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L_q-L_r—Estimates for Non-Stationary Stokes Equations in an Aperture Domain

H. Abels

Abstract. This article deals with asymptotic estimates of strong solutions of Stokes equations in aperture domains. An aperture domain is a domain, which outside a bounded set is identical to two half spaces separated by a wall and connected inside the bounded set by one or more holes in the wall. It is known that the corresponding Stokes operator generates a bounded analytic semigroup in the closed subspace $J_q(\Omega)$ of divergence free vector fields of $L_q(\Omega)^n$. We deal with L_q-L_r-estimates for the semigroup, which are known for $\mathbb{R}^n$, the half space and exterior domains.

Keywords: *Stokes equations, aperture domains, asymptotic behavior, asymptotic expansions*

AMS subject classification: 35Q30, 76D07, 35B40, 35C20

1. Introduction and main results

Suppose that $\Omega \subset \mathbb{R}^n \quad (n \geq 3)$ is an aperture domain (see Figure 1) with smooth boundary, i.e.

$$\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0) \qquad (r > 0)$$

with

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$$
$$\mathbb{R}^n_- = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n < -d\} \quad (d > 0).$$

We consider the homogeneous non-stationary Stokes equations in $(0, \infty) \times \Omega$ concerning the velocity field $u(t, x)$ and the scalar pressure $p(t, x)$:

$$\partial_t u - \Delta u + \nabla p = f \qquad \text{in } (0, \infty) \times \Omega \tag{1}$$

H. Abels: FB Math. der Techn. Univ. Darmstadt, Schloßgartenstr. 7, D-64289 Darmstadt
abels@mathematik.tu-darmstadt.de

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$$\begin{aligned} \operatorname{div} u &= 0 && \text{in } (0,\infty)\times\Omega && (2)\\ u|_{\partial\Omega} &= 0 && \text{on } (0,\infty)\times\partial\Omega && (3)\\ \Phi(u) &= \alpha && \text{in } (0,\infty) && (4)\\ u|_{t=0} &= u_0 && \text{in } \Omega && (5) \end{aligned}$$

where $\partial_t = \frac{\partial}{\partial t}$, $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)^T$,

$$\Phi(u(t)) = \int_M N \cdot u(t,x)\, d\sigma(x) = \alpha(t)$$

is the flux through a smooth, bounded $(n-1)$-dimensional manifold M with normal vector N directed downwards dividing Ω into two unbounded connected components. This flux has to be prescribed in order to get a unique solution with $u(t) \in L_q(\Omega)$ with $\frac{n}{n-1} < q < \infty$. In the case $1 < q \leq \frac{n}{n-1}$ the flux has to vanish, i.e. $\Phi(u) = 0$ (see [4] for the corresponding resolvent problem).

Figure 1: An aperture domain

In this paper we only deal with the case $f = 0$ and $\Phi(u) = 0$. We consider the asymptotic behaviour of the solutions $u(t)$. The general case can be derived from this case depending on the asymptotic behaviour of $f(t)$ and $\alpha(t)$. Since the Stokes operator A_q generates a bounded semigroup in $J_q(\Omega) = \overline{\{u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0\}}^{\|\cdot\|_q}$ the estimate $\|u(t)\|_q \leq C\|u_0\|_q$ holds.

The goal of this paper is to prove the following decay rate measuring $u(t)$ and u_0 in the norm of L_q for different $1 < q < \infty$.

Theorem 1.1. *Let $1 < q \leq r < \infty$. Then there is a constant $C = C(\Omega, q, r)$ such that*

$$\|u(t)\|_{L_r(\Omega)} \leq C t^{-\sigma} \|u_0\|_{L_q(\Omega)} \tag{6}$$

with $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$ for all $t > 0$ and $u_0 \in J_q(\Omega)$.

Theorem 1.2. *Let $1 < q \le r < n$. Then there is a constant $C = C(\Omega, q, r)$ such that*

$$\|\nabla u(t)\|_{L_r(\Omega)} \le C t^{-\sigma-\frac{1}{2}} \|u_0\|_{L_q(\Omega)} \tag{7}$$

with $\sigma = \frac{n}{2}(\frac{1}{q} - \frac{1}{r})$ for all $t > 0$ and $u_0 \in J_q(\Omega)$.

These inequalities are known for other unbounded domains. In [12] Ukai showed these estimates for $1 < q < \infty$ if the domain is the half-space $\mathbb{R}^n_+$. This is done by using an explicit solution formula in terms of Riesz operators and the heat kernel in $\mathbb{R}^n_+$. In the case of an exterior domain, Iwashita [8] showed the validity of (6) for $1 < q \le r < \infty$ and that of (7) for $1 < q \le r \le n$.

The proof of Theorems 1.1 and 1.2 uses a similar technique as in [8]. It consists of first showing a local estimate of the L_q-norm of $u(t)$ and then comparing the full L_q-norm with suitable solutions of the non-stationary Stokes equations in $\mathbb{R}^n_+$. The local estimate is derived from an asymptotic expansion of the resolvent of the Stokes operator in the aperture domain around 0 in special weighted L_q-spaces. The resolvent expansion is constructed by using a similar resolvent expansion of the Stokes operator in the half-space $\mathbb{R}^n_+$. For the latter expansion we combine Ukai's solution formula [12] with an resolvent expansion of the Laplace operator Δ in $\mathbb{R}^n$, based on the results of Murata [9].

Remark 1.3. With the methods of this article we can not prove Theorem 1.2 for the case $r = n$, which is done by Iwashita in the case of the exterior domain. This is due to a slightly weaker estimate of the local part of the L_q-norm (see Corollary 6.2 and [8: Theorem 1.2/(i)]). We get this condition because we have to deal with weighted L_q-spaces of the kind $L_q(\Omega; \omega^{sq})$ such that ω^{sq} is a Muckenhoupt weight (see preliminaries); this condition on the weights is not needed in [8].

The L_q-L_r-estimate can be used to construct solutions of the instationary Navier-Stokes equations with arbitrary flux $\Phi(u)$ as perturbation of steady-state solution. For the case $n = 2$ this problem is still unsolved. Unfortunately, the used approach can not be applied to a two-dimensional aperture domain. The reason is that we can not prove Theorem 4.1 since there is no number σ with $1 < \sigma < \frac{n}{2}$, $n = 2$. The restriction $\sigma < \frac{n}{2}$ is due to the restriction to Muckenhoupt weights. The condition $\sigma > 1$ is necessary for the perturbation argument used in the proof of Theorem 4.1. – We have to assure that the resolvent of the Stokes operator in $\mathbb{R}^n_+$ considered as map between different weighted L_q-spaces exists for $z = 0$.

2. Preliminaries and notation

We will consider the resolvent expansion in a scale of weighted L_q-spaces

$$L_q(\Omega;\omega^{sq}) = \left\{ f : \Omega \to \mathbb{R} \text{ measurable} \Big| \, \|f\|_{L_q(\Omega;\omega^{sq})} < \infty \right\} \qquad (s \in \mathbb{R})$$

where

$$\|f\|_{L_q(\Omega;\omega^{sq})} = \left(\int_\Omega |f(x)|^q \omega^{sq}(x)\, dx \right)^{\frac{1}{q}}.$$

Analogously we define the weighted Sobolev spaces as

$$W^m_q(\Omega;\omega^{sq}) = \left\{ f \in L_{1,loc}(\overline{\Omega}) \Big| \, D^\alpha f \in L_q(\Omega;\omega^{sq})\ \forall\, |\alpha| \le m \right\}$$

and

$$W^m_{0,q}(\Omega;\omega^{sq}) = \overline{C_0^\infty(\Omega)}^{W^m_q(\Omega;\omega^{sq})}.$$

Recall that $f \in L_{1,loc}(\overline{\Omega})$ means that $f \in L_1(\Omega \cap B)$ for all balls B with $\Omega \cap B \neq \emptyset$. Moreover,

$$D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f(x) \qquad (\alpha \in \mathbb{N}_0^n).$$

By $\dot{W}^m_q(\Omega;\omega^{sq})$ we denote the corresponding homogeneous Sobolev space of $L_{1,loc}$-functions f with $D^\alpha f \in L_q(\Omega;\omega^{sq})$ for all $|\alpha| = m$. Finally,

$$J_q(\Omega;\omega^{sq}_n) = \overline{\{u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0\}}^{L_q(\Omega;\omega^{sq}_n)}.$$

For simplicity we often will skip the exponent n if we deal with spaces of vector fields, e.g. we write $f \in L_q(\Omega)$ instead of $f \in L_q(\Omega)^n$. If X and Y are two Banach spaces, we denote by $\mathcal{L}(X,Y)$ the space of all bounded linear maps $T : X \to Y$. Furthermore, $\mathcal{L}(X) = \mathcal{L}(X,X)$.

