get the estimate

$$\begin{aligned} \|H_j(\mathbf{u}_n) - H_j(\mathbf{u})\|_{V_{\eta,p}^0(\Omega)}^p &= \int_{\Omega} r^{\eta p} |H_j(\mathbf{u}_n) - H_j(\mathbf{u})|^p d\mathbf{x} \\ &\leq \int_{\Omega} r^{\eta p} d\mathbf{x} \|H_j(\mathbf{u}_n) - H_j(\mathbf{u})\|_{\mathbf{C}(\bar{\Omega})}^p \\ &\leq C \|H_j(\mathbf{u}_n) - H_j(\mathbf{u})\|_{\mathbf{C}(\bar{\Omega})}^p \end{aligned}$$

since the exponents of  $r$  are chosen in appropriate manner ( $\eta + \frac{n}{p} > 0$ ).

Let  $d' > 0$  and the sequence  $(\mathbf{u}_n)$  be defined as before. We assume for simplicity that the functions  $\mathbf{u} = u$  and  $\mathbf{u}_n = u_n$  have only one component, i.e. we take  $k = 1$ . In this case Faa di Bruno formula (29) simplifies to

$$D^\gamma(g \circ u) = \sum_{q=1}^{|\gamma|} \frac{g^{(q)}(u)}{q!} \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \gamma}} \prod_{i=1}^q \frac{D^{j_i} u}{j_i!}. \quad (31)$$

The case  $k > 1$  is treated similarly using the general form (29) of the chain rule.

We have to show that

$$\|r^{-d'+|\alpha|+\eta} \{D^\alpha(h_j \circ u_n) - D^\alpha(h_j \circ u)\}\|_{L_p(\Omega)} \rightarrow 0 \quad (|\alpha| \leq d'). \quad (32)$$

The case  $\alpha = 0$  is treated as before since  $(-d' + \eta)p + n > 0$  is equivalent to the condition  $d - d' - \gamma + \gamma' > 0$ . For  $|\alpha| \geq 1$ , using Faa di Bruno's formula, the norm in (32) can be written as

$$\begin{aligned} & \left\| r^{-d'+|\alpha|+\eta} \left\{ \sum_{q=1}^{|\alpha|} (h_j^{(q)} \circ u_n) \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} \prod_{i=1}^q c_{iq} D^{j_i} u_n \right. \right. \\ & \quad \left. \left. - \sum_{q=1}^{|\alpha|} (h_j^{(q)} \circ u) \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} \prod_{i=1}^q c_{iq} D^{j_i} u \right\} \right\|_{L_p(\Omega)} \\ & \leq \sum_{q=1}^{|\alpha|} \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} \left\| r^{-d'+|\alpha|+\eta} \left\{ (h_j^{(q)} \circ u_n) \prod_{i=1}^q c_{iq} D^{j_i} u_n - (h_j^{(q)} \circ u) \prod_{i=1}^q c_{iq} D^{j_i} u \right\} \right\|_{L_p(\Omega)} \\ & \leq \sum_{q=1}^{|\alpha|} \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} c_{iq} \left\{ \left\| r^{-d'+|\alpha|+\eta} (h_j^{(q)} \circ u_n - h_j^{(q)} \circ u) \prod_{i=1}^q D^{j_i} u_n \right\|_{L_p(\Omega)} \right. \end{aligned} \quad (33)$$

$$\left. + \left\| r^{-d'+|\alpha|+\eta} (h_j^{(q)} \circ u) \sum_{k=1}^q (D^{j_k} u_n - D^{j_k} u) \prod_{i>k} D^{j_i} u_n \cdot \prod_{i<k} D^{j_i} u \right\|_{L_p(\Omega)} \right\} \quad (34)$$

with some real constants  $c_{iq} > 0$ . We now remark that the regularity  $h_j \in C^{d'}(\mathbb{R}^k)$  and (28) yield  $h_j^{(q)} \circ u_n \rightarrow h_j^{(q)} \circ u$  in  $C(\bar{\Omega})$  for all  $q = 0, \dots, d'$ . By the multiplication Theorem 4.1 it follows that

$$r^{-d'+|\alpha|+\eta} \prod_{i=1}^q D^{j_i} u_n \in L_p(\Omega)$$

provided that  $d' - \eta - \frac{n}{p} \leq q(d - \gamma - \frac{n}{p})$  for  $q = 1, \dots, |\alpha|$ . This is indeed satisfied since  $d' - \gamma' + \gamma - d < d - \gamma - \frac{n}{p}$ . Thus (33) converges to 0 because it can be estimated by

$$\sum_{q=1}^{|\alpha|} \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} c_{iq} \|h_j^{(q)} \circ u_n - h_j^{(q)} \circ u\|_{C(\bar{\Omega})} \left\| r^{-d'+|\alpha|+\eta} \prod_{i=1}^q D^{j_i} u_n \right\|_{L_p(\Omega)}.$$

Let us estimate (34). We remark that (16) yields  $|h_j^{(q)}(\mathbf{x})| \leq c|\mathbf{x}|^{s-1-q}$  for  $q = 1, \dots, d'$  and  $|\mathbf{x}| \leq 1$ . Applying inequality (28) to  $|u|^{s-1-q}$  and to  $D^{j_k}(u_n - u)$



we obtain

$$\begin{aligned} & \sum_{q=1}^{|\alpha|} \sum_{\substack{j_1, \dots, j_q \in \mathbb{N}^n \\ j_1 + \dots + j_q = \alpha}} \sum_{k=1}^q c_{iq} \left| r^{-d' + |\alpha| + \eta} (h_j^{(q)} \circ u) \prod_{i>k} D^{j_i} u_n \cdot \prod_{i<k} D^{j_i} u \cdot D^{j_k} (u_n - u) \right| \\ & \leq C \left| r^{-d' + |\alpha| + \eta + (d - \frac{n}{p} - \gamma)(s-1-q)} \prod_{i>k} D^{j_i} u_n \cdot \prod_{i<k} D^{j_i} u \cdot r^{d - \frac{n}{p} - \gamma - j_k} \right| \|u_n - u\|_{V_{\gamma, p}^d(\Omega)} \end{aligned}$$

