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Stably Solvable Maps are Unstable under Small Perturbations

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Abstract. We show that the set of stably solvable maps from an infinite dimensional Banach space E into itself is not open in the topological space $C(E)$ of the continuous selfmaps of E. The question of whether or not this set is open is related to nonlinear spectral theory and was posed in [7].

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1. Introduction

Let $(E, \|\cdot\|)$ be an infinite dimensional Banach space over a field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). Given a (continuous) map $g : E \to E$, the extended real number

$$|g| = \limsup_{\|x\|\to\infty} \frac{\|g(x)\|}{\|x\|}$$

is called the *quasinorm* of g (see [10]). If $|g| < \infty$, then g is said to be *quasibounded*. Clearly, for bounded linear operators the quasinorm and the standard operator norm coincide.

A map $f : E \to E$ is said to be *stably solvable* (see [6]) if the equation

$$f(x) = h(x)$$

has a solution whenever $h : E \to E$ is a completely continuous map with $|h| = 0$. In particular, any stably solvable map is onto. The converse is true for bounded linear operators (see [7]).

Stably solvable maps play an important role in the notion of spectrum for nonlinear operators introduced in [7]. In that paper, to show that the

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spectrum $\sigma(f)$ of any map $f : E \to E$ is a closed subset of $\mathbb{K}$, a suitable topology in the space $C(E)$ of continuous selfmaps of E was considered. Such a topology coincides with the standard one in the subspace $L(E)$ of $C(E)$ of bounded linear operators from E into itself. Since, as well known, the set of surjective bounded linear operators from E into itself is open in $L(E)$, it is natural to ask wether or not the set of stably solvable maps of E is open in $C(E)$. This question, which was posed in [7], has a negative answer, as we show here with an example of a stably solvable map which is the limit (in $C(E)$) of non-surjective maps.

We need first some preliminaries.

The *Kuratowski measure of non-compactness* (see [11]) is a non-negative real function α which assigns to any bounded metric space X the number

$$\alpha(X) = \inf \left\{ r > 0 \,\middle|\, \begin{array}{l} X \text{ admits a finite covering made} \\ \text{of sets with diameter less than } r \end{array} \right\}.$$

Clearly, $\alpha(X) = 0$ if and only if X is totally bounded, which implies that X is compact whenever it is complete. These are the main properties of the measure α for bounded subsets of E (recall that E is an infinite dimensional Banach space):

- $\alpha(A) = \alpha(\bar{A})$, where $\bar{A}$ stands for the closure of A.
- $\alpha(A) = 0$ if and only if $\bar{A}$ is compact.
- $\alpha(\lambda A) = |\lambda|\alpha(A)$, for any $\lambda \in \mathbb{K}$.
- $\alpha(A + B) \le \alpha(A) + \alpha(B)$.
- $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
- $\alpha(\mathrm{co}(A)) = \alpha(A)$, where $\mathrm{co}(A)$ is the convex hull of A (see [3]).
- $\alpha(S) = 2$, where S is the unit sphere of E (see [9, 12]).

Given a continuous map $f : E \to E$, consider the extended real number

$$\alpha(f) = \sup_{\alpha(A)>0} \frac{\alpha(f(A))}{\alpha(A)}.$$

Observe that f is completely continuous if and only if $\alpha(f) = 0$. Moreover, when f is of Lipschitz type with constant k, then $\alpha(f) \le k$. Thus, in particular, for a bounded linear operator $L \in L(E)$ one has $\alpha(L) \le \|L\|$, where $\|L\|$ is the standard operator norm of L.

Let $C(E)$ denote the space of continuous selfmaps of E with the following topology introduced in [7]. Given $\epsilon > 0$, let

$$U_\epsilon = \Big\{ f \in C(E) : \alpha(f) \le \epsilon,\ \|f(x)\| < \epsilon(1 + \|x\|) \text{ for all } x \in E \Big\}$$

and take the family $\{U_\epsilon : \epsilon > 0\}$ as a fundamental system of neighborhoods of the origin of the vector space $C(E)$. By translation we get a fundamental system of neighborhoods of any point of $C(E)$, and this makes $C(E)$ into a topological space. With this topology a sequence $\{f_n\}$ in $C(E)$ converges to f if and only if the following conditions are verified:

- $f_n \to f$, uniformly on bounded subsets of E.
- $|f_n - f| \to 0$.
- $\alpha(f_n - f) \to 0$.

We observe that the topology of $C(E)$ induces the standard one in the subspace $L(E)$ of the bounded linear operators.