In [8, 9] the simple weight $\omega(x) = \langle x \rangle := (1+|x|^2)^{\frac{1}{2}}$ is used. For $-\frac{n}{q} < s < \frac{n}{q'}$ the weight $\langle x \rangle^{sq}$ is an element of the Muckenhoupt class $\mathcal{A}_q$. This is the class of all measurable functions $\omega : \mathbb{R}^n \to [0,\infty)$ with

$$\frac{1}{|B|} \int_B \omega(x)\, dx \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{q'}{q}}\, dx \right)^{\frac{q}{q'}} \le A < \infty$$

where B is an arbitrary ball in $\mathbb{R}^n$ and A is independent of B. The weights $\omega \in \mathcal{A}_q$ have the important property that singular integral operators like the Riesz transforms

$$R_j f(x) := \mathcal{F}^{-1}\left[\frac{i\xi_j}{|\xi|} \hat{f}(\xi) \right] = c_n \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy$$

$(j = 1,\dots,n)$ are continuous on $L_q(\mathbb{R}^n;\omega)$ into itself. Here $\mathcal{F}[u](\xi) = \hat{u}(\xi)$ denotes the Fourier transform with respect to x. See, for example, [11: Chapter V,§4.2/Theorem 2] for the continuity and [10: Chapter III, Section 1] for Riesz transforms.

We will also use the partial Riesz transforms

$$S_j f(x) = \mathcal{F}^{-1}_{\xi'\mapsto x'}\left[\frac{i\xi_j}{|\xi'|}\tilde{f}(\xi',x_n)\right] = c_{n-1}\lim_{\varepsilon\to 0}\int_{\mathbb{R}^{n-1}\setminus B_\varepsilon(x')}\frac{x'_j-y'_j}{|x'-y'|^n}f(y',x_n)\,dy$$

$\big(j = 1,\dots,n-1;\ x = (x',x_n), \xi = (\xi',\xi_n)\big)$ for functions f defined on $\mathbb{R}^n_+$ or $\mathbb{R}^n$. These partial Riesz transforms are used in Ukai's solution formula.

Unfortunately, the weight $\langle x\rangle^{sq}$ considered for fixed x_n as weight in $\mathbb{R}^{n-1}$ is in the class $\mathcal{A}_q$ only if $-\frac{n-1}{q} < s < \frac{n-1}{q'}$. Therefore we will use the slightly weaker weight

$$\omega_n(x) = \prod_{i=1}^n \langle x_i\rangle^{\frac{1}{n}}.$$

For this weight $\omega_n(x)^{sq}$ considered for fixed x_n is in $\mathcal{A}_q$ on $\mathbb{R}^n$ for $-\frac{n}{q} < s < \frac{n}{q'}$. This is easily derived from the special product structure and the fact that $\langle x_i\rangle^{\frac{s}{n}}$ is a one-dimensional weight in $\mathcal{A}_q$.

Therefore we get

Lemma 2.1. *Let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, $1 < q < \infty$, $-\frac{n}{q} < s < \frac{n}{q'}$ and $\omega_n(x) = \prod_{i=1}^n\langle x_i\rangle^{\frac{1}{n}}$. Then the (partial) Riesz transforms are continuous from $L_q(\Omega;\omega_n^{sq})$ into itself.*

Moreover, we introduce

$$\Sigma_\delta = \{z\in\mathbb{C}\setminus\{0\} : |\arg z| < \delta\}$$
$$\Sigma_{\delta,\varepsilon} = \Sigma_\delta\cap B_\varepsilon(0).$$

Recall the Helmholtz decomposition of a vector field $f\in L_q(\Omega;\omega_n^{sq})^n$, i.e. the unique decomposition $f = f_0 + \nabla p$ with $f_0\in J_q(\Omega;\omega_n^{sq})$ and $p\in \dot{W}^1_q(\Omega;\omega_n^{sq})$. The existence and continuity of the corresponding Helmholtz projection

$$P_q : L_q(\Omega;\omega_n^{sq})^n \to J_q(\Omega;\omega_n^{sq}),\qquad f\mapsto P_q f = f_0$$

is proved in [3: Theorem 5] for the case that $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, or that Ω is a bounded domain. For the case of an aperture domain and $s = 0$ the result is proved in [4: Theorem 2.6].

Furthermore, we define the Stokes operator

$$A_q = -P_q\Delta$$

in $J_q(\Omega)$ with $\mathcal{D}(A_q) = W_q^2(\Omega) \cap W_{0,q}^1(\Omega) \cap J_q(\Omega)$. Note that the resolvent of A_q satisfies the estimate

$$\|(z + A_q)^{-1} f\|_{L_q(\Omega)} \leq C_\delta |z|^{-1} \|f\|_{L_q(\Omega)} \tag{8}$$

for $z \in \Sigma_\delta$ $(\delta \in (0,\pi))$ if Ω is an aperture domain (see [9: Theorem 2.5]). Therefore $-A_q$ generates an analytic semigroup.

3. The resolvent expansion in $\mathbb{R}^n_+$

We consider the resolvent equations system

$$(z - \Delta)u + \nabla p = f \qquad \text{in } \mathbb{R}^n_+ \tag{9}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n_+ \tag{10}$$

$$u|_{\partial\mathbb{R}^n_+} = 0 \qquad \text{on } \partial\mathbb{R}^n_+. \tag{11}$$

Let $R_0(z) = (z - \Delta)^{-1}$ denote the resolvent of the Laplace operator in $\mathbb{R}^n$.

Lemma 3.1. *Let $1 \leq p \leq \infty$, $0 < \delta < \pi$, $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq 2$, $\frac{|\alpha|}{2} < \sigma < \frac{n+|\alpha|}{2}$, $-\frac{n}{p} < s' < s < \frac{n}{p'}$ and $s' = s - 2\sigma + |\alpha|$. Then*

$$D^\alpha R_0(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_{0j} + G_{0r}(z)$$

where

$$G_{0r}(z) = O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}\big(W_p^m(\mathbb{R}^n; \omega_n^{sp}), W_p^{m+2-|\alpha|}(\mathbb{R}^n; \omega_n^{s'p})\big)$$

for $z \to 0$ with $z \in \Sigma_\delta$.

Proof. The proof is the same as [9: Lemma 2.3/(i)]. It is based on the estimate for the convolution operator with the heat kernel $E_0(t)$

$$\|D^\alpha E_0(t)\|_{\mathcal{L}\left(L_p(\mathbb{R}^n;\omega^{sp}), L_p(\mathbb{R}^n;\omega^{s'p})\right)} \leq |t|^{-\frac{|\alpha|}{2}} \langle t\rangle^{-\sigma} \tag{12}$$

for $\omega(x) = \omega_n(x)$, $t \in \Sigma_{\delta_0}$, $0 < \delta_0 < \frac{\pi}{2}$, $\alpha \in \mathbb{N}_0^n$, $0 \leq \sigma < \frac{n}{2}$ and $-\frac{n}{p} < s' < s < \frac{n}{p'}$ with $s' = s - 2\sigma$. This estimate is proved in [9: Lemma 2.2] for the case $\omega(x) = \langle x \rangle$. But this case implies the estimate for $\omega(x) = \omega_n(x)$ since

$$
\begin{aligned}
&\|D^\alpha E_0(t) f\|_{L_p(\mathbb{R}^n;\omega_n^{s'p})} \\
&\quad \leq \Bigg\| \int_{\mathbb{R}^{n-1}} \left| D^{\alpha'} \frac{e^{-\frac{|x'-y'|^2}{4t}}}{(4\pi t)^{\frac{n-1}{2}}} \right| \\
&\qquad \times \left\| \int_{\mathbb{R}} \partial_{x_n}^{\alpha_n} \frac{e^{-\frac{|x_n-y_n|^2}{4t}}}{\sqrt{4\pi t}} f(y', y_n)\, dy_n \right\|_{L_p\left(\mathbb{R};\langle x_n\rangle^{\frac{s'p}{n}}\right)} dy' \Bigg\|_{L_p\left(\mathbb{R}^{n-1};\omega_{n-1}^{s'p\frac{n-1}{n}}(x')\right)} \\
&\quad \leq C|t|^{-\frac{\alpha_n}{2}} \langle t\rangle^{-\frac{\sigma}{n}} \\
&\qquad \times \left\| \int_{\mathbb{R}^{n-1}} \left| D^{\alpha'} \frac{e^{-\frac{|x'-y'|^2}{4t}}}{(4\pi t)^{\frac{n-1}{2}}} \right| \|f(y', \cdot)\|_{L_p\left(\mathbb{R};\langle x_n\rangle^{\frac{sp}{n}}\right)} dy' \right\|_{L_p\left(\mathbb{R}^{n-1};\omega_{n-1}^{s'p\frac{n-1}{n}}(x')\right)} \\
&\quad \leq C \left(\prod_{i=1}^{n} |t|^{-\frac{\alpha_i}{2}} \langle t\rangle^{-\frac{\sigma}{n}} \right) \|f\|_{L_p(\mathbb{R}^n;\omega_n^{sp})} \\
&\quad = C|t|^{-\frac{|\alpha|}{2}} \langle t\rangle^{-\sigma} \|f\|_{L_p(\mathbb{R}^n;\omega_n^{sp})}
\end{aligned}
$$