for some constant  $C > 0$ . Thus (34) converges to 0 because it can be estimated by

$$\left\| r^{-d' + |\alpha| + \eta + (d - \frac{n}{p} - \gamma)(s-1-q) + d - \frac{n}{p} - \gamma - j_k} \prod_{i>k} D^{j_i} u_n \cdot \prod_{i<k} D^{j_i} u \right\|_{L_p(\Omega)} \|u_n - u\|_{V_{\gamma, p}^d(\Omega)}$$

and

$$r^{-d' + |\alpha| + \eta + (d - \frac{n}{p} - \gamma)(s-1-q) + d - \frac{n}{p} - \gamma - j_k} \prod_{i>k} D^{j_i} u_n \cdot \prod_{i<k} D^{j_i} u \in L_p(\Omega)$$

due to the multiplication Theorem 4.1 because  $d' - \eta - \frac{n}{p} < (s-1)(d - \gamma - \frac{n}{p})$  for  $s \geq 2$  ■

**5.1.2 Straight polyhedral domains.** Firstly, we prove the multiplication Theorem 4.4.

**Proof of Theorem 4.4.** In standard Sobolev spaces this is already a classical result (see, e.g., [18, 48]). Therefore we localize the problem in an appropriate neighbourhood  $\mathcal{V}$  of a vertex  $P$  and work in the cone  $C_P$ . First using spherical coordinates, we easily show that

$$\|v\|_{H_{\alpha, \beta}^d(C_P)}^2 \cong \sum_{l=0}^d \int_0^\infty r^{2\alpha - 2d + 3} \left\| \left( r \frac{\partial}{\partial r} \right)^l v \right\|_{V_\beta^{d-l}(G_P)}^2 \frac{dr}{r}$$

where  $V_\beta^{d-l}(G_P)$  is the weighted Sobolev space of Kondrat'ev's type on  $G_P$ , with weight being the distance to the corners of  $G_P$  (which is equivalent to  $\theta$ ); here for shortness we write  $r$  instead of  $r_P$ . Secondly, performing Euler Transformation (26) we get

$$\|v\|_{H_{\alpha, \beta}^d(C_P)}^2 \cong \sum_{l=0}^d \int_{-\infty}^\infty e^{t(2\alpha - 2d + 3)} \left\| \left( \frac{\partial}{\partial t} \right)^l \mathcal{T}v \right\|_{V_\beta^{d-l}(G_P)}^2 dt. \quad (35)$$

For  $\alpha = \alpha_1 + \alpha_2 + d - d_1 - d_2 + \frac{3}{2}$  we get by Leibniz's rule

$$\begin{aligned} & e^{t(\alpha-d+\frac{3}{2})} \left( \frac{\partial}{\partial t} \right)^l (\mathcal{T}u_1 \mathcal{T}u_2) \\ &= \sum_{j=0}^l \binom{l}{j} e^{t(\alpha_1-d_1+\frac{3}{2})} \frac{\partial^j \mathcal{T}u_1}{\partial t^j} e^{t(\alpha_2-d_2+\frac{3}{2})} \frac{\partial^{l-j} \mathcal{T}u_2}{\partial t^{l-j}}. \end{aligned} \quad (36)$$

Since  $u_i \in H_{\alpha_i, \beta_i}^{d_i}(C_P)$  ( $i = 1, 2$ ), it follows from (35) that  $\frac{\partial^j \mathcal{T}u_1}{\partial t^j}$  belongs to  $V_{\beta_1}^{d_1-j}(G_P)$  and  $\frac{\partial^{l-j} \mathcal{T}u_2}{\partial t^{l-j}}$  belongs to  $V_{\beta_2}^{d_2-(l-j)}(G_P)$ . Therefore by Theorem 4.1 (with  $n = 2$ ) we obtain

$$\frac{\partial^j \mathcal{T}u_1}{\partial t^j} \frac{\partial^{l-j} \mathcal{T}u_2}{\partial t^{l-j}} \in V_{\beta}^{d-l}(G_P).$$

Furthermore, there exists a constant  $C > 0$  such that

$$\left\| \frac{\partial^j \mathcal{T}u_1}{\partial t^j} \frac{\partial^{l-j} \mathcal{T}u_2}{\partial t^{l-j}} \right\|_{V_{\beta}^{d-l}(G_P)} \leq C \left\| \frac{\partial^j \mathcal{T}u_1}{\partial t^j} \right\|_{V_{\beta_1}^{d_1-j}(G_P)} \left\| \frac{\partial^{l-j} \mathcal{T}u_2}{\partial t^{l-j}} \right\|_{V_{\beta_2}^{d_2-(l-j)}(G_P)}.$$

Using this estimate in (36) and relation (35), we readily get the results. For  $q \geq 3$  we use an iterative argument. Finally, we derive estimate (18) from the localized estimates by means of a partition of unity ■

To prove the composition result we first need an asymptotic estimate as in (28) (see [30: Lemma 1.1]).

**Lemma 5.1.** *Let  $2 \leq d \in \mathbb{N}_0$ . Then for all  $\alpha, \beta \in \mathbb{R}$ , there exists a constant  $C > 0$  such that, for all  $u \in H_{\alpha, \beta}^d(\Omega)$ ,*

$$|u(\mathbf{x})| \leq Cr^{d-\frac{3}{2}-\alpha} \theta^{d-1-\beta} \|u\|_{H_{\alpha, \beta}^d(\Omega)} \quad (\mathbf{x} \in \Omega).$$

*In particular, if  $d-\frac{3}{2}-\alpha$  and  $d-1-\beta$  are non-negative, we have the continuous embedding*

$$H_{\alpha, \beta}^d(\Omega) \hookrightarrow C(\bar{\Omega}).$$

**Proof.** If the support of an element  $u \in H_{\alpha, \beta}^d$  does not contain edges and corners, then this is a consequence of the Sobolev imbedding theorem. Using (35) and a localization near a corner  $P$  and an edge  $E$  by a cut-off function  $\chi_E$ , we have

$$e^{t(\alpha-d+\frac{3}{2})} e^{s(\beta-d+1)} (\chi_E u)(e^t, e^s, \varphi_E) \in H^d(\mathbb{R}^2 \times (0, \omega_{0E}))$$

with  $e^s = \theta_E$  and

$$\left\| e^{t(\alpha-d+\frac{3}{2})} e^{s(\beta-d+1)} (\chi_E u)(e^t, e^s, \varphi_E) \right\|_{H^d(\mathbb{R}^2 \times (0, \omega_{0E}))} \leq C \|u\|_{H_{\alpha, \beta}^d(\Omega)}.$$

