Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 1, 215–231

On the Optimality for Cascade Connection of Passive Scattering Systems and the Best Minorant Outer Function

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Abstract. In this paper we study passive scattering systems in the framework introduced by Arov. The main purpose is to find conditions for conserving the optimality of a cascade connection of passive scattering systems in terms of the best minorant outer function and to characterize optimal passive scattering systems which have the same transfer function.

Keywords: *Passive scattering system, optimality, controllability, best minorant outer function*

AMS subject classification: 49J15, 93B05, 93B20

1. Introduction

In linear dynamic system theory the transfer function $\theta(z)$ is an important characteristic. For any analytic contractive operator function $\theta(z)$ there is a unique outer function called the best minorant outer function corresponding to it [11]. Some important properties such as observability, controllability, minimality and optimality are not always conserved through the cascade connection of two systems. In Section 2 we list some notations and results that will be used in the following. In Section 3 we study conditions for the conservation of the optimality of a cascade connection of passive scattering systems in terms of the best minorant outer functions corresponding to their transfer functions. This study has developed the trend of research of D. Z. Arov and D. C.Khanh which was started since 1988 [1 - 3, 6]. We obtain a necessary and sufficient condition for a cascade connection of optimal (resp. optimal minimal) passive scattering system to be an optimal (resp. optimal minimal) passive scattering system. In Section 4 we will show that, although an optimal

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ISSN 0232-2064 / $ 2.50

passive scattering system is not uniquely determined by a given transfer function, two optimal passive scattering systems having the same transfer function will be partially equivalent.

2. Notations and some results

2.1 In this paper, we study a linear dynamic system

$$\alpha = (X, U, V, A, B, C, D)$$

modeled by the equations

$$\left.\begin{aligned} x_{n+1} &= Ax_n + Bu_n \\ v_n &= Cx_n + Du_n \end{aligned}\right\}$$

where x_n, u_n, v_n belong to separable Hilbert spaces X, U, V, respectively, and

$$\begin{aligned} A &: X \to X \\ B &: U \to X \\ C &: X \to V \\ D &: U \to V \end{aligned}$$

are linear bounded operators. The operator function of a complex variable

$$\theta(z) = D + zC(I - zA)^{-1}B$$

is called the *transfer function* of the system α. The subspaces

$$\begin{aligned} X_\alpha^C &= \vee_0^\infty A^k BU \\ X_\alpha^0 &= \vee_0^\infty A^{*k} C^* V \end{aligned}$$

are called *controllable* and *observable subspaces* of α, respectively. The system α is said to be

- *controllable*, if $X = X_\alpha^C$
- *observable*, if $X = X_\alpha^0$
- *simple*, if $X = X_\alpha^C \vee X_\alpha^0$
- *minimal*, if $X = X_\alpha^C = X_\alpha^0$.

By Arov [2] the linear system α is called a *passive scattering system*, if the operator

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \oplus U \to X \oplus V$$

is contractive. If this operator is unitary, then the system is called *unitary* [4]. The transfer function of a passive scattering system belongs to the class $\mathcal{B}(U,V)$ of all analytic functions in the unit disk $\mathcal{D}=\{z\in\mathbb{C}:\,|z|<1\}$, where values are contractive operators from U to V [2].

In [2], for a passive scattering system α there exist operators

$$M:\,X\to W$$
$$N:\,U\to W$$
$$Q:\,W_*\to X$$
$$R:\,W_*\to V$$

such that

$$\left.\begin{aligned} I-A^*A-C^*C&=M^*M\\ -A^*B-C^*D&=M^*N\\ I-B^*B-D^*D&=N^*N\\ I-AA^*-BB^*&=QQ^*\\ -AC^*-BD^*&=QR^*\\ I-CC^*-DD^*&=RR^*\end{aligned}\right\}.$$

For

$$\psi(z)=N+zM(I-zA)^{-1}B \qquad K(z)=(I-zA)^{-1}B$$
$$\psi_*(z)=R+zC(I-zA)^{-1}Q \qquad H(z)=C(I-zA)^{-1}$$

we have the equalities

$$I-\theta(z)^*\theta(z')=\psi(z)^*\psi(z')+(1-\bar{z}z')K(z)^*K(z') \tag{2.1}$$
$$I-\theta(z)\theta(z')^*=\psi_*(z)\psi_*(z')^*+(1-z\bar{z}')H(z)H(z')^* \tag{2.2}$$

where z,z' belong to the unit disk $\mathcal{D}$.