with $\alpha = (\alpha', \alpha_n)$ ∎

Remark 3.2. The operators G_{0j} and $G_{0r}(z)$ are given by

$$G_{0j} = \int_0^\infty E_0(t) \frac{(-t)^j}{j!}\, dt \tag{13}$$

$$G_{0r}(z) = \int_0^\infty E_0(t) f_{[\sigma]}(zt)\, dt \quad \text{with} \quad f_{[\sigma]}(zt) = e^{-zt} - \sum_{j=0}^{[\sigma]-1} \frac{(-zt)^j}{j!}. \tag{14}$$

We recall Ukai's solution formula for the homogeneous non-stationary Stokes equations in $\mathbb{R}^n_+$ (see [13]), i.e. (1) - (3) and (5) for $\Omega = \mathbb{R}^n_+$, $f = 0$ with compatibility condition $\operatorname{div} u_0 = 0$ in $\mathbb{R}^n_+$ and $u_0^n = 0$, $u_0 = (u_0', u_0^n)$ on $\partial\mathbb{R}^n_+$. Let R_j and S_j be as above. Moreover, let $rf = f|_{\mathbb{R}^n_+}$, $\gamma f = f|_{\partial\mathbb{R}^n_+}$ and e be the extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ with value 0. Finally, let $E(t)$ be the solution operator for the heat equation in $\mathbb{R}^n_+$, which is derived from $E_0(t)$ by odd extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$. Then the solution $(u(t), p(t))$ of the non-stationary Stokes equations in $\mathbb{R}^n_+$ is

$$
\begin{aligned}
u(t) &= WE(t)Vu_0 \\
p(t) &= -D\gamma\partial_n E(t)V_1 u_0
\end{aligned}
$$

where

$$W = \begin{pmatrix} I & -SU \\ 0 & U \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} V_2 \\ V_1 \end{pmatrix}$$

with

$$\begin{aligned} S &= (S_1, \dots, S_{n-1})^T \\ U &= rR' \cdot S(R' \cdot S + R_n)e \\ V_1 u_0 &= -S \cdot u_0' + u_0^n \\ V_2 u_0 &= u_0' + S u_0^n \\ R' &= (R_1, \dots, R_{n-1})^T \end{aligned}$$

and D is the Poisson operator for the Dirichlet problem of the Laplace equation in $\mathbb{R}^n_+$.

Using this result, we get:

Theorem 3.3. *Let* $1 < q < \infty, 0 < \delta < \pi$, $n \geq 3$, $\frac{|\alpha|}{2} < \sigma < \frac{n+|\alpha|}{2}$, $\alpha \in \mathbb{N}_0^n$ *with* $|\alpha| \leq 2, -\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$ *and* $s' = s - 2\sigma + |\alpha|$. *Then there exist operators* $R_+(z)$ *and* $P_+(z)$ *with*

$$\begin{aligned} D^\alpha R_+(z) &\in \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'q})\big) \\ P_+(z) &\in \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'q})\big) \end{aligned}$$

depending continuously on $z \in \Sigma_\delta \cup \{0\}$ *such that:*

1. $u = R_+(z)f$ *and* $p = P_+(z)f$ *with* $f \in L_q(\mathbb{R}^n_+; \omega_n^{sq})$ *is a solution of problem* (9) – (11) *for* $z \in \Sigma_\delta$.

2. $R_+(z) \in \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^2(\mathbb{R}^n_+)\big)$ *and* $P_+(z) \in \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+)\big)$ *for every* $z \in \Sigma_\delta$.

3. *The asymptotic expansions*

$$D^\alpha R_+(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j + O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'q})\big)$$

$$P_+(z) = \sum_{j=0}^{[\sigma]-1} z^j P_{+,j} + O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}\big(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'q})\big) \text{ if } |\alpha| = 2$$

hold for $z \to 0, z \in \Sigma_\delta$.

Proof. Because of the Helmholtz decomposition in weighted L_q-Spaces (see [5: Theorem 5]) we can assume without loss of generality that $f \in J_q(\Omega; \omega^{sq})$. Therefore the asymptotic expansion for $R_+(z)$ simply follows from the expansion of $R_0(z)$, equations (13) - (14), the continuity of the Riesz transforms S_j and R_j in $L_q(\mathbb{R}^n; \omega_n^{sq})$ and $L_q(\mathbb{R}^n_+; \omega_n^{sq})$ if $-\frac{n}{q} < s < \frac{n}{q'}$ and the fact

$$R_+(z)f = \int_0^\infty e^{-tz} W E(t) V f \, dt.$$

In order to get the result for $D^\alpha R_+(z)$ $(|\alpha| \le 2)$ we use the relations

$$\begin{aligned} \partial_n U &= (I-U)|\nabla'| = -(I-U)\textstyle\sum_{i=1}^{n-1} S_i\partial_i \\ \partial_i S &= S\partial_i \qquad (i=1,\dots,n) \\ \partial_i U &= U\partial_i \qquad (i=1,\dots,n-1) \end{aligned}$$

and prove the expansion in the same way as in the case $\alpha = 0$. We note that the first equation is a consequence of

$$\mathcal{F}_{x'\mapsto\xi'}[Uf](\xi',x_n) = |\xi'| \int_0^{x_n} e^{-|\xi|(x_n-y_n)} \tilde{f}(\xi',x_n)\,dy_n \tag{15}$$

(see the proof of [12: Theorem 1.1]); the other equations are obvious. Finally, we get the expansion of $\nabla P_+(z)$ in the same way using $|\nabla'|D\gamma = \partial_n U - U\partial_n$ ∎

Because of estimate (12) and Ukai's formula we also easily get

Lemma 3.4. *Let $u(t) = WE(t)Vu_0$ with $u_0 \in J_q(\mathbb{R}^n_+;\omega_n^{sq})$ denote the solution of the homogeneous non-stationary Stokes equations* $(1)-(3),(5)$ *for $\Omega = \mathbb{R}^n_+$ and $f = 0$. Then*

$$\|u(t)\|_{L_q(\mathbb{R}^n_+;\omega_n^{s'q})} \le C(1+t)^{-\sigma}\|u_0\|_{L_q(\mathbb{R}^n_+;\omega_n^{sq})}$$

with $1 < q < \infty$, $-\frac{n}{q} < s' \le 0 \le s < \frac{n}{q'}$, $s' = s - 2\sigma$ and $t \ge 0$.

8. Resolvent expansions in aperture domains

We consider the resolvent equations system

$$\begin{aligned} (z-\Delta)u + \nabla p &= f \qquad \text{in } \Omega \qquad &(16)\\ \operatorname{div} u &= 0 \qquad \text{in } \Omega \qquad &(17)\\ u|_{\partial\Omega} &= 0 \qquad \text{on } \partial\Omega \qquad &(18)\\ \Phi(u) &= 0 \qquad &(19) \end{aligned}$$

for an aperture domain Ω.

Theorem 4.1. *Let $1 < q < \infty, 0 < \delta < \pi$, $n \ge 3$, $1 < \sigma < \frac{n}{2}$, $\sigma \notin \mathbb{Z}$, $-\frac{n}{q} < s' \le 0 \le s < \frac{n}{q'}$ and $s' := s - 2\sigma$. Then there are an $\varepsilon > 0$ and operators*

$$\begin{aligned} R(z) &\in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q})\big) \\ P(z) &\in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), \dot{W}_q^1(\Omega;\omega_n^{s'q})\big) \end{aligned}$$

depending continuously on $z \in \Sigma_{\delta,\varepsilon} \cup \{0\}$ with the following properties:

1. *The pair $u = R(z)f$ and $p = P(z)f$ is a solution of problem (16)–(19).*
2. *$R(z) \in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega)\big)$ for every $z \in \Sigma_{\delta,\varepsilon}$.*
3. *The operator-valued function $R(z)$ $(z \in \Sigma_{\delta,\varepsilon_0})$ has an expansion*

$$R(z) = \sum_{j=0}^{[\sigma]-1} z^j G_j + G_r(z)$$

in $\mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q})\big)$ where $G_r(z) = O(z^{\sigma-1})$ for $z \to 0$.