The Sobolev imbedding Theorem then yields

$$\sup_{t, s \in \mathbb{R}, \varphi_E \in (0, \omega_{0E})} \left| e^{t(\alpha-d+\frac{3}{2})} e^{s(\beta-d+1)} (\chi_E u)(e^t, e^s, \varphi_E) \right| \leq C \|u\|_{H_{\alpha, \beta}^d(\Omega)}.$$

Going back to  $C_P$ , we get the conclusion ■

**Proof of Theorem 4.5.** For the sake of simplicity, we give the proof in the case  $k = 1$ , the case  $k \geq 2$  can be treated similarly. We first remark that if  $d - \frac{3}{2} - \alpha$  and  $d - 1 - \beta$  are non-negative, then Lemma 5.1 implies that  $u \in H_{\alpha,\beta}^d(\Omega)$  is bounded in  $\bar{\Omega}$ , consequently  $g$  satisfies (13) for all  $x \in [-\|u\|_\infty, \|u\|_\infty]$ , with a function  $C$  which depends continuously on  $\|u\|_\infty$ . Conversely, if  $d - \frac{3}{2} - \alpha$  or  $d - 1 - \beta$  is negative, then  $u \in H_{\alpha,\beta}^d(\Omega)$  may be unbounded in  $\Omega$ . Therefore we have imposed (13) for all  $x \in \mathbb{R}$ . This means that in both cases there exists a constant  $C(u)$  depending continuously on  $\|u\|_{H_{\alpha,\beta}^d(\Omega)}$  such that, for all  $j = 0, \dots, d'$ ,

$$|g^{(j)}(u(\mathbf{x}))| \leq C(u)r^{(d-\frac{3}{2}-\alpha)(s-j)}\theta^{(d-1-\beta)(s-j)}\|u\|_{H_{\alpha,\beta}^d(\Omega)}^{s-j} \quad (\mathbf{x} \in \Omega). \quad (37)$$

For all  $|\gamma| \leq d'$  we shall show that this implies

$$r^{|\gamma|-d'+\alpha'}\theta^{|\gamma|-d'+\beta'}D^\gamma(g \circ u) \in L_2(\Omega) \quad (38)$$

$$\|r^{|\gamma|-d'+\alpha'}\theta^{|\gamma|-d'+\beta'}D^\gamma(g \circ u)\|_{L_2(\Omega)} \leq C(u)\|u\|_{H_{\alpha,\beta}^d(\Omega)}^s. \quad (39)$$

i) For  $|\gamma| = 0$ , owing to (37) with  $j = 0$ , we may write

$$|r^{-d'+\alpha'}\theta^{-d'+\beta'}(g \circ u)(\mathbf{x})| \leq C(u)r^{-d'+\alpha'+(d-\frac{3}{2}-\alpha)s}\theta^{-d'+\beta'+(d-1-\beta)s}\|u\|_{H_{\alpha,\beta}^d(\Omega)}^s \quad \blacksquare$$

for all  $\mathbf{x} \in \Omega$ . Integrating the square of this estimate over  $\Omega$ , one obtains

$$\|r^{-d'+\alpha'}\theta^{-d'+\beta'}(g \circ u)\|_{L_2(\Omega)} \leq C(u)\|u\|_{H_{\alpha,\beta}^d(\Omega)}^s$$

since conditions (19) - (20) guarantee the convergence of

$$\int_{\Omega} r^{2(-d'+\alpha')+(d-\frac{3}{2}-\alpha)2s}\theta^{2(-d'+\beta')+(d-1-\beta)2s}d\mathbf{x}.$$

This yields (38) - (39) for  $\gamma = 0$ .

ii) For  $|\gamma| \geq 1$  we use Faa di Bruno's formula (31). We shall show that each term of this right-hand side multiplied by the weight  $r^{|\gamma|-d'+\alpha'}\theta^{|\gamma|-d'+\beta'}$  belongs to  $L_2(\Omega)$ . Therefore let us fix  $q = 1, \dots, |\gamma|$  and  $j_1, \dots, j_q \in \mathbb{N}^n$  such that  $j_1 + \dots + j_q = \gamma$ . Then our goal reduces to prove that

$$w = r^{|\alpha|-d'+\alpha'}\theta^{|\alpha|-d'+\beta'}g^{(q)}(u)\prod_{i=1}^q D^{j_i}u$$

belongs to  $L_2(\Omega)$  and that

$$\|w\|_{L_2(\Omega)} \leq C(u)\|u\|_{H_{\alpha,\beta}^d(\Omega)}^s. \quad (40)$$

But from (37) we have for every  $\mathbf{x} \in \Omega$

$$|w(\mathbf{x})| \leq C(u) \|u\|_{H_{\alpha,\beta}^d(\Omega)}^{s-q} r^{|\gamma|-d'+\alpha'+(d-\frac{3}{2}-\alpha)(s-q)} \\ \times \theta^{|\gamma|-d'+\beta'+(d-1-\beta)(s-q)} \left| \prod_{i=1}^q D^{j_i} u(x) \right|.$$

Consequently,  $w \in L_2(\Omega)$  if

$$\prod_{i=1}^q D^{j_i} u \in H_{\delta,\eta}^0(\Omega) \quad (41)$$

for

$$\delta = |\gamma| - d' + \alpha' + (d - \frac{3}{2} - \alpha)(s - q) \\ \eta = |\gamma| - d' + \beta' + (d - 1 - \beta)(s - q).$$

Moreover, we will have the estimate

$$\|w\|_{L_2(\Omega)} \leq C(u) \|u\|_{H_{\alpha,\beta}^d(\Omega)}^{s-q} \left\| \prod_{i=1}^q D^{j_i} u \right\|_{H_{\delta,\eta}^0(\Omega)}. \quad (42)$$