2.2. It is well known that a simple unitary system is determined on the basis of the transfer function up to unitary similarity, and for any $\theta(z)\in\mathcal{B}(U,V)$ there exists a simple unitary system having $\theta(z)$ as the transfer function [5].

In this paper, we shall use the following functional model type of Sz.Nagy and Foias for a simple unitary system α [5], constructed by the given transfer function $\theta(z)\in\mathcal{B}(U,V)$:

$$\begin{aligned} X&=\big[H^2(V)\oplus\overline{\Delta L_2(U)}\big]\ominus\big\{\theta\omega\oplus\Delta\omega:\,\omega\in H^2(U)\big\}\\ A\big(f(t),g(t)\big)&=\big(e^{-it}(f(t)-f(0)),e^{-it}g(t)\big)\\ Bu&=\big(e^{-it}(\theta(e^{it})-\theta(0))u,e^{-it}\Delta(e^{it})u\big)\\ C\big(f(t),g(t)\big)&=f(0)\\ Du&=\theta(0)u\end{aligned}$$

where $\Delta(e^{it}) = \big(I - \theta(e^{it})^*\theta(e^{it})\big)^{\frac{1}{2}}$ and $H^2(U)$ stands for the Hardy space with values in U. For this model, the controllable subspace has the form

$$X_\alpha^C = \vee\left\{\left(\frac{\theta(e^{it}) - \theta(z)}{e^{it} - z}u, \frac{\Delta(e^{it})}{e^{it} - z}u\right) \,\middle|\, z \in \mathcal{D} \text{ and } u \in U\right\}. \tag{2.3}$$

2.3 Let

$$\alpha_k = (X_k, U_k, V_k, A_k, B_k, C_k, D_k) \qquad (k = 1, 2)$$

be two linear systems satisfying $V_1 = U_2$. Then the system

$$\alpha = \Big(X_1 \oplus X_2, U_1, V_2, A_1P_1 + A_2P_2 + B_2C_1P_1, B_1 + B_2D_1, D_2C_1P_1 + C_2P_2, D_2D_1\Big)$$

where P_k is the orthoprojection from $X = X_1 \oplus X_2$ onto X_k $(k = 1, 2)$ is called a *cascade connection* of α_1 and α_2 and is written as $\alpha = \alpha_2\alpha_1$. We have the following result [5]: if $\alpha = \alpha_2\alpha_1$, then $\theta_\alpha(z) = \theta_{\alpha_2}(z)\theta_{\alpha_1}(z)$, and if α_1 and α_2 are passive or unitary, then $\alpha = \alpha_2\alpha_1$ is passive or unitary, respectively.

If α_1 and α_2 are simple unitary systems constructed by the model of Sz.Nagy and Foias and $\alpha = \alpha_2\alpha_1$, then

$$\begin{aligned} X_\alpha^C = \vee\Bigg\{&\left(\frac{\theta_1(e^{it}) - \theta_1(z)}{e^{it} - z}u, \frac{\Delta_1(e^{it})}{e^{it} - z}u\right) \\ &\oplus \left(\frac{\theta_2(e^{it}) - \theta_2(z)}{e^{it} - z}\theta_1(z)u, \frac{\Delta_2(e^{it})}{e^{it} - z}\theta_1(z)u\right) \,\Bigg|\, z \in \mathcal{D} \text{ and } u \in U\Bigg\}. \end{aligned} \tag{2.4}$$