Proof. We use the technique used in the proof of [8: Theorem 3.1]. Let $\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0)$. We choose $b, R \in \mathbb{R}$ such that $b > R > r+3$ and denote $\mathbb{R}^n_\pm = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$, $\Omega_\pm = \Omega \cap \mathbb{R}^n_\pm$ and $\Omega_b = \Omega \cap B_b(0)$. Let $\varphi, \psi \in C^\infty(\Omega)$ be cut-off functions with

$$\varphi(x) = \begin{cases} 1 & \text{for } |x| > R \\ 0 & \text{for } |x| < R-1 \end{cases} \qquad \text{and} \qquad \psi(x) = \begin{cases} 1 & \text{for } |x| > R-2 \\ 0 & \text{for } |x| < R-3. \end{cases}$$

We identify ψf with its extension by 0 to $\mathbb{R}^n_\pm$. Moreover, we define

$$R_\pm(z):\ L_q(\mathbb{R}^n_\pm;\omega_n^{sq}) \to W_q^2(\mathbb{R}^n_\pm;\omega_n^{s'q})$$

by

$$R_\pm(z)g(x) = \begin{cases} R_+(z)(g|_{\mathbb{R}^n_+})(x) & \text{if } x \in \mathbb{R}^n_+ \\ R_-(z)(g|_{\mathbb{R}^n_-})(x) & \text{if } x \in \mathbb{R}^n_-. \end{cases}$$

The operator

$$P_\pm(z):\ L_q(\mathbb{R}^n_\pm;\omega_n^{sq}) \to \dot{W}_q^1(\mathbb{R}^n_\pm;\omega_n^{s'q})$$

is defined analogously. Let $f_b := f|_{\Omega_b}$ and

$$(L,P) : L_q(\Omega_b)^n \to W_q^2(\Omega_b)^n \times \dot{W}_q^1(\Omega_b)$$

be the solution operator of the Stokes equation in the bounded domain Ω_b. Define

$$R_1(z) \in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q})\big)$$

by

$$R_1(z)f = \varphi R_\pm(z)(\psi f) + (1-\varphi)Lf_b.$$

Similarly, define

$$\Pi(z) \in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), \dot{W}_q^1(\Omega;\omega_n^{s'q})\big)$$

by

$$\Pi(z)f = \varphi P_\pm(z)(\psi f) + (1-\varphi)Pf_b.$$

Obviously, the operator $R_1(z)$ has the same type of expansion as $R_\pm(z)$. Let

$$P_\pm(z) = \sum_{j=0}^{[\sigma]-1} z^j P_{\pm,j} + P_{\pm,r}(z)$$

with

$$P_{\pm,r}(z) = O(z^{\sigma-1}) \quad \text{in} \quad \mathcal{L}\big(L_q(\mathbb{R}^n_\pm;\omega_n^{sq}), \dot{W}^1_q(\mathbb{R}^n_\pm;\omega_n^{s'q})\big)$$

be the expansion for $P_\pm(z)$. We choose $P_{\pm,j}f, P_{\pm,r}f \in \dot{W}^1_q(\mathbb{R}^n_\pm)$ such that

$$\int_{D_R\cap\Omega} P_{\pm,0}f\,dx = \int_{D_R\cap\Omega} Pf_b\,dx$$
$$\int_{D_R\cap\Omega} P_{\pm,r}(z)f\,dx = 0, \qquad \int_{D_R\cap\Omega} P_{\pm,j}f\,dx = 0 \quad (j=1,\dots,[\sigma]-1)$$

where $D_R = \{x\in\Omega:\ R-1<|x|<R\}$. Applying Poincaré's inequality

$$\|f\|_q \le C\bigg(\|\nabla f\|_q + \bigg|\int_D f(x)\,dx\bigg|\bigg)$$

for a bounded domain D with C^0-boundary (see [2: Chapter 5/Theorem 4.19]) it follows that

$$\|P_{\pm,0}f - Pf_b\|_{L_q(D_R\cap\Omega)} \le C\left(\|\nabla P_{\pm,0}f\|_{L_q(D_R\cap\Omega)} + \|\nabla Pf_b\|_{L_q(\Omega_b)}\right) \le C\|f\|_{L_q(\Omega;\omega_n^{sq})}$$
$$\|P_{\pm,j}f\|_{L_q(D_R\cap\Omega)} \le C\|\nabla P_{\pm,j}f\|_{L_q(D_R\cap\Omega)} \le C\|f\|_{L_q(\Omega;\omega_n^{sq})}$$
$$\|P_{\pm,r}(z)f\|_{L_q(D_R\cap\Omega)} \le C\|\nabla P_{\pm,r}(z)f\|_{L_q(D_R\cap\Omega)} \le C|z|^{\sigma-1}\|f\|_{L_q(\Omega;\omega_n^{sq})}.$$

Because of these inequalities and the identity

$$\nabla\Pi(z)f = \varphi\nabla P_\pm(z)(\psi f) + (1-\varphi)\nabla Pf_b + (\nabla\varphi)(P_\pm(z)(\psi f) - Pf)$$

the operator $\Pi(z)$ has the same type of expansion as $P_\pm(z)$.

It remains to correct the divergence of $R_1(z)f$. For this we apply Bogovskii's Theorem (see, e.g., [6: Theorem 3.2]) to $\operatorname{div}(R_1(z)f) = \nabla\varphi\cdot\{R_\pm(z)(\psi f) - Lf_b\}$, which has compact support in D_R. We note that

$$\begin{aligned}\int_{D_R}\operatorname{div}(R_1(z)f) &= -\int_{B_R\cap\mathbb{R}^n_\pm}\operatorname{div}\big((1-\varphi)R_\pm(z)(\psi f)\big)dx - \int_{\Omega_b}\operatorname{div}(\varphi Lf_b)\,dx\\ &= -\int_{\partial(B_R\cap\mathbb{R}^n_\pm)} N\cdot(1-\varphi)R_\pm(z)(\psi f)\,d\sigma - \int_{\partial\Omega_b} N\cdot\varphi Lf_b\,d\sigma\\ &= 0.\end{aligned}$$

Since $\mathrm{div} R_1(z)f \in W^2_q(D_R) \cap W^1_{0,q}(D_R)$, we get a compact operator $Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^2_{0,q}(D_R)$ with $\mathrm{div} Q(z)f = \mathrm{div} R_1(z)f$. The operator $Q(z)$ depends continuously on $z \in \Sigma_\delta \cup \{0\}$.

We identify $Q(z)f$ with its extension by zero to a function $Q(z)f \in W^2_{0,q}(\Omega;\omega_n^{s'q})$. Now let

$$R_2(z) := R_1(z) - Q(z) \in \mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W^2_q(\Omega;\omega_n^{s'q})\big).$$

Then $R_2(z)f$ solves

$$\begin{aligned} (z-\Delta)R_2(z)f + \nabla\Pi(z)f &= f + S(z)f && \text{in } \Omega \\ \mathrm{div} R_2(z)f &= 0 && \text{in } \Omega \\ R_2(z)f &= 0 && \text{on } \partial\Omega \end{aligned}$$

for all $f \in L_q(\Omega;\omega_n^{sq})$, where

$$\begin{aligned} S(z)f = &-\big\{2(\nabla\varphi)\cdot\nabla + (\Delta\varphi)\big\}\big\{R_\pm(z)(\psi f) - Lf_b\big\} \\ &+ z(1-\varphi)Lf_b + (\Delta - z)Q(z)f + \nabla\varphi\big(P_\pm(z)(\psi f) - Pf_b\big). \end{aligned}$$

Since $\mathrm{supp} S(z)f \subseteq \overline{D_R}$, we conclude $S(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$. The term $(\Delta - z)Q(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is a compact operator since $Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^2_{0,q}(D_R)$ is compact. Furthermore, $S(z) - (\Delta - z)Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^1_q(D_R)$ is continuous, so $S(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is a compact operator. Moreover, $S(z)$ is continuous in $z \in \Sigma_\delta \cup \{0\}$ and has the same type of expansion in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ as $R_\pm(z)$ in $\mathcal{L}\big(L_q(\Omega;\omega_n^{sq}), W^2_q(\Omega;\omega_n^{s'q})\big)$.