The proof of inclusion (41) follows from Theorem 4.4 (except the case  $q = 1$  which is easily treated). Indeed, owing to that theorem, the inclusions  $D^{j_i} u \in H_{\alpha,\beta}^{l-|j_i|}(\Omega)$  ( $i = 1, \dots, q$ ) imply

$$\prod_{i=1}^q D^{j_i} u \in H_{\delta',\eta'}^0(\Omega) \quad (43)$$

where

$$\delta' = q\alpha - ql + |\gamma| + (q-1)\frac{3}{2} \\ \eta' = q\beta - ql + |\gamma| + (q-1)$$

with the estimate

$$\left\| \prod_{i=1}^q D^{j_i} u \right\|_{H_{\delta',\eta'}^0(\Omega)} \leq C \|u\|_{H_{\alpha,\beta}^d(\Omega)}^q. \quad (44)$$

It can be easily seen that condition (19) is equivalent to  $\delta > \delta'$  and (20) is equivalent to  $\eta > \eta'$ . Thus we have the imbedding

$$H_{\delta',\eta'}^0(\Omega) \hookrightarrow H_{\delta,\eta}^0(\Omega)$$

and consequently (41) follows from (43). This imbedding, (44) and (42) yield (40) ■

The proof of Theorem 4.6 is similar to that of Theorem 4.3 and is therefore omitted.

**5.2 Existence, uniqueness and asymptotic behaviour of the solution of the quasilinear problem.** We write problem (1) in the form of an operator equation

$$\mathbf{N}\mathbf{u} = (\mathbf{f}, \mathbf{h}, \mathbf{g})$$

with

$$\mathbf{f} = (f_1, \dots, f_k)$$

$$\mathbf{h} = (h_1, \dots, h_k)$$

$$\mathbf{g} = (g_1, \dots, g_k)$$

and the operator  $\mathbf{N}$  defined by

$$\begin{aligned} (\mathbf{N}\mathbf{u})_\tau = & \left( -\partial_j [a_{ij\sigma\tau}(\mathbf{u})\partial_i u_\sigma + b_{j\tau}(\mathbf{u})] + c_{i\sigma\tau}(\mathbf{u})\partial_i u_\sigma + d_\tau(\mathbf{u}), \right. \\ & \left. [a_{ij\sigma\tau}(\mathbf{u})\partial_i u_\sigma + b_{j\tau}(\mathbf{u})]n_j|_{\Gamma^N}, u_\tau|_{\Gamma^D} \right). \end{aligned} \quad (45)$$

In the case of straight polyhedral domains, we omit the boundary terms in the above definition.

We decompose the operator  $\mathbf{N}$  into the linearized part, defined by (3), and a remaining one. From assumptions (A1) and (A2) the following Taylor expansions and corresponding estimates are valid (see, e.g., [24]):

$$\begin{aligned} a_{ij\sigma\tau}(\mathbf{x}) &= a_{ij\sigma\tau}(\mathbf{0}) + \langle \partial_{\mathbf{x}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{x} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{x}) \\ |D^s \tilde{a}_{ij\sigma\tau}(\mathbf{x})| &\leq c|\mathbf{x}|^{2-|s|} \quad (|s| \leq 2, |\mathbf{x}| \leq 1) \end{aligned} \quad (46)$$

$$\begin{aligned} b_{j\tau}(\mathbf{x}) &= \langle \partial_{\mathbf{x}} b_{j\tau}(\mathbf{0}), \mathbf{x} \rangle + \tilde{b}_{j\tau}(\mathbf{x}) \\ |D^s \tilde{b}_{j\tau}(\mathbf{x})| &\leq c|\mathbf{x}|^{2-|s|} \quad (|s| \leq 2, |\mathbf{x}| \leq 1) \end{aligned} \quad (47)$$

$$\begin{aligned} c_{i\sigma\tau}(\mathbf{x}) &= c_{i\sigma\tau}(\mathbf{0}) + \tilde{c}_{i\sigma\tau}(\mathbf{x}) \\ |\tilde{c}_{i\sigma\tau}(\mathbf{x})| &\leq c|\mathbf{x}| \quad (|\mathbf{x}| \leq 1) \end{aligned} \quad (48)$$

$$\begin{aligned} d_\tau(\mathbf{x}) &= \langle \partial_{\mathbf{x}} d_\tau(\mathbf{0}), \mathbf{x} \rangle + \tilde{d}_\tau(\mathbf{x}) \\ |D^s \tilde{d}_\tau(\mathbf{x})| &\leq c|\mathbf{x}|^{2-|s|} \quad (|s| \leq 2, |\mathbf{x}| \leq 1). \end{aligned} \quad (49)$$

Inserting these Taylor expansions into (45) we get the decomposition

$$\begin{aligned} (\mathbf{N}\mathbf{u})_\tau = & \left\{ (\mathbf{A}\mathbf{u})_\tau - \partial_j \left[ \{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{u}) \} \partial_i u_\sigma + \tilde{b}_{j\tau}(\mathbf{u}) \right] \right. \\ & + \tilde{c}_{i\sigma\tau}(\mathbf{u})\partial_i u_\sigma + \tilde{d}_\tau(\mathbf{u}), (\mathbf{C}^N \mathbf{u})_\tau \\ & \left. + \left[ \{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{u}) \} \partial_i u_\sigma + \tilde{b}_{j\tau}(\mathbf{u}) \right] n_j|_{\Gamma^N}, u_\tau|_{\Gamma^D} \right\} \end{aligned}$$

shortly written as

$$\mathbf{N}\mathbf{u} = (\mathbf{A}, \mathbf{C}^N, \mathbf{I})\mathbf{u} + \tilde{\mathbf{N}}\mathbf{u}. \quad (50)$$

**5.2.1 Domains with conical points.** We investigate the mapping properties and the Fréchet differentiability of the operator  $\tilde{\mathbf{N}}$  as well as the continuity of the operator  $\mathbf{v} \mapsto \tilde{\mathbf{N}}'(\mathbf{v})$  at  $\mathbf{0}$ . Furthermore, we prove the main result on the existence, uniqueness and asymptotic behaviour of the solution of quasilinear problem (1).