2.4 Let $\theta(z) : U \to V$ be a contractive operator function analytic on the unit disk $\mathcal{D}$. According to [11], there exists an outer function $\varphi(z)$ on $\mathcal{D}$, whose values are operators from U to an auxiliary space F such that

$$\varphi(e^{it})^*\varphi(e^{it}) \le I - \theta(e^{it})^*\theta(e^{it}) \qquad \text{a.e.,}$$

and if $\phi(z)$ is an analytic contractive operator function such that $\phi(e^{it})^*\phi(e^{it}) \le I - \theta(e^{it})^*\theta(e^{it})$ a.e., then $\phi(e^{it})^*\phi(e^{it}) \le \varphi(e^{it})^*\varphi(e^{it})$ a.e. The function $\varphi(z)$ is unique up to a constant unitary factor on the left and is called the *best minorant outer function* of $I - \theta(z)^*\theta(z)$.

3. Optimality and *-optimality for a cascade connection

A passive scattering system $\alpha = (X, U, V, A, B, C, D)$ is said to be *optimal* [3] if for any passive scattering system $\alpha' = (X', U, V, A', B', C', D')$ with the same transfer function we have

$$\left\| \sum_{k=0}^{n} A^k B u_k \right\| \le \left\| \sum_{k=0}^{n} {A'}^k B' u_k \right\| \qquad (n \in \mathbb{N}, u_k \in U).$$

Further, a passive scattering system α is said to be **-optimal* if for any passive scattering system α' with $\theta_{\alpha'}(z) = \theta_\alpha(z)$ we have

$$\left\| \sum_{k=0}^{n} A^{*k} C^* v_k \right\| \le \left\| \sum_{k=0}^{n} {A'}^{*k} {C'}^* v_k \right\| \qquad (n \in \mathbb{N}, v_k \in V).$$

Obviously, if a system $\alpha = (X, U, V, A, B, C, D)$ is optimal, then the dual system $\alpha_* = (X, V, U, A^*, C^*, B^*, D^*)$ is *-optimal.

In [2, 3] it is proved that an optimal minimal passive scattering system is determined on the basis of the transfer function up to unitary similarity, and for any $\theta(z) \in \mathcal{B}(U, V)$ there exists an optimal minimal passive scattering system having $\theta(z)$ as transfer function.

Let $\bar\theta(z) = \binom{\theta(z)}{\varphi(z)} : U \to V \oplus F$, where $\varphi(z)$ is the best minorant outer function of $I - \theta(z)^*\theta(z)$. Of course, $\bar\theta(z)$ is contractive since $\varphi(z)^*\varphi(z) \le I - \theta(z)^*\theta(z)$. Let

$$\bar\alpha = (X, U, V \oplus F, \bar A, \bar B, \bar C, \bar D)$$

be the simple unitary system having $\bar\theta(z)$ as transfer function and denote

$$\mathring{A} = \bar A, \quad \mathring{B} = \bar B, \quad \mathring{C} = P_V \bar C, \quad \mathring{D} = P_V \bar D \tag{3.1}$$

where P_V is the orthoprojection from $V \oplus F$ onto V. It is proved [1] that the system

$$\mathring{\alpha} = (X, U, V, \mathring{A}, \mathring{B}, \mathring{C}, \mathring{D})$$

is an optimal passive scattering system with transfer function $\theta_{\mathring{\alpha}}(z) = \theta(z)$ and

$$\mathring{N} = P_F \bar D, \quad \mathring{M} = P_F \bar C, \quad \psi_{\mathring{\alpha}}(z) = \mathring{N} + z\mathring{M}(I - z\mathring{A})^{-1}\mathring{B} = \varphi(z). \tag{3.2}$$

Theorem 3.1. *Let $\alpha_k = (X_k, U_k, V_k, A_k, B_k, C_k, D_k)$ be an optimal passive scattering systems having $\varphi_k(z) : U_k \to F_k$ as best minorant outer functions corresponding to transfer functions $\theta_k(z)$ $(k = 1, 2)$ and satisfying $U_2 = V_1$. Then the passive scattering system $\alpha = \alpha_2\alpha_1$ is optimal if* *and only if* $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ *is the best minorant outer function corresponding to* $\theta_2(z)\theta_1(z)$.