2. The example

Let, as before, E denote an infinite dimensional Banach space and define $f : E \to E$ by

$$f(x) = \|x\|x. \tag{2.1}$$

We will show that the map f is stably solvable and there exists a sequence $\{f_n\}$ of non-surjective maps which converges to f in $C(E)$. Since stably solvable maps are surjective, this implies that the set $S(E)$ of stably solvable maps from E onto itself is not open in $C(E)$, whenever the Banach space E is infinite dimensional.

Consider any completely continuous map $h : E \to E$ such that $|h| = 0$. To prove that f is stably solvable we need to show that the equation

$$f(x) = h(x) \tag{2.2}$$

admits at least one solution. Observe first that f is invertible, with inverse given by

$$f^{-1}(y) = \begin{cases} \frac{y}{\sqrt{\|y\|}} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0. \end{cases}$$

Thus f^{-1} is continuous and quasibounded (with $|f^{-1}| = 0$). Now, equation (2.2) is equivalent to $x = f^{-1}(h(x))$ which is solvable, since the identity is stably solvable and the composite map $f^{-1} \circ h$ is completely continuous with $|f^{-1} \circ h| \leq |f^{-1}|\,|h| = 0$. Thus f is stably solvable, as claimed.

Let r be a Lipschitz retraction of the closed unit ball D of E onto its boundary $S = \partial D$. The existence of such a retraction is ensured by a general result of Benyamini and Sternfeld (see [2]). An explicit construction in the space $C[0, 1]$ of continuous functions satisfying $\alpha(r) \leq 9$ has been given in [5] (see also [15] for a general discussion on the smallest possible value of $\alpha(r)$). Define $g : E \to E$ by

$$g(x) = \begin{cases} r(x) & \text{if } \|x\| \le 1 \\ x & \text{if } \|x\| \ge 1 \end{cases}$$

and let $\{f_n\}$ be the sequence of continuous selfmaps of E given by

$$f_n(x) = f\big(\tfrac{1}{n}\, g(nx)\big).$$

Clearly, the maps f_n are not surjective and, in particular, not stably solvable. We will show that $\{f_n\}$ converges to f in $C(E)$. We have

$$f_n(x) = \begin{cases} \frac{1}{n^2}\, r(nx) & \text{if } \|x\| \le \frac{1}{n} \\ f(x) & \text{if } \|x\| \ge \frac{1}{n}. \end{cases}$$

Thus $\{f_n\}$ converges uniformly to f, because for $\|x\| \le \frac{1}{n}$ one has

$$\big\|f_n(x) - f(x)\big\| = \big\|\tfrac{1}{n^2}\, r(nx) - \|x\|x\big\| \le \tfrac{1}{n^2} + \|x\|^2 \le \tfrac{2}{n^2}$$

and for $\|x\| \ge \frac{1}{n}$ the maps f_n and f coincide. Moreover, $|f_n - f| = 0$ for all $n \in \mathbb{N}$, as f_n and f coincide outside a bounded set. Thus, to show that $\{f_n\}$ converges to f in $C(E)$, we are reduced to proving that $\alpha(f_n - f) \to 0$ as $n \to \infty$. To this end, given a bounded subset A of E and $n \in \mathbb{N}$, let us estimate the α-measure of non-compactness of the set $(f_n - f)(A)$. Since the map $f_n - f$ is zero outside the open ball $B_n = \{x \in E : \|x\| < \frac{1}{n}\}$, we may assume $A \subseteq B_n$. The inclusion

$$(f_n - f)(A) \subseteq f_n(A) - f(A)$$

implies

$$\alpha\big((f_n - f)(A)\big) \le \alpha(f_n(A)) + \alpha(f(A)).$$

Thus, we will estimate $\alpha(f_n(A))$ and $\alpha(f(A))$. Since $A \subseteq B_n$ and the map r is of Lipschitz type with some constant k, one has

$$\alpha(f_n(A)) = \alpha\big(\tfrac{1}{n^2}\, r(nA)\big) = \tfrac{1}{n^2}\, \alpha(r(nA)) \le \tfrac{k}{n^2}\, \alpha(nA) = \tfrac{k}{n}\, \alpha(A).$$

Regarding $\alpha(f(A))$, the inclusion $A \subseteq B_n$ implies $f(A) \subseteq \big[0, \frac{1}{n}\big] \cdot A \subseteq \operatorname{co}\big(\{0\} \cup \frac{1}{n}A\big)$ which yields

$$\alpha(f(A)) \le \alpha\big(\operatorname{co}(\{0\} \cup \tfrac{1}{n}A)\big) = \alpha\big(\{0\} \cup \tfrac{1}{n}A\big) = \tfrac{1}{n}\, \alpha(A).$$

We have proved the inequality $\alpha\big((f_n - f)(A)\big) \le \frac{1+k}{n}\, \alpha(A)$ for all $A \subseteq B_n$, which implies the same inequality for any bounded subset A of E since, we recall, $f_n - f$ vanishes outside B_n. We may conclude that $\alpha(f_n - f) \le \frac{1+k}{n}$ and, consequently, the sequence $\{f_n\}$ converges to f in $C(E)$.