**Theorem 5.2.** *Let assumptions (A1) - (A3) be satisfied. Then  $\tilde{\mathbf{N}}$  maps  $\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)$  into  $Y_{\vec{\mu}, p}^d$  and is differentiable in a neighbourhood  $\mathcal{U}(\mathbf{0}) \subset \mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)$  with*

$$\begin{aligned} & (\tilde{\mathbf{N}}'(\mathbf{v})\mathbf{u})_\tau \\ &= \left\{ -\partial_j \left[ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}) \right] \partial_i u_\sigma + \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma \right. \\ & \quad \left. + \langle \partial_{\mathbf{u}} \tilde{b}_{j\tau}(\mathbf{v}), \mathbf{u} \rangle \right] + \tilde{c}_{i\sigma\tau}(\mathbf{v}) \partial_i u_\sigma + \langle \partial_{\mathbf{u}} c_{i\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma + \langle \partial_{\mathbf{u}} \tilde{d}_\tau(\mathbf{v}), \mathbf{u} \rangle, \\ & \quad \left[ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}) \right] \partial_i u_\sigma \\ & \quad \left. + \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma + \langle \partial_{\mathbf{u}} \tilde{b}_{j\tau}(\mathbf{v}), \mathbf{u} \rangle \right] n_j|_{\Gamma^N}, 0|_{\Gamma^D} \Big\}. \end{aligned} \quad (51)$$

In particular,  $\tilde{\mathbf{N}}'(\mathbf{0}) = (\mathbf{0}, \mathbf{0}, \mathbf{0})$ .

**Proof.** Firstly, we investigate  $\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma$  in  $V_{\vec{\mu}, p}^{d+1}(\Omega)$  (by composition with  $\partial_j$  we will get the result in  $V_{\vec{\mu}, p}^d(\Omega)$  for the first two terms of the right-hand side of (51)). Let us start with the expression  $\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \partial_i u_\sigma$ . Since  $\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \in V_{\vec{\nu}, p}^{d+2}(\Omega)$  and  $\partial_i u_\sigma \in V_{\vec{\nu}, p}^{d+1}(\Omega)$ , the multiplication Theorem 4.1 yields  $\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \partial_i u_\sigma \in V_{\vec{\mu}, p}^{d+1}(\Omega)$  provided  $\mu_P > 2\nu_P - d - 2 + \frac{n}{p}$ , i.e.  $\delta_P < 2\alpha_P$  for all  $P \in \mathcal{P}$ . Since the Fréchet derivative of a linear operator coincides with the operator itself, it follows from the product rule that  $\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \partial_i u_\sigma$  is Fréchet differentiable and its Fréchet derivative at  $\mathbf{v}$  is

$$\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \partial_i v_\sigma + \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v} \rangle \partial_i u_\sigma.$$

The expression  $\tilde{a}_{ij\sigma\tau}(\mathbf{u})$  belongs to  $V_{\vec{\mu}-\vec{\nu}+d+1-n/p, p}^{d+1}(\Omega)$  due to the composition Theorem 4.2 (set  $s = 2$ , use the assumption  $\mu_P > 2\nu_P - d - 2 + \frac{n}{p}$  and estimate (A)) and is Fréchet differentiable due to Theorem 4.3. Its derivative at  $\mathbf{v}$  is given by  $\langle \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle$  (see (17)). It follows from the multiplication Theorem 4.1 that  $\tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma \in V_{\vec{\mu}, p}^{d+1}(\Omega)$ . The product rule shows that the Fréchet derivative of  $\tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma$  at  $\mathbf{v}$  coincides with

$$\langle \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma + \tilde{a}_{ij\sigma\tau}(\mathbf{v}) \partial_i u_\sigma.$$

Thus the derivative of

$$\{\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{u})\} \partial_i u_\sigma$$

is given by

$$\{\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v})\} \partial_i u_\sigma + \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma.$$

The expression  $\tilde{b}_{j\tau}(\mathbf{u})$  belongs to  $V_{\bar{\mu},p}^{d+1}(\Omega)$  due to composition Theorem 4.2 for  $s = 2$  and assumption (47). It is only necessary to require that  $\mu_P > 2\nu_P - d - 3 + \frac{n}{p}$ , i.e.  $\delta_P < 2\alpha_P + 1$ . Theorem 4.3 gives the Fréchet differentiability of  $\tilde{b}_{j\tau}(\mathbf{u})$ .

Similar arguments are applied to the expressions  $\tilde{c}_{i\sigma\tau}(\mathbf{u}) \partial_i u_\sigma$  and  $\tilde{d}_\tau(\mathbf{u})$  setting  $s = 1$  in Theorem 4.2. The mapping properties and the Fréchet differentiability of the boundary expressions follow immediately from the above considerations ■

**Theorem 5.3.** *Let assumptions (A1) - (A3) be satisfied. Then the mapping*

$$\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega) \ni \mathbf{v} \mapsto \tilde{\mathbf{N}}'(\mathbf{v}) \in \mathcal{L}(\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega), Y_{\bar{\mu},p}^d)$$

is continuous at  $\mathbf{v} = \mathbf{0}$ .