Proof. Let $\mathring{\alpha}_1$ and $\mathring{\alpha}_2$ be optimal passive scattering systems having $\theta_2(z)$ and $\theta_1(z)$ as transfer functions, respectively, constructed as in (3.1). Putting $\theta(z) = \theta_2(z)\theta_1(z)$ we have that $\theta(z)$ is the transfer function of $\alpha = \alpha_2\alpha_1$.

Since $\alpha = \alpha_2\alpha_1$, we have for every $z \in \mathcal{D}$ and $u \in U$

$$\left\|(I - zA)^{-1}Bu\right\|^2 = \left\|(I - zA_1)^{-1}B_1u\right\|^2 + \left\|(I - zA_2)^{-1}B_2\theta_1(z)u\right\|^2.$$

Let $\mathring{\alpha}$ be an optimal passive scattering system having $\theta(z)$ as transfer function and let $\varphi(z)$ be the best minorant outer function corresponding to $\theta(z)$. Since α and α_1, α_2 are optimal, from (2.1) and (3.2) we deduce the equalities

$$\begin{aligned}
\left\|(I - z\mathring{A})^{-1}\mathring{B}u\right\|^2 &= \left\|(-z\mathring{A}_1)^{-1}\mathring{B}_1u\right\|^2 + \left\|(I - z\mathring{A}_2)^{-1}\mathring{B}_2\theta_1(z)u\right\|^2 \\
\big\langle\big(I - \theta(z)^*\theta(z) - \varphi(z)^*\varphi(z)\big)u, u\big\rangle &= \big\langle\big(I - \theta_1(z)^*\theta_1(z) - \varphi_1(z)^*\varphi_1(z)\big)u, u\big\rangle + \\
&\quad \big\langle\big(I - \theta_2(z)^*\theta_2(z) - \varphi_2(z)^*\varphi_2(z)\big)\theta_1(z)u, \theta_1(z)u\big\rangle \\
&= \Big\langle\Big(I - \theta(z)^*\theta(z) - \varphi_1(z)^*\varphi_1(z) \\
&\quad - \theta_1(z)^*\varphi_2(z)^*\varphi_2(z)\theta_1(z)\Big)u, u\Big\rangle.
\end{aligned}$$

This implies

$$\Big\langle\varphi(z)^*\varphi(z) - \varphi_1(z)^*\varphi_1(z) - \theta_1(z)^*\varphi_2(z)^*\varphi_2(z)\theta_1(z)u, u\Big\rangle = 0.$$

Therefore

$$\varphi(z)^*\varphi(z) = \begin{pmatrix}\varphi_2(z)\theta_1(z)\\ \varphi_1(z)\end{pmatrix}^* \begin{pmatrix}\varphi_2(z)\theta_1(z)\\ \varphi_1(z)\end{pmatrix}. \tag{3.3}$$

Besides, as was proved [8], there exists an operator function μ such that $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)} \oplus \mu(z)$ is the best minorant outer function of $I - \theta(z)^*\theta(z)$. Then from (3.3) it follows that $\mu(z) = 0$ and thus $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ is the best minorant outer function of $I - \theta(z)^*\theta(z)$.

Conversely, let $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ be the best minorant outer function of $I - \theta(z)^*\theta(z)$. Put $\varphi(z) = \binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ and let $\mathring{\alpha}$ be the optimal passive scattering system constructed by (3.1), corresponding to the best minorant outer function $\varphi(z)$. We have for all $n \in \mathbb{N}$, $z_k \in \mathcal{D}$ and $u_k \in U$