3. Some observations

We make several observations on the above example. First of all, we mention that stably solvable maps play a prominent role in spectral theory for nonlinear operators. For instance, an important part of the nonlinear spectrum introduced in [7] is the subspectrum

$$\sigma_\delta(f) = \Big\{\lambda \in \mathbb{K} : \lambda I - f \text{ is not stably solvable}\Big\}$$

where I denotes the identity in E. It was stated as an open problem in [7] whether or not this is a closed subset of the scalar field $\mathbb{K}$. Unfortunately, our example does not allow us to solve this problem. As a matter of fact, any class of maps which *is* closed in $C(E)$ (such as, for example, the class of maps f satisfying $\alpha(f) = 0$ or $|f| = 0$) generates a corresponding closed subspectrum, since the map $\lambda \mapsto \lambda I - f$ is continuous, but not vice versa.

Next, there is a relation to the class of so-called strictly stably solvable maps introduced in [1]. A map $f : E \to E$ is called *k-stably solvable* ($k \geq 0$) if, given any continuous map $h : E \to E$ such that $\alpha(h) \leq k$ and $|h| \leq k$, we may solve equation (2.2) in E. Moreover, f is called *strictly stably solvable* if f is k-stably solvable for some $k > 0$. Obviously, 0-stably solvable maps are then nothing else but stably solvable maps in the sense of [6].

Now, function (2.1) may serve as an example of a stably solvable map which is not strictly stably solvable. In fact, suppose that f is k-stably solvable for some $k > 0$, and let $h = f - f_n$, where n is so large that $\alpha(h) \leq k$. Since $|h| = 0$, by what we have proved above, equation (2.2) admits a solution $\hat{x} \in E$, i.e. $f_n(\hat{x}) = 0$. But this is impossible, since $\|f_n(x)\| \geq \frac{1}{n^2}$ for any $x \in E$.

Finally, function (2.1) may be used to solve another open problem. Let E be a Banach space and $\Omega \subset E$ open, connected and bounded. Following [13], we call a continuous map $f : \overline{\Omega} \to E$ a *k-epi map* ($k \geq 0$) if $f(x) \neq 0$ on $\partial\Omega$ and, for any continuous map $h : \overline{\Omega} \to E$ satisfying $\alpha(h) \leq k$ and $h|_{\partial\Omega} \equiv 0$, equation (2.2) has a solution in Ω. In case $k = 0$ (i.e., for *compact* right-hand sides h), one gets the class of 0-epi maps introduced in [8]. Loosely speaking, the concept of k-epi (in particular, 0-epi) maps is a "local analogue" to the "asymptotic" concept of k-stably solvable (in particular, stably solvable) maps; this class is also quite useful in nonlinear spectral theory (see [4]).

In [13] the authors claim to present an example of a map which is 0-epi but not k-epi for any $k > 0$; unfortunately, this is not true. Our function (2.1), however, has this property on the closed unit ball D of any infinite dimensional Banach space E. Indeed, if $h : D \to E$ is completely continuous with $h|_{\partial D} \equiv 0$, the trivial extension of h to the whole space is also completely continuous and satisfies $|h| = 0$, and so equation (2.2) has a solution $\hat{x} \in E$. Obviously, this solution must belong to D, since otherwise $1 < \|\hat{x}\|^2 = \|f(\hat{x})\| = \|h(\hat{x})\| = 0$.

To see that f is not k-epi on D for any $k > 0$, we may choose again $h = f - f_n$ as above and follow the same reasoning.

We point out that the so-called *lower measure of non-compactness* of map (2.1) on D, i.e.

$$\beta(f) = \inf_{\alpha(A)>0} \frac{\alpha(f(A))}{\alpha(A)}$$

is zero, as may be easily seen by considering spheres of small radius. This is not accidental. In fact, a remarkable and highly non-trivial coincidence theorem due to Väth (see [14]) states that, whenever a continuous map f is 0-epi on some domain and satisfies $\beta(f) > 0$, then f is also k-epi for $0 < k < \beta(f)$.

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