**Proof.** Let  $\{\mathbf{v}_m\}_{m=1}^\infty \subset \mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)$  be a sequence with property  $\|\mathbf{v}_m\|_{\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)} \rightarrow 0$  as  $m \rightarrow \infty$ . We have to show that  $\|\tilde{\mathbf{N}}'(\mathbf{v}_m)\|_{\mathcal{L}(\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega), Y_{\bar{\mu},p}^d)} \rightarrow 0$ , i.e.

$$\sup_{\|\mathbf{u}\|_{\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)}=1} \|\tilde{\mathbf{N}}'(\mathbf{v}_m)\mathbf{u}\|_{Y_{\bar{\mu},p}^d} \rightarrow 0$$

as  $m \rightarrow \infty$ . We start with the term

$$-\partial_j [\{\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m)\} \partial_i u_\sigma]$$

to get

$$\begin{aligned} & \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)}=1} \left\| -\partial_j [\{\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m)\} \partial_i u_\sigma] \right\|_{V_{\bar{\mu},p}^d(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)}=1} \left\| \{\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m)\} \partial_i u_\sigma \right\|_{V_{\bar{\mu},p}^{d+1}(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\bar{\nu},p}^{d+2}(\Omega)}=1} \left\| \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{V_{\bar{\xi},p}^{d+1}(\Omega)} \cdot \|\partial_i u_\sigma\|_{V_{\bar{\nu},p}^{d+1}(\Omega)} \\ & \leq \left\| \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{V_{\bar{\xi},p}^{d+1}(\Omega)} \end{aligned} \tag{52}$$

where  $\vec{\xi} = (\xi_P)_{P \in \mathcal{P}}$  with

$$\xi_P = \mu_P - \nu_P + d + 1 - \frac{n}{p} = d + 1 + \alpha_P - \delta_P - \frac{n}{p}.$$

Here we used multiplication Theorem 4.1 and the continuity of the differential operators  $\partial_j : V_{\vec{\mu}, p}^{d+1}(\Omega) \rightarrow V_{\vec{\mu}, p}^d(\Omega)$  and  $\partial_i : V_{\vec{\nu}, p}^{d+2}(\Omega) \rightarrow V_{\vec{\nu}, p}^{d+1}(\Omega)$ . Furthermore, we have

$$\|\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle\|_{V_{\vec{\xi}, p}^{d+1}(\Omega)} \leq c \|\mathbf{v}_m\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)}$$

due to the continuity of the imbedding  $\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega) \rightarrow \mathbf{V}_{\vec{\xi}, p}^{d+1}(\Omega)$ , which is valid for  $\nu_P - d - 2 < \xi_P - d - 1$ , i.e.  $\delta_P < 2\alpha_P$  (see Lemma 3.4), and

$$\|\tilde{a}_{ij\sigma\tau}(\mathbf{v}_m)\|_{V_{\vec{\xi}, p}^{d+1}(\Omega)} \rightarrow 0$$

due to the continuity of the composition operator

$$\tilde{A}_{ij\sigma\tau} : \mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega) \rightarrow V_{\vec{\xi}, p}^{d+1}(\Omega), \quad \tilde{A}_{ij\sigma\tau} \mathbf{u} = \tilde{a}_{ij\sigma\tau} \circ \mathbf{u}$$

at  $\mathbf{0}$  which is guaranteed by Theorem 4.2 provided  $\delta_P < 2\alpha_P$ . Thus (52) tends to 0 as  $\|\mathbf{v}_m\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)} \rightarrow 0$ .

Let us investigate the term

$$-\partial_j [\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}_m), \mathbf{u} \rangle \partial_i v_{m,\sigma}].$$

Here we write  $\mathbf{v}_m = (v_{m,1}, \dots, v_{m,k})$ . There holds

$$\begin{aligned} & \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)}=1} \left\| -\partial_j [\langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}_m), \mathbf{u} \rangle \partial_i v_{m,\sigma}] \right\|_{V_{\vec{\mu}, p}^d(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)}=1} \left\| \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}_m), \mathbf{u} \rangle \partial_i v_{m,\sigma} \right\|_{V_{\vec{\mu}, p}^{d+1}(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)}=1} \left\| \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}) + \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{\mathbf{V}_{\vec{\xi}, p}^{d+1}(\Omega)} \\ & \quad \times \|\mathbf{u}\|_{\mathbf{V}_{\vec{\nu}, p}^{d+2}(\Omega)} \cdot \|\partial_i v_{m,\sigma}\|_{V_{\vec{\nu}, p}^{d+1}(\Omega)} \\ & \leq \left\| \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}) + \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{\mathbf{V}_{\vec{\xi}, p}^{d+1}(\Omega)} \cdot \|v_{m,\sigma}\|_{V_{\vec{\nu}, p}^{d+2}(\Omega)} \end{aligned} \quad (53)$$

where  $\vec{\zeta} = (\zeta_P)_{P \in \mathcal{P}}$  with

$$\zeta_P = \mu_P - 2\nu_P + 2d + 3 - 2\frac{n}{p} = d + 1 + 2\alpha_P - \delta_P - \frac{n}{p}.$$



Firstly, the Nemytskij operator

$$\mathbf{v} \rightarrow \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau} \circ \mathbf{v}$$

is continuous from  $\mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega)$  into  $V_{\vec{\zeta},p}^{d+1}(\Omega)$ . This follows from Theorems 4.2 and 4.3 provided

$$d + 1 - \zeta_P - \frac{n}{p} < 2(d + 2 - \nu_P - \frac{n}{p}),$$

i.e.  $\delta_P < 4\alpha_P$ . Secondly, the constant  $\partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0})$  belongs to  $V_{\vec{\zeta},p}^{d+1}(\Omega)$  provided  $\delta_P < 2\alpha_P$ . These two properties guarantee that

$$\left\| \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}) + \partial_{\mathbf{u}} \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{\mathbf{V}_{\vec{\zeta},p}^{d+1}(\Omega)}$$

remains bounded and therefore (53) tends to 0. The remaining terms in (51) can be treated analogously ■

In the proofs of Theorems 5.2 and 5.3 we did not exploit the definitions (21) - (22) where the numbers  $\alpha_P$  and  $\delta_P$  are connected with the eigenvalues of  $\mathcal{A}_P$ . In fact, the proofs remain valid if  $\alpha_P$  and  $\delta_P$  are arbitrary real numbers satisfying assumption (A3). Thus the assumptions of Theorems 5.2 and 5.3 can be relaxed.

**Theorem 5.4.** *Let  $\mu'_P = d + 2 - \frac{n}{p} - \beta_P$  with  $\beta_P \geq 0$  and let assumptions (A1) - (A2) be satisfied. Then  $\tilde{\mathbf{N}}$  maps  $\mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega) \rightarrow Y_{\vec{\mu}',p}^d$  and is differentiable in a neighbourhood  $\mathcal{U}(\mathbf{0}) \subset \mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega)$ . Furthermore, the mapping*

$$\mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega) \ni \mathbf{v} \mapsto \tilde{\mathbf{N}}'(\mathbf{v}) \in \mathcal{L}(\mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega), Y_{\vec{\mu}',p}^d)$$

*is continuous at  $\mathbf{v} = \mathbf{0}$ .*

Now we are in position to prove the main results for domains with conical points.