In the following Lemma 4.2 we show that $I + S(0)$ is injective. Since $S(0)$ is compact, the Fredholm alternative yields that $(I+S(0))^{-1} \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ exists. Therefore $(I + S(z))^{-1}$ exists for all $z \in \Sigma_{\delta,\varepsilon}$ for some $\varepsilon > 0$. More precisely,

$$(I + S(z))^{-1} = (I + S(0))^{-1} \sum_{k=0}^{\infty} \big[(S(0) - S(z))(I + S(0))^{-1}\big]^k$$

for all $z \in \Sigma_{\delta,\varepsilon_0}$, where $\varepsilon_0 > 0$ is chosen so small that

$$\|S(z) - S(0)\| \leq \frac{1}{2\|(I + S(0))^{-1}\|} \qquad (z \in \Sigma_{\delta,\varepsilon_0}).$$

Since $S(z)$ and therefore all powers $(S(0) - S(z))^k$ have an expansion in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ of the same type as $R_\pm(z)$, the inverse $(I + S(z))^{-1}$ has the same.

If we now set $R(z) = R_2(z)(I+S(z))^{-1}$ and $P(z) = \Pi(z)(I+S(z))^{-1}$, we get the solution operators of the resolvent problem with the desired expansion ∎

Lemma 4.2. *Let $S(z)$ denote the same operator as in the proof of Theorem 4.1. Then $I+S(0)\in\mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is injective.*

Proof. It is known [3, 4] that the Stokes equations in an aperture domain have a unique solution $(u,\tilde{p})\in\big[\dot{W}_p^2(\Omega)\cap\dot{W}_{p^*}^1(\Omega)\big]^n\times\dot{W}_p^1(\Omega)$ $(\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}$ with $1<p<n)$ for given force $f\in L_p(\Omega)$ and prescribed flux $\Phi(u)=\alpha\in\mathbb{R}$.

We calculate the flux of $R_2(0)$. Since $M\subset B_r$, the identity $R_2(0)f(x)=Lf_b(x)$ holds for all $x\in M$. Denote by B_+ the connected component of $B_r(0)\setminus M$ "above" M. Then we conclude that

$$0=\int_{B_+}\operatorname{div}Lf_b\,dx=\int_{\partial B_+}Lf_b\cdot N\,d\sigma=\int_M Lf_b\cdot N\,d\sigma=\int_M R_2(0)f\cdot N\,d\sigma.$$

Therefore we get $R_2(0)f=0$ and $\Pi(0)=$ const if we show that $R_2(0)f\in\big[\dot{W}_p^2(\Omega)\cap\dot{W}_{p^*}^1(\Omega)\big]^n$ and $\Pi(0)f\in\dot{W}_p^1(\Omega)$.

Let $(I+S(0))f=0$. That means $f=-S(0)f$, and therefore the support of f is contained in $\overline{\Omega}_b$. This implies $f\in L_p(\Omega;\omega_n^{sp})$ for all $s\in\mathbb{R}$ and $1\le p\le q$.

Claim. $\nabla^2R_2(0)f,\nabla\Pi(0)f\in L_p(\Omega)$ *for all* $1<p\le q$ *and* $\nabla R_2(0)f\in L_{p^*}(\Omega)$ *with* $\frac{1}{p^*}=\frac{1}{p}-\frac{1}{n}$ *and* $1<p<\min\{q,n\}$.

Proof of claim. For $i,j\in\{1,\dots,n\}$ there holds

$$\begin{aligned}\partial_i\partial_jR_2(0)f&=\varphi\partial_i\partial_jR_\pm(0)(\psi f)+\partial_i\partial_j[(1-\varphi)Lf_b]+(\partial_i\varphi)\partial_jR_\pm(0)(\psi f)\\&\quad+(\partial_j\varphi)\partial_iR_\pm(0)(\psi f)+(\partial_i\partial_j\varphi)R_\pm(0)(\psi f)-\partial_i\partial_jQ(0)f.\end{aligned}$$

The support of every term except the first one is contained in $\overline{\Omega}_b$. Therefore each of these function is an element of $L_p(\Omega)$ for every $1\le p\le q$.

Considering the first term, Theorem 3.3 tells us that

$$\partial_i\partial_jR_\pm(0)\in\mathcal{L}\big(L_p(\mathbb{R}^n_\pm;\omega_n^{sp}),L_p(\Omega,\omega_n^{s'p})\big)$$

for all $-\frac{n}{p}<s'\le 0\le s<\frac{n}{p'}$, $s'=s-2\sigma+2$ and $1<\sigma<\frac{n}{2}$. Since $f\in L_p^s(\Omega)$ for arbitrary $s\in\mathbb{R}$ and $1\le p\le q$, we can apply Theorem 3.3 for $s'=0$ and $s=2\sigma-2$. Therefore we choose $1<\sigma<\frac{n}{2}$ such that $\frac{n}{n-2\sigma+2}<p$ which is equivalent to $2\sigma-2<\frac{n}{p'}$. Thus we get $\partial_i\partial_jR_\pm(0)(\psi f)\in L_p(\Omega)$ for every $1<p\le q$. With the same choice of s and s' we see that $\nabla\Pi(0)f\in L_p(\Omega)$ for all $1<p\le q$.

The same argumentation can be applied to

$$\partial_iR_2(0)f=\varphi\partial_iR_\pm(0)(\psi f)+\partial_i[(1-\varphi)Lf_b]+(\partial_i\varphi)R_\pm(0)(\psi f)-\partial_iQ(0)f.$$

In this case

$$\partial_iR_\pm(0)\in\mathcal{L}\big(L_r(\Omega;\omega_n^{sr}),L_r(\Omega;\omega_n^{s'r})\big)$$

holds for all $-\frac{n}{r} < s' \leq 0 \leq s < \frac{n}{r'}$, $s' := s - 2\sigma + 1$, $1 < \sigma < \frac{n}{2}$. The choice of $s' = 0$ and $s = 2\sigma - 1$ yields the condition $2\sigma - 1 < \frac{n}{r'}$. Since $\frac{1}{r} + \frac{1}{n} = \frac{1}{p}$, this condition is equivalent to $2\sigma - 2 < \frac{n}{p'}$ which is equivalent to $p > \frac{n}{n-2\sigma+2}$. This proves the claim.

Thus $R_2(0)f = 0$ and $\nabla\Pi(0)f = 0$. Since $\mathrm{supp}Q(0) \subseteq \{x : R-1 \leq |x| \leq R\}$, it is obvious that for $x \in \Omega$

$$R_2(0)f(x) = \begin{cases} R_\pm(0)(\psi f)(x) = 0 & \text{if } |x| \geq R \\ Lf_b(x) = 0 & \text{if } |x| \leq R-1 \end{cases}$$

$$\nabla\Pi(0)f(x) = \begin{cases} \nabla P_\pm(0)(\psi f)(x) = 0 & \text{if } |x| \geq R \\ \nabla Pf_b(x) = 0 & \text{if } |x| \leq R-1. \end{cases}$$

This implies $f = 0$ for $|x| \geq R$ since

$$\Delta R_\pm(0)(\psi f) + \nabla P_\pm(0)(\psi f) = \psi f \qquad \text{in } \mathbb{R}^n_\pm.$$

Similarly we get $f = 0$ for $x \in \Omega$ with $|x| \leq R-1$ since $-\Delta Lf_b + \nabla Pf_b = f_b$ in Ω_b. The support of $\big(R_\pm(0)(\psi f), P_\pm(0)(\psi f)\big)$ and of (Lf_b, Pf_b) is contained in $\widetilde{D} = \{x \in \Omega : R-1 < |x| < b\}$. Therefore both terms solve the Stokes problem

$$\begin{aligned} -\Delta u + \nabla p = f \qquad & \text{in } \widetilde{D} \\ \mathrm{div} u = 0 \qquad & \text{in } \widetilde{D} \\ u = 0 \qquad & \text{on } \partial\widetilde{D}. \end{aligned}$$

This implies that $R_\pm(0)(\psi f) = Lf_b$ and $\nabla P_\pm(0)(\psi f) = \nabla Pf_b$ in $\widetilde{D}$ because of the unique solvability of the Stokes equations in a bounded domain. Hence $Q(z)f = 0, Lf_b = R_2(0)f = 0$ and $\nabla Pf_b = \nabla\Pi(0)f = 0$ in $\widetilde{D}$ and finally $f = 0$ in the whole domain ∎

5. Decay of the semigroup in weighted spaces

Let $A_q = -P_q\Delta$ denote the Stokes operator for an aperture domain Ω.