$$\begin{aligned}
&\left\|\sum_{k=1}^{n}(I - z_k \mathring{A})^{-1}\mathring{B}u_k\right\|^2 \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle K_{\mathring{\alpha}}(z_j)^* K_{\mathring{\alpha}}(z_i)u_i, u_j\right\rangle \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\bar{z}_j z_i)^{-1}\left\langle \big(I - \theta(z_j)^*\theta(z_i) - \varphi(z_j)^*\varphi(z_i)\big)u_i, u_j\right\rangle \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\bar{z}_j z_i)^{-1}\Big\langle \Big(I - \theta(z_j)^*\theta(z_i) - \varphi_1(z_j)^*\varphi_1(z_i) \\
&\qquad - \theta_1(z_j)^*\varphi_2(z_j)^*\varphi_2(z_i)\theta_1(z_i)\Big)u_i, u_j\Big\rangle \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\bar{z}_j z_i)^{-1}\Big\langle \big(I - \theta_1(z_j)^*\theta_1(z_i) - \varphi_1(z_j)^*\varphi_1(z_i)\big)u_i, u_j\Big\rangle \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n}(1-\bar{z}_j z_i)^{-1}\Big\langle \theta_1(z_j)^*\big(I - \theta_2(z_j)^*\theta_2(z_i) - \varphi_2(z_j)^*\varphi_2(z_i)\big)\theta_1(z_i)u_i, u_j\Big\rangle \\
&\quad= \sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle K_{\mathring{\alpha}_1}(z_j)^* K_{\mathring{\alpha}_1}(z_i)u_i, u_j\right\rangle \\
&\qquad + \sum_{i=1}^{n}\sum_{j=1}^{n}\left\langle \theta_1(z_j)^* K_{\mathring{\alpha}_2}(z_j)^* K_{\mathring{\alpha}_2}(z_i)\theta_1(z_i)u_i, u_j\right\rangle \\
&\quad= \left\|\sum_{k=1}^{n}(I - z_k \mathring{A}_1)^{-1}\mathring{B}_1 u_k\right\|^2 + \left\|\sum_{k=1}^{n}(I - z_k \mathring{A}_2)^{-1}\mathring{B}_2\theta_1(z_k)u_k\right\|^2 \\
&\quad= \left\|\sum_{k=1}^{n}(I - z_k A_1)^{-1}B_1 u_k\right\|^2 + \left\|\sum_{k=1}^{n}(I - z_k A_2)^{-1}B_2\theta_1(z_k)u_k\right\|^2 \\
&\quad= \left\|\sum_{k=1}^{n}(I - z_k A)^{-1}B u_k\right\|^2.
\end{aligned}$$

From the last equality we have that α is optimal and the proof is complete ∎

By duality, we introduced an analogue notion for the *-best minorant outer function corresponding to $\theta(z) : U \to V$. The *-outer function $\varphi_*(z) \in \mathcal{B}(F', V)$ is called *-*best minorant outer function* of $I - \theta(z)\theta(z)^*$ if

$$\varphi_*(e^{it})\varphi_*(e^{it})^* \leq I - \theta(e^{it})\theta(e^{it})^* \qquad \text{a.e.}$$

and if $\phi_*(z)$ is an analytic contractive operator function such that a.e. the implication

$$\phi_*(e^{it})\phi_*(e^{it})^* \leq I - \theta(e^{it})\theta(e^{it})^* \quad \Longrightarrow \quad \phi_*(e^{it})\phi_*(e^{it})^* \leq \varphi_*(e^{it})\varphi_*(e^{it})^*$$

holds. It is easy to see that the function $\varphi_*(z)$ is *-outer if the function $\tilde{\varphi}_*(z) \in \mathcal{B}(V, F')$ is outer. Obviously, $\varphi_*(z)$ is the *-best minorant outer function of $I - \theta(z)\theta(z)^*$ if and only if $\tilde{\varphi}_*(z)$ is the best minorant outer function of $I - \tilde{\theta}(z)^*\tilde{\theta}(z)$ where $\tilde{\varphi}_*(z) = \varphi_*(\bar{z})^*$ and $\tilde{\theta}(z) = \theta(\bar{z})^*$.