**Proof of Theorem 4.7.** According to Theorem 5.3 the nonlinear operator

$$\tilde{\mathbf{N}} : \mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega) \rightarrow Y_{\vec{\mu},p}^d$$

is Fréchet differentiable in a neighbourhood of  $\mathbf{u} = \mathbf{0}$ . The same property is valid for the operator  $\tilde{\mathbf{N}}$  acting between the spaces

$$\tilde{\mathbf{N}} : \mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega) \rightarrow Y_{\vec{\mu},p}^d$$

because the space  $\mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega)$  is continuously imbedded into  $\mathbf{V}_{\vec{\nu},p}^{d+2}(\Omega)$  due to Lemma 3.13. By (50) and the fact that the linear part is Fréchet differentiable between the above spaces, the operator  $\mathbf{N}$  is Fréchet differentiable from  $\mathbf{D}_{\vec{\mu},p}^{d+2}(\Omega)$  into  $Y_{\vec{\mu},p}^d$ . From Theorem 5.2 we know that the Fréchet derivative  $\mathbf{N}'(\mathbf{0})$  coincides with the operator  $(\mathbf{A}, \mathbf{C}^{\mathbf{N}}, \mathbf{I})$  which is an isomorphism due to Theorem 3.14. Furthermore, Theorem 5.3 shows that the mapping  $\mathbf{v} \mapsto \tilde{\mathbf{N}}'(\mathbf{v})$  and therefore also the mapping  $\mathbf{v} \mapsto \mathbf{N}'(\mathbf{v})$  are continuous at  $\mathbf{0}$ . Thus we can apply Local Invertibility Theorem 2.2 and obtain the assertion ■

**Proof of Theorem 4.9.** The proof is the same as for Theorem 4.7 applying Theorem 5.4 instead of Theorems 5.2 and 5.3, taking into account that the operator  $\mathbf{N}'(0) : \mathbf{V}_{\vec{\mu},p}^{d+2}(\Omega) \rightarrow Y_{\vec{\mu}',p}^d$  is an isomorphism thanks to the assumption on  $\beta_P$  due to Theorem 3.7 and the assumption (U) ■

**Remark 5.5.** Theorem 4.9 can be used for an alternative proof of Theorem 4.7. Applying Theorem 4.9 we can proceed as in [28] to get an asymptotic expansion of the solution for  $(\mathbf{f}, \mathbf{h}, \mathbf{g}) \in Y_{\vec{\mu},p}^d$ . Namely,  $\mathbf{u} \in \mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega)$  solution of problem (1) is also a solution of

$$\mathbf{N}'(0)\mathbf{u} = (\mathbf{f}, \mathbf{h}, \mathbf{g}) - \tilde{\mathbf{N}}\mathbf{u}.$$

In order to apply Theorem 3.7 we have to assure that  $\tilde{\mathbf{N}}$  maps  $\mathbf{V}_{\vec{\mu}',p}^{d+2}(\Omega)$  into  $Y_{\vec{\mu},p}^d$ . Actually, this property follows from assumptions (A1) - (A3) (see Theorem 5.2), which cannot be weakened due to the term

$$\partial_j \{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{u} \rangle \partial_i u_\sigma \}.$$

Thus the results of Theorem 4.7 can not be improved in this way.

**5.2.2 Straight polyhedral domains.** Similarly to Subsection 5.2.1 we derive the main results for polyhedral domains.

**Theorem 5.6.** *Assume that (A1'') and (A2) hold and  $\alpha < \frac{1}{4}$  and  $\beta < \frac{1}{2}$ . Then  $\tilde{\mathbf{N}}$  maps  $\mathbf{H}_{\alpha,\beta}^2(\Omega) \cap \mathring{H}^1(\Omega)$  into  $\mathbf{L}_2(\Omega)$  and is differentiable in a neighbourhood of  $\mathbf{0}$  with*

$$\begin{aligned} (\tilde{\mathbf{N}}'(\mathbf{v})\mathbf{u})_\tau &= -\partial_j \left[ \{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v} \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}) \} \partial_i u_\sigma \right. \\ &\quad \left. + \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma + \langle \partial_{\mathbf{u}} \tilde{b}_{j\tau}(\mathbf{v}), \mathbf{u} \rangle \right] \\ &\quad + \tilde{c}_{i\sigma\tau}(\mathbf{v}) \partial_i u_\sigma + \langle \partial_{\mathbf{u}} c_{i\sigma\tau}(\mathbf{v}), \mathbf{u} \rangle \partial_i v_\sigma + \langle \partial_{\mathbf{u}} \tilde{d}_\tau(\mathbf{v}), \mathbf{u} \rangle. \end{aligned}$$

In particular,  $\tilde{\mathbf{N}}'(\mathbf{0}) = \mathbf{0}$ .

**Proof.** As in Theorem 5.2 we have to show that the mappings

$$\begin{aligned} \mathbf{u} &\rightarrow u_\nu \partial_i u_\sigma \\ \mathbf{u} &\rightarrow \tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma \\ \mathbf{u} &\rightarrow \tilde{b}_{j\tau}(\mathbf{u}) \end{aligned}$$

are differentiable in a neighbourhood of  $\mathbf{0}$  as operators from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $H_{0,0}^1(\Omega)$ , and similarly that

$$\begin{aligned} \mathbf{u} &\rightarrow \tilde{c}_{i\sigma\tau}(\mathbf{u}) \partial_i u_\sigma \\ \mathbf{u} &\rightarrow \tilde{d}_\tau(\mathbf{u}) \end{aligned}$$

are differentiable in a neighbourhood of  $\mathbf{0}$  as operators from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $L_2(\Omega)$ . Under the assumptions  $\alpha < \frac{1}{4}$  and  $\beta < \frac{1}{2}$  the multiplication Theorem 4.4 guarantees that the expression  $u_\nu \partial_i u_\sigma$  belongs to  $H_{0,0}^1(\Omega)$ , the differentiability following from the fact that the product of two differentiable mappings is differentiable.