Theorem 5.1. *Let $n \geq 3$, $1 < \sigma < \frac{n}{2}$, $1 < q < \infty$, $-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$ and $s' = s - 2\sigma$. Then there exists a constant $C = C(q, s, s')$ such that*

$$\|e^{-tA_q} f\|_{L_q(\Omega;\omega_n^{s'q})} \leq C(1+t)^{-\sigma} \|f\|_{L_q(\Omega;\omega_n^{sq})} \qquad (t \geq 0)$$

for all $f \in J_q(\Omega) \cap L_q(\Omega;\omega_n^{sq})$. Furthermore,

$$\|e^{-tA_q} f\|_{W_q^2(\Omega;\omega_n^{s'q})} \leq C(1+t)^{-\sigma} \max\left\{\|f\|_{W_q^2(\Omega)}, \|f\|_{L_q(\Omega;\omega_n^{sq})}\right\} \qquad (t \geq 0)$$

for all $f \in \mathcal{D}(A_q) \cap L_q(\Omega; \omega_n^{sq})$.

Proof. The proof of the inequalities is nearly the same as the proof of [8: Theorem 1.1]. So we give only a sketch.

Since the semigroup e^{-tA_q} is bounded in $J_q(\Omega)$, the first estimate is satisfied for $0 < t < 1$. The second estimate holds for $0 < t < 1$ because of the estimates

$$\|f\|_{W_q^2(\Omega)} \leq c\|(I + A_q)f\|_{L_q(\Omega)} \leq C\|f\|_{W_q^2(\Omega)} \tag{20}$$

for all $f \in \mathcal{D}(A_q)$ (the first inequality is a consequence of [4: Theorem 2.1], the second inequality is obvious). For $t \geq 1$ consider the representation of the semigroup

$$e^{-tA_q} = \tfrac{1}{2\pi i} \int_\Gamma e^{tz}(z + A_q)^{-1} dz$$

where the curve Γ coincides outside a ball $B_\varepsilon(0)$ $(0 < \varepsilon < \varepsilon_0)$ with the rays $e^{\pm\phi i}\tilde{t}$ $(\tilde{t} > 0)$ with $\frac{\pi}{2} < \phi < \delta$ (δ and ε_0 are the same numbers as in Theorem 4.1). We split the curve Γ into two parts

$$\Gamma_1 = \big\{ z \in \Gamma : \, 0 < |z| < \varepsilon \big\}$$
$$\Gamma_2 = \big\{ z \in \Gamma : \, \varepsilon \leq |z| \big\}.$$

So we get

$$e^{-tA_q} f = \tfrac{1}{2\pi i} \int_{\Gamma_1} e^{tz} R(z) f \, dz + \tfrac{1}{2\pi i} \int_{\Gamma_2} e^{tz}(z + A_q)^{-1} f \, dz$$

for all $f \in J_q(\Omega) \cap L_q(\Omega; \omega_n^{sq})$ since $R(z)f = (z + A_q)^{-1} f$ for $z \in \Sigma_{\delta,\varepsilon}$. Using the resolvent estimate $\|(z + A_q)^{-1} f\|_q \leq C|z|^{-1}\|f\|_q$ we easily get

$$\begin{aligned} \left\| \tfrac{1}{2\pi i} \int_{\Gamma_2} e^{tz}(z + A_q)^{-1} dz f \right\|_{L_q(\Omega; \omega_n^{s'q})} &\leq C \int_\varepsilon^\infty \frac{e^{ts\cos\phi}}{s} \, ds \, \|f\|_{L_q(\Omega)} \\ &\leq C(\varepsilon, \phi) \frac{e^{-ct}}{t} \|f\|_{L_q(\Omega; \omega_n^{sq})} \end{aligned}$$

with some constant $C = C(\varepsilon, \phi) > 0$. Analogously we get

$$\begin{aligned} \left\| \tfrac{1}{2\pi i} \int_{\Gamma_2} e^{tz}(z + A_q)^{-1} dz f \right\|_{W_q^2(\Omega; \omega_n^{s'q})} &\leq C \int_\varepsilon^\infty \frac{e^{ts\cos\phi}}{s} \, ds \, \|f\|_{W_q^2(\Omega)} \\ &\leq C(\varepsilon, \phi) \frac{e^{-ct}}{t} \|f\|_{W_q^2(\Omega)} \end{aligned}$$

if we use (20) for $f \in \mathcal{D}(A_q)$.

We use the resolvent expansion of Theorem 4.1 to estimate the first integral. Since $\sum_{j=0}^{[\sigma]-1} z^j G_j$ is holomorphic in $\mathbb{C}$, there holds

$$\left\| \sum_{j=0}^{[\sigma]-1} \int_{\Gamma_1} e^{tz} z^j G_j dz \right\|_{\mathcal{L}(L_q(\omega_n^{sq}),W_q^2(\omega_n^{s'q}))} \leq C e^{\varepsilon t \cos(\phi)} = C e^{-ct}$$

with $C > 0$. In order to estimate the remainder term we deform the curve Γ_1 to a curve Γ^* which coincides with $z = e^{\pm\phi i}\tilde{t}$ $(\tilde{t} \in [0,\varepsilon])$. Therefore

$$\left\| \tfrac{1}{2\pi i} \int_{\Gamma_1} e^{tz} G_r(z)\, dz \right\|_{\mathcal{L}(L_q(\omega_n^{sq}),W_q^2(\omega_n^{s'q}))} \leq C \int_0^\infty e^{\lambda t \cos(\phi)} \lambda^{\sigma-1} d\lambda = C' t^{-\sigma}.$$

Collecting all estimates we proved the theorem ∎

6. The L_q-L_r-estimate

In order to get an estimate of $\|e^{-tA_q}f\|_{L_q(\Omega_b)}$ where $\Omega_b = \Omega \cap B_b(0)$, we need the following

Lemma 6.1. *Let $1 < q < \infty$ and $-\frac{n}{q} < s' < 0$. Then*

$$\|e^{-tA_q}f\|_{L_q(\Omega;\omega_n^{s'q})} \leq C(1+t)^{\frac{s'}{2}} \|f\|_{L_q(\Omega)}$$

for all $f \in J_q(\Omega)$ and

$$\|e^{-tA_q}f\|_{W_q^2(\Omega;\omega_n^{s'q})} \leq C(1+t)^{\frac{s'}{2}} \|f\|_{W_q^2(\Omega)}$$

for all $f \in \mathcal{D}(A_q)$.

Corollary 6.2. *Let $1 < q < \infty$. Then for every $0 \leq s < \frac{n}{2q}$ there is a constant $C = C(s,q,\Omega)$ with*

$$\|e^{-tA_q}f\|_{L_q(\Omega_b)} \leq C(1+t)^{-s} \|f\|_{L_q(\Omega)}$$

for all $f \in J_q(\Omega)$ and

$$\|e^{-tA_q}f\|_{W_q^2(\Omega_b)} \leq C(1+t)^{-s} \|f\|_{W_q^2(\Omega)}$$

for all $f \in \mathcal{D}(A_q)$.

Proof of Lemma 6.1. If $1 < p < \frac{n}{2}$, then $\frac{n}{p} > 2$. So we can we apply Theorem 5.1 with $s = 0$. Therefore we get

$$\|e^{-tA_p} f\|_{W_p^m(\Omega;\omega_n^{\tilde{s}'p})} \le C(1+t)^{\frac{\tilde{s}'}{2}} \|f\|_{W_p^m(\Omega)} \tag{21}$$

for $m = 0, 2$, $f \in J_p(\Omega)$ resp. $f \in \mathcal{D}(A_p)$ and $-\frac{n}{p} < \tilde{s}' < -2$. In order to get the statement of the lemma we interpolate estimates (21) and

$$\|e^{-tA_r} f\|_{W_r^m(\Omega)} \le C\|f\|_{W_r^m(\Omega)} \qquad \big(m = 0, 2;\ f \in J_r(\Omega) \text{ resp. } \mathcal{D}(A_r)\big) \tag{22}$$

for suitable p close to 1 and large r. For this we need the statement about complex interpolation

$$\big(L_p(\Omega;\omega_n^{\tilde{s}'p}), L_r(\Omega)\big)_{[\theta]} = L_q(\Omega;\omega_n^{\tilde{s}'p(1-\theta)})$$

with $0 < \theta < 1$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ (see, for example, [1: Theorem 5.5.3]).

Now let $1 < q < \infty$ and $-\frac{n}{q} < s' < 0$ be given as in the assumptions. We set $\tilde{s}' = \frac{s'}{1-\theta}$ and $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ for $0 < \theta < 1$. Then we choose $0 < \theta < 1$ such that

$$-\frac{n}{p}(1-\theta) < s' < -2(1-\theta) \quad \Longleftrightarrow \quad -\frac{n}{p} < \tilde{s}' < -2$$

which exists if $1 < p < \min\{\frac{n}{2}, q\}$. If we furthermore use $(J_p(\Omega), J_r(\Omega))_{[\theta]} = J_q(\Omega)$ (see Appendix), we get with these chosen θ and p and the corresponding r that

$$\|e^{-tA_q} f\|_{L_q(\Omega;\omega_n^{s'q})} \le C\big[(1+t)^{\frac{\tilde{s}'}{2}}\big]^{1-\theta} \|f\|_{L_q(\Omega)} = C(1+t)^{\frac{s'}{2}} \|f\|_{L_q(\Omega)}$$

for $f \in J_q(\Omega)$. Complex interpolation with the same parameters yields the estimate for $f \in \mathcal{D}(A_q)$. For this we use the second estimate of Theorem 5,1 and $\big(\mathcal{D}(A_p), \mathcal{D}(A_r)\big)_{[\theta]} = \mathcal{D}(A_q)$. The latter equation will be proved in Appendix ∎

Proof of Theorem 1.1. The proof is similar to that of [8: Theorem 1.2] but a little bit shorter. It is sufficient to show the statement for $0 < \sigma < \frac{1}{2}$ since we can reduce the general case to this statement (choose $q = q_0 < q_1 < \ldots < q_k = r$ such that $\sigma_i := \frac{n}{2}(\frac{1}{q_i} - \frac{1}{q_{i+1}}) < \frac{1}{2}$ and apply the statement to q_i and q_{i+1}).

Step 1: *The inequality holds for* $t \ge 2$. Let $\tilde{u}_0 := e^{-A_q} u_0$. Then $\tilde{u}_0 \in \mathcal{D}(A_q)$ and $\|\tilde{u}_0\|_{W_q^2(\Omega)} \le C\|u_0\|_{L_q(\Omega)}$. Moreover, let $\tilde{u}(t) := e^{-tA_q}\tilde{u}_0$ and $\tilde{p}(t) \in \dot{W}_q^1(\Omega)$ be the pressure corresponding to $\tilde{u}(t)$. Let $\Omega \cup B_r(0) = \mathbb{R}_+^n \cup \mathbb{R}_-^n \cup B_r(0)$ and $b > r + 1$. We choose a cut-off function $\psi \in C^\infty(\Omega)$ with

$\psi(x) = 1$ for $|x| \geq b$ and $\psi(x) = 0$ for $|x| \leq b-1$. Then $\operatorname{div}(\psi\tilde{u}(t)) = \nabla\psi\cdot\tilde{u}(t) \in W^1_{0,q}(D_b)$ with $D_b = \{x \in \Omega : b-1 < |x| < b\}$ and $\int_{D_b} \nabla\psi \cdot \tilde{u}(t)\, dx = 0$. Applying Bogovskii's theorem [6: Theorem 3.2] we know that there exists a $v_0(t) \in W^2_{0,q}(D_b)$ with $\operatorname{div} v_0(t) = \operatorname{div}(\psi\tilde{u}(t))$ and

$$\|v_0(t)\|_{W^2_q(D_b)} \leq C\|\tilde{u}(t)\|_{W^1_q(D_b)}. \tag{23}$$

Therefore we have

$$\|\partial_t v_0(t)\|_{W^1_q(D_b)} \leq C\|e^{-tA_q}A_q\tilde{u}_0\|_{L_q(D_b)} \leq C(1+t)^{-\tilde{s}}\|\tilde{u}_0\|_{W^2_q(\Omega)} \tag{24}$$

with an arbitrary $0 \leq \tilde{s} < \frac{n}{2q}$. If we define $v_1(t) = \psi\tilde{u}(t) - v_0(t)$, it solves the equations

$$\partial_t v_1(t) - \Delta v_1(t) + \nabla(\psi\tilde{p}(t)) = h(t) \quad \text{in } (0,\infty)\times\mathbb{R}^n_\pm \tag{25}$$

$$\operatorname{div} v_1(t) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^n_\pm \tag{26}$$

$$v_1(t)|_{\partial\mathbb{R}^n_\pm} = 0 \quad \text{in } (0,\infty) \tag{27}$$

$$v_1(0) = v_1 \tag{28}$$

with $v_1 = \psi\tilde{u}_0 - v_0(0)$ and

$$h(t) = -\{2(\nabla\psi)\cdot\nabla + (\Delta\psi)\}\tilde{u}(t) - (\partial_t - \Delta)v_0(t) + (\nabla\psi)\tilde{p}(t).$$

Moreover, $\operatorname{supp} h(t) \subseteq \overline{D}_b$. We choose the pressure $\tilde{p}(t)$ such that $\int_{D_b} \tilde{p}(t)\, dx = 0$. If we now apply (23) - (24), Poincaré's inequality [2: Theorem 4.19] and Corollary 6.2, we get

$$\begin{aligned}
\|h(t)\|_{L_q(D_b)} &\leq C\left(\|\tilde{u}(t)\|_{W^1_q(D_b)} + \|v_0(t)\|_{W^2_q(D_b)} + \|\partial_t v_0(t)\|_{L_q(D_b)} + \|\tilde{p}(t)\|_{L_q(D_b)}\right)\\
&\leq C\left((1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W^2_q(\Omega)} + \|\nabla\tilde{p}(t)\|_{L_q(\Omega_b)}\right)\\
&\leq C\left((1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W^2_q(\Omega)} + \|\partial_t\tilde{u}(t)\|_{L_q(D_b)} + \|\tilde{u}(t)\|_{W^2_q(D_b)}\right)\\
&\leq C(1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W^2_q(\Omega)}
\end{aligned}$$

with an arbitrary $\tilde{s}$ such that $0 \leq \tilde{s} < \frac{n}{q}$.

Let $E_\pm(t)$ denote the semigroup of the Stokes operator in $\mathbb{R}^n_\pm$ and $P_\pm$ denote the Helmholtz projection in $L_q(\mathbb{R}^n_\pm;\omega_n^{sq})$. Since $v_1(t)$ solves (25) - (28), the identity

$$v_1(t) = E_\pm(t)v_1 + \int_0^t E_\pm(t-\tau)P_\pm h(\tau)\, d\tau$$

holds. Because of Corollary 3.4 and the $L_q - L_r$-estimate in the half space [12: Theorem 3.1] the semigroup $E_\pm(t)$ satisfies

$$\|E_\pm(t)f\|_{L_r(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\|f\|_{L_q(\mathbb{R}^n_\pm)}$$
$$\|E_\pm(t)f\|_{L_q(\mathbb{R}^n_\pm)} \le C(1+t)^{-\frac{s}{2}}\|f\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}$$

with $1 < q \le r < \infty$, $0 \le s < \frac{n}{q'}$ and $\sigma = \frac{n}{2}(\frac{1}{q} - \frac{1}{r})$ for all $t > 0$ and $f \in J_q(\mathbb{R}^n_\pm)$ resp. $f \in J_q(\mathbb{R}^n_\pm;\omega_n^{sq})$. Using both inequalities we get

$$\|E_\pm(t)f\|_{L_r(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\left\|E_\pm\left(\frac{t}{2}\right)f\right\|_{L_q(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}(1+t)^{-\frac{s}{2}}\|f\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}$$

for $f \in J_q(\mathbb{R}^n_\pm;\omega_n^{sq})$ and $t > 0$. Therefore we conclude

$$\|E_\pm(t)v_1\|_{L_r(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\|v_1\|_{L_q(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\|\tilde{u}_0\|_{L_q(\Omega)}$$

and

$$\begin{aligned}
&\left\|\int_0^t E_\pm(t-\tau)P_\pm h(\tau)\,d\tau\right\|_{L_r(\mathbb{R}^n_\pm)} \\
&\le C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}\underbrace{\|P_\pm h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}}_{\le C\|h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}}d\tau \\
&\le C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}\|h(\tau)\|_{L_q(D_b)}d\tau \\
&\le C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}(1+\tau)^{-\frac{\tilde{s}}{2}}d\tau\;\|\tilde{u}_0\|_{W_q^2(\Omega)}.
\end{aligned}$$

We now choose $0 \le s < \frac{n}{q'}$ and $\sigma \le \frac{\tilde{s}}{2} < \frac{n}{2q}$ such that $\frac{s}{2} + \frac{\tilde{s}}{2} > 1$, $\frac{s}{2} + \sigma \ne 1$ and $\frac{\tilde{s}}{2} \ne 1$ (this is possible since $\frac{n}{2q} + \frac{n}{2q'} = \frac{n}{2} > 1$). If we apply Lemma A.2 (see Appendix) with this choice of s and $\tilde{s}$, we get

$$\left\|\int_0^t E_\pm(t-\tau)P_\pm h(\tau)\,d\tau\right\|_{L_r(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\|\tilde{u}_0\|_{W_q^2(\Omega)}$$

and therefore

$$\|v_1(t)\|_{L_r(\mathbb{R}^n_\pm)} \le Ct^{-\sigma}\|\tilde{u}_0\|_{W_q^2(\Omega)}.$$

Since $u(t,x) = v_1(t,x)$ for all $x \in \Omega \setminus \Omega_b$, the previous estimates, Corollary 6.2 and Sobolev's embedding theorem imply that

$$\begin{aligned}
\|\tilde{u}(t)\|_{L_r(\Omega)} &\le \|\tilde{u}(t)\|_{L_r(\Omega_b)} + \|v_1(t)\|_{L_r(\Omega\setminus\Omega_b)} \\
&\le C\big(\|\tilde{u}(t)\|_{W_q^2(\Omega_b)} + \|v_1(t)\|_{L_r(\Omega\setminus\Omega_b)}\big) \\
&\le Ct^{-\sigma}\|\tilde{u}_0\|_{W_q^2(\Omega)} \\
&\le Ct^{-\sigma}\|f\|_{L_q(\Omega)}.
\end{aligned}$$

Since $\tilde{u}(t) = e^{-(t+1)A_q}u_0$, we have proved the theorem for $t \geq 2$

Step 2: *The inequality holds for $t < 2$.* The case $t < 2$ is proved in the same way as in the proof of [8: Theorem 1.2] using Sobolev's embedding theorem and an interpolation method ∎

Proof of Theorem 1.2. Because of the semigroup property of e^{-tA_q} and Theorem 1.1 it suffices to prove the statement for $\sigma = 0$, i.e. $1 < q = r < n$. The proof for the case $t < 2$ uses the same interpolation method as in the proof of Theorem 1.2.

So let $t \geq 2$ and $v_1(t)$, $v_0(t)$, $h(t)$ be the functions used in the proof of Theorem 1.1. Then

$$\nabla v_1(t) = \nabla E_\pm(t)v_1 + \int_0^t \nabla E_\pm(t-\tau)P_\pm h(\tau)\,d\tau.$$

The estimate for the Stokes semigroup in $\mathbb{R}^n_\pm$ yields

$$\|\nabla E_\pm(t)v_1\|_{L_q(\mathbb{R}^n_\pm)} \leq Ct^{-\frac{1}{2}}\|v_1\|_{L_q(\mathbb{R}^n_+)}.$$

Now we choose $0 \leq s < \frac{n}{q'}$ and $1 \leq \tilde{s} < \frac{n}{q}$ with $\frac{s}{2} + \frac{\tilde{s}}{2} > 1$, $\frac{\tilde{s}}{2} \neq 1$ and $\frac{1}{2} + \frac{s}{2} \neq 1$. So we get because of Corollary 6.2 and Lemma A.2 (see Appendix)

$$\begin{aligned}
&\left\|\int_0^t \nabla E_\pm(t-\tau)P_\pm h(\tau)\,d\tau\right\|_{L_q(\mathbb{R}^n_\pm)} \\
&\quad\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{s}{2}}\|P_\pm h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega^{sq})}d\tau \\
&\quad\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{s}{2}}\|h(\tau)\|_{L_q(\Omega_b)}d\tau \\
&\quad\leq C\int_0^t (t-\tau)^{-\frac{1}{2}}(1+t-\tau)^{-\frac{s}{2}}(1+\tau)^{-\frac{\tilde{s}}{2}}d\tau\ \|\tilde{u}_0\|_{W^2_q(\Omega)} \\
&\quad\leq Ct^{-\frac{1}{2}}\|\tilde{u}_0\|_{W^2_q(\Omega)}.
\end{aligned}$$

Moreover, let $\tilde{s} = 1 < \frac{n}{q}$. Therefore we get for $t \geq 1$

$$\begin{aligned}
\|\nabla e^{-(t+1)A_q}f\|_{L_q(\Omega)} &\leq C\big(\|\nabla\tilde{u}(t)\|_{L_q(\Omega_b)} + \|\nabla v_1(t)\|_{L_q(\mathbb{R}^n_\pm)}\big) \\
&\leq C\big((1+t)^{-\frac{\tilde{s}}{2}} + t^{-\frac{1}{2}}\big)\|\tilde{u}_0\|_{W^2_q(\Omega)} \\
&\leq Ct^{-\frac{1}{2}}\|f\|_{L_q(\Omega)}.
\end{aligned}$$

Thus the theorem is also true for $t \geq 2$ ∎

A. Appendix

It remains to prove the necessary technical lemma used in the last section.

Lemma A.1. *Let* $1 < p, q, r < \infty$, $\theta \in (0,1)$ *with* $\frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{p}$ *and let* Ω *be an aperture domain. Then*

$$(\mathcal{D}(A_r), \mathcal{D}(A_p))_{[\theta]} = \mathcal{D}(A_q)$$
$$(J_r(\Omega), J_p(\Omega))_{[\theta]} = J_q(\Omega).$$

Proof. To prove the first equality we define a continuous projection $P_q : W_q^2(\Omega)^n \to \mathcal{D}(A_q)$ for arbitrary $1 < q < \infty$. For a function $u \in W_q^2(\Omega)^n$ let $(v, p) \in W_q^2(\Omega)^n \times \dot{W}_q^1(\Omega)$ denote the unique solution of the resolvent equations (16) - (19) with right-hand side $f = (z - \Delta)u$ for some fixed $z \in \Sigma_\delta$ (see [9: Theorem 2.1]). We set $P_q u = v$. Then

$$\|v\|_{W_q^2(\Omega)} \leq C\|(z-\Delta)u\|_{L_q(\Omega)} \leq C\|u\|_{W_q^2(\Omega)}.$$

If $u \in \mathcal{D}(A_q)$, $(u, 0)$ is the unique solution of these equations. Therefore P_q is a continuous projection on $\mathcal{D}(A_q)$.

If $u \in W_r^2(\Omega)^n \cap W_q^2(\Omega)^n$, the corresponding solutions in $W_r^2(\Omega)^n$ and $W_q^2(\Omega)^n$ coincide (see [3: Lemma 3.2]). Therefore we can extend P_q and P_r to a well-defined projection $P(u_r + u_q) = P_r u_r + P_q u_q$ on $W_r^2(\Omega)^n + W_p^2(\Omega)^n$ with $P|_{W_r^2(\Omega)^n} = P_r$ and $P|_{W_p^2(\Omega)^n} = P_p$. Therefore we conclude

$$\begin{aligned}
\mathcal{D}(A_q) &= P\big(W_r^2(\Omega)^n, W_p^2(\Omega)^n\big)_{[\theta]} \\
&= \big(PW_r^2(\Omega)^n, PW_p^2(\Omega)^n\big)_{[\theta]} \\
&= \big(\mathcal{D}(A_r), \mathcal{D}(A_p)\big)_{[\theta]}.
\end{aligned}$$

The second equality immediately follows from the fact that $P_q = P_r$ on $J_q(\Omega) \cap J_r(\Omega)$ (see [4: Lemma 3.2]) ∎

Lemma A.2. *Let* $0 \leq \alpha < 1$, $\beta \geq 0$, $\alpha \leq \gamma$, $\beta + \gamma > 1$, $\alpha + \beta \neq 1$ *and* $\gamma \neq 1$. *Then*

$$\int_0^t (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds \leq Ct^{-\alpha}.$$

Proof. The case $t \in (0,1)$ is trivial. For $t > 1$ we simply estimate

$$\begin{aligned}
\int_0^{\frac{t}{2}} (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds &\leq Ct^{-\alpha-\beta} \int_0^{\frac{t}{2}} (1+s)^{-\gamma} ds \\
&\leq Ct^{-\alpha-\beta} \begin{cases} t^{1-\gamma} & \text{if } \gamma < 1 \\ 1 & \text{if } \gamma > 1 \end{cases} \\
&\leq Ct^{-\alpha}.
\end{aligned}$$

Similarly we get

$$\int_{\frac{t}{2}}^{t} (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds \leq Ct^{-\gamma} \begin{cases} t^{1-\alpha-\beta} & \text{if } \alpha+\beta<1 \\ 1 & \text{if } \alpha+\beta>1 \end{cases}$$
$$\leq Ct^{-\alpha}$$

and the proof is finished ∎

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