Let

$$\bar{\theta}_*(z) = (\theta(z), \varphi_*(z)) : U \oplus F' \to V$$

where $\varphi_*(z)$ is the *-best minorant outer function of $I - \theta(z)\theta(z)^*$ and $\mathring{\alpha}_*$ is the optimal passive scattering system with the corresponding operator function

$$\tilde{\bar{\theta}}_*(z) = \begin{pmatrix} \tilde{\theta}(z) \\ \tilde{\varphi}_*(z) \end{pmatrix} : V \to U \oplus F'.$$

By applying the proof of Theorem 3.1 to the dual systems

$$\begin{aligned} \alpha_* &= (X, V, U, A^*, C^*, B^*, D^*) \\ \alpha_{k*} &= (X_k, V_k, U_k, A_k^*, C_k^*, B_k^*, D_k^*) \quad (k = 1, 2) \end{aligned}$$

and by using the model optimal passive scattering $\mathring{\alpha}_*, \mathring{\alpha}_{1*}, \mathring{\alpha}_{2*}$ constructed as in (3.1) corresponding to systems $\alpha_*, \alpha_{1*}, \alpha_{2*}$ and equality (2.2), we obtain the following duality of Theorem 3.1:

Theorem 3.2. *Let $\alpha_k = (X_k, U_k, V_k, A_k, C_k, B_k, D_k)$ be *-optimal passive scattering systems having $\varphi_{*k}(z)$ as *-best minorant outer functions of $I - \theta_k(z)\theta_k(z)^*$, where $\theta_k(z) = \theta_{\alpha k}(z)$ $(k = 1, 2)$ and $U_2 = V_1$. Then the passive scattering system $\alpha = \alpha_2\alpha_1$ is *-optimal if and only if $\left(\begin{smallmatrix} \varphi_{*2}(z) \\ \theta_2(z)\varphi_{*1}(z) \end{smallmatrix}\right)$ is the *-best minorant outer function of $I - \theta(z)\theta(z)^*$ $(\theta(z) = \theta_\alpha(z))$.*

From the optimal passive scattering system defined by (3.1) one constructs the optimal minimal passive scattering system $\alpha_0 = (X_0, U, V, A_0, B_0, C_0, D_0)$ as follows [1]:

$$X_0 = X^C_{\mathring{\alpha}}(X^C_{\bar{\alpha}}), \quad A_0 = \mathring{A}|_{X_0}, \quad B_0 = \mathring{B}, \quad C_0 = \mathring{C}|_{X_0}, \quad D_0 = \mathring{D}. \qquad (3.4)$$

According to the functional model of Sz.Nagy and Foias for a simple unitary system with transfer function $\bar{\theta}(z) = \left(\begin{smallmatrix} \theta(z) \\ \varphi(z) \end{smallmatrix}\right)$ and (2.3) and by some simple computations we have the following explicit description of the system α_0:

$$X_0 = X_{\bar{\alpha}}^C = \vee\left\{\left(\frac{\theta(e^{it})-\theta(z)}{e^{it}-z}u, \frac{\varphi(e^{it})-\varphi(z)}{e^{it}-z}u, \frac{\bar{\Delta}(e^{it})}{e^{it}-z}u,\right)\,\middle|\, z\in\mathcal{D}, u\in U\right\}$$

$$\text{where } \bar{\Delta}(e^{it}) = \big(I-\bar{\theta}(e^{it})^*\bar{\theta}(e^{it})\big)^{\frac{1}{2}} = \big(I-\theta(e^{it})^*\theta(e^{it})-\varphi(e^{it})^*\varphi(e^{it})\big)^{\frac{1}{2}}$$

$$A_0\big(f(t),g(t),h(t)\big) = \big(e^{-it}(f(t)-f(0)), e^{-it}(g(t)-g(0)), e^{-it}h(t)\big) \tag{3.5}$$

$$B_0u = \big(e^{-it}(\theta(e^{it})-\theta(0))u, e^{-it}(\varphi(e^{it})-\varphi(0))u, e^{-it}\bar{\Delta}(e^{it})u\big)$$

$$C_0(f(t),g(t),h(t)) = f(0)$$

$$D_0u = \theta(0)u.$$

Lemma 3.1. *Let*

$$\alpha_{10} = (X_{10}, U_1, V_1, A_{10}, B_{10}, C_{10}, D_{10})$$
$$\alpha_{20} = (X_{20}, U_2, V_2, A_{20}, B_{20}, C_{20}, D_{20})$$

with $U_2 = V_1$ be optimal minimal passive scattering systems modeled as (3.5), *having respectively $\theta_1(z)$ and $\theta_2(z)$ as transfer functions, and let $\alpha = \alpha_{20}\alpha_{10}$. Then*

$$X_\alpha^C = \vee\left\{\left(\frac{\theta_1(e^{it})-\theta_1(z)}{e^{it}-z}u, \frac{\varphi_1(e^{it})-\varphi_1(z)}{e^{it}-z}u, \frac{\bar{\Delta}_1(e^{it})}{e^{it}-z}u,\right)\oplus \left(\frac{\theta_2(e^{it})-\theta_2(z)}{e^{it}-z}\theta_1(z)u, \frac{\varphi_2(e^{it})-\varphi_2(z)}{e^{it}-z}\theta_1(z)u, \frac{\bar{\Delta}_2(e^{it})}{e^{it}-z}\theta_1(z)u,\right)\,\middle|\, z\in\mathcal{D}, u\in U\right\}.$$

Proof. By the definition of the controllable subspace, we have

$$X_\alpha^C = \vee\big\{(I-zA)^{-1}Bu|\, z\in\mathcal{D} \text{ and } u\in U\big\}.$$

Put

$$(I-zA)^{-1}Bu = (f_1,g_1,h_1)\oplus(f_2,g_2,h_2) \qquad \text{with } \begin{cases}(f_1,g_1,h_1)\in X_{10}\\(f_2,g_2,h_2)\in X_{20}.\end{cases}$$

It follows that

$$\begin{aligned}Bu &= (I-zA)\big((f_1,g_1,h_1)\oplus(f_2,g_2,h_2)\big)\\&= \big(I-z(A_{10}P_1+A_{20}P_2+B_{20}C_{10}P_1)\big)\big((f_1,g_1,h_1)\oplus(f_2,g_2,h_2)\big)\\&= \left((e^{it}-z)\frac{f_1(e^{it})}{e^{it}}+z\frac{f_1(0)}{e^{it}}, (e^{it}-z)\frac{g_1(e^{it})}{e^{it}}+z\frac{g_1(0)}{e^{it}}, (e^{it}-z)\frac{h_1(e^{it})}{e^{it}}\right)\\&\quad\oplus\left((e^{it}-z)\frac{f_2(e^{it})}{e^{it}}+z\frac{f_2(0)}{e^{it}}-z\frac{\theta_2(e^{it})-\theta_2(0)}{e^{it}}f_1(0), (e^{it}-z)\frac{g_2(e^{it})}{e^{it}}\right.\\&\quad\left.+z\frac{g_2(0)}{e^{it}}-z\frac{\varphi_2(e^{it})-\varphi_2(0)}{e^{it}}f_1(0), (e^{it}-z)\frac{h_2(e^{it})}{e^{it}}-z\frac{\bar{\Delta}_2(e^{it})}{e^{it}}f_1(0)\right).\end{aligned} \tag{3.6}$$

On the other hand we have

$$\begin{aligned} Bu &= B_{10}u + B_{20}D_{10}u \\ &= \Big(e^{-it}(\theta_1(e^{it}) - \theta_1(0))u, e^{-it}(\varphi_1(e^{it}) - \varphi_1(0))u, e^{-it}\bar{\Delta}_1(e^{it})u\Big) \oplus \\ &\quad \Big(e^{-it}(\theta_2(e^{it}) - \theta_2(0))\theta_1(0)u, e^{-it}(\varphi_2(e^{it}) - \varphi_2(0))\theta_1(0)u, e^{-it}\bar{\Delta}_2(e^{it})\theta_1(0)u\Big). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) we have

$$(e^{it} - z)f_1(e^{it}) + zf_1(0) = \big(\theta_1(e^{it}) - \theta_1(0)\big)u. \tag{3.8}$$

This equality may be rewritten as

$$(\xi - z)f_1(\xi) + zf_1(0) = \big(\theta_1(\xi) - \theta_1(0)\big)u \qquad (\xi \in \mathcal{D}).$$

By choosing $\xi = z$, we obtain $zf_1(0) = (\theta_1(z) - \theta_1(0))u$. Substituting this expression of $zf_1(0)$ into (3.8) we get

$$f_1(e^{it}) = \frac{\theta_1(e^{it}) - \theta_1(z)}{e^{it} - z}u.$$

Similarly, by simple computations we also obtain

$$\begin{aligned} g_1(e^{it}) &= \frac{\varphi_1(e^{it}) - \varphi_1(z)}{e^{it} - z}u \\ h_1(e^{it}) &= \frac{\bar{\Delta}_1(e^{it})}{e^{it} - z}u \\ f_2(e^{it}) &= \frac{\theta_2(e^{it}) - \theta_2(z)}{e^{it} - z}\theta_1(z)u \\ g_2(e^{it}) &= \frac{\varphi_2(e^{it}) - \varphi_2(z)}{e^{it} - z}\theta_1(z)u \\ h_2(e^{it}) &= \frac{\bar{\Delta}_2(e^{it})}{e^{it} - z}\theta_1(z)u. \end{aligned}$$

Thus X_α^C has the desired form and the proof is complete ∎

In the rest of this section we will study conditions for conserving the optimal minimal property of a cascade connection of optimal minimal passive scattering systems. It is easy to see that if an optimal passive scattering system is controllable, then it is minimal. So we first consider the conditions for the connection of optimal controllable passive scattering system to be controllable.

In [6] the notion of (-)regular factorization was introduced. Let $\theta_k(z) \in \mathcal{B}(U_k, V_k)$ $(k = 1, 2)$ with $U_2 = V_1$. The factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is said to be (-)*regular* if the operator

$$\Gamma^- : \Delta_* h \to \Delta_{1*}\theta_2^* h \oplus \Delta_{2*} h \qquad (h \in L_2^-(V_2))$$

can be continuously extended to a unitary operator from

$$\overline{\Delta_* L_2^-(V_2)} \quad \text{onto} \quad \overline{\Delta_{1*} L_2^-(V_1)} \oplus \overline{\Delta_{2*} L_2^-(V_2)}$$

where

$$\Delta_{k*}(e^{it}) = \big(I - \theta_k(e^{it})\theta_k(e^{it})^*\big)^{\frac{1}{2}}.$$

It was proved that if α_1 and α_1 are controllable simple unitary systems having respectively $\theta_1(z)$ and $\theta_2(z)$ as transfer functions, then the system $\alpha = \alpha_2\alpha_1$ is controllable if and only if the factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is (-)regular (see [6]).

Let

$$\theta_k(z) : U_k \to V_k \qquad \text{and} \qquad \varphi_k(z) : U_k \to F_k$$

be best minorant outer functions of

$$I - \theta_k(z)^*\theta_k(z), \qquad \bar{\theta}_k(z) = \begin{pmatrix} \theta_k(z) \\ \varphi_k(z) \end{pmatrix} : U_k \to V_k \oplus F_k$$

and α_{k0} be the optimal minimal passive scattering system constructed as (3.4) $(k = 1, 2)$ with $U_2 = V_1$. Denote

$$\begin{aligned} \eta_1(z) &= \bar{\theta}_1(z) : U_1 \to V_1 \oplus F_1 \\ \eta_2(z) &= \begin{pmatrix} \bar{\theta}_2(z) & 0 \\ 0 & 1 \end{pmatrix} : V_1 \oplus F