For the expression  $\tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma$ , we first show that the mapping

$$\mathbf{u} \rightarrow \tilde{a}_{ij\sigma\tau}(\mathbf{u})$$

is differentiable from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $H_{\alpha+\varepsilon,\beta+\varepsilon}^2(\Omega)$  with  $\varepsilon > 0$  as small as we want. For this purpose, we decompose  $\tilde{a}_{ij\sigma\tau}$  as

$$\tilde{a}_{ij\sigma\tau}(\mathbf{x}) = \sum_{\gamma,\delta=1}^k t_{\gamma,\delta} x_\gamma x_\delta + \hat{a}_{ij\sigma\tau}(\mathbf{x})$$

where  $t_{\gamma,\delta}$  are real constants and  $\hat{a}_{ij\sigma\tau}$  satisfies

$$|D^\gamma \hat{a}_{ij\sigma\tau}(\mathbf{x})| \leq C|\mathbf{x}|^{3-|\gamma|} \quad (|\gamma| \leq 3).$$

The first mapping

$$\mathbf{u} \rightarrow \sum_{\gamma,\delta=1}^k t_{\gamma,\delta} u_\gamma u_\delta$$

is differentiable from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $H_{\alpha,\beta}^2(\Omega)$  due to Theorem 4.4. Thus it is differentiable from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $H_{\alpha+\varepsilon,\beta+\varepsilon}^2(\Omega)$  since  $\varepsilon > 0$ . For the remaining part

$$\mathbf{u} \rightarrow \hat{a}_{ij\sigma\tau}(\mathbf{u})$$

we simply use Theorem 4.6 with  $d = d' = 2$ ,  $\alpha' = \alpha + \varepsilon$ ,  $\beta' = \beta + \varepsilon$  and  $s = 3$ . Theorem 4.4 allows then to conclude that the mapping

$$\mathbf{u} \rightarrow \tilde{a}_{ij\sigma\tau}(\mathbf{u}) \partial_i u_\sigma$$

is differentiable from  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$  into  $H_{0,0}^1(\Omega)$ . The other mappings are treated similarly ■

**Theorem 5.7.** *Under the assumptions of Theorem 5.6, the mapping*

$$\mathbf{D}_{0,0}(\Omega) \ni \mathbf{v} \mapsto \tilde{\mathbf{N}}'(\mathbf{v}) \in \mathcal{L}(D_{0,0}(\Omega), \mathbf{L}_2(\Omega))$$

*is continuous at  $\mathbf{v} = \mathbf{0}$ .*

**Proof.** Let  $\{\mathbf{v}_m\}_{m=1}^\infty \subset \mathbf{D}_{0,0}(\Omega)$  be a sequence with property  $\|\mathbf{v}_m\|_{\mathbf{D}_{0,0}(\Omega)} \rightarrow 0$  as  $m \rightarrow \infty$ . For the sake of shortness, we analyze only the term

$$-\partial_j [\{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \} \partial_i u_\sigma],$$

the other terms being managed in the same way using the results from Subsection 4.1.2. We get

$$\begin{aligned} & \sup_{\|\mathbf{u}\|_{D_{0,0}(\Omega)}=1} \left\| -\partial_j [\{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \} \partial_i u_\sigma] \right\|_{L_2(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{D_{0,0}(\Omega)}=1} \left\| \{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \} \partial_i u_\sigma \right\|_{H_{0,0}^1(\Omega)} \\ & \leq \sup_{\|\mathbf{u}\|_{D_{0,0}(\Omega)}} \left\| \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \right\|_{H_{\alpha+\varepsilon,\beta+\varepsilon}^2(\Omega)} \|\partial_i u_\sigma\|_{H_{\alpha,\beta}^1(\Omega)} \end{aligned}$$

for  $\varepsilon > 0$  small enough due to Theorem 4.4 since  $\alpha < \frac{1}{4}$  and  $\beta < \frac{1}{2}$ . Since Theorem 3.15 shows that  $\mathbf{D}_{0,0}(\Omega)$  is continuously embedded into  $\mathbf{H}_{\alpha,\beta}^2(\Omega)$ , the previous estimate becomes

$$\begin{aligned} & \sup_{\|\mathbf{u}\|_{\mathbf{D}_{0,0}(\Omega)}=1} \left\| -\partial_j [\{ \langle \partial_{\mathbf{u}} a_{ij\sigma\tau}(\mathbf{0}), \mathbf{v}_m \rangle + \tilde{a}_{ij\sigma\tau}(\mathbf{v}_m) \} \partial_i u_\sigma] \right\|_{L_2(\Omega)} \\ & \leq C (\|\mathbf{v}_m\|_{\mathbf{H}_{\alpha,\beta}^2(\Omega)} + \|\tilde{a}_{ij\sigma\tau}(\mathbf{v}_m)\|_{H_{\alpha+\varepsilon,\beta+\varepsilon}^2(\Omega)}) \end{aligned}$$

for some constant  $C > 0$ . Theorem 5.6 implies that this right-hand side tends to 0 as  $m \rightarrow \infty$ . Therefore the assertion follows ■

We are now ready to give the

**Proof of Theorem 4.1.** We first apply Theorem 5.6 to the nonlinear operator  $\tilde{\mathbf{N}}$  with  $\alpha = \frac{1}{2} - \alpha_0 + \varepsilon$  and  $\beta = 1 - \beta_0 + \varepsilon$ , with an arbitrarily small  $\varepsilon > 0$ . By Theorem 3.15, we deduce that  $\mathbf{N}$  is Fréchet differentiable from  $\mathbf{D}_{0,0}(\Omega)$  into  $\mathbf{L}_2(\Omega)$  in the neighbourhood of  $\mathbf{u} = \mathbf{0}$  with  $\mathbf{A}'(\mathbf{0}) = \mathbf{A}$ . Moreover, Theorem 5.7 shows the continuity of the Fréchet derivative at  $\mathbf{0}$ . Thus the assertion follows due to Theorem 2.2 ■

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Received 31.08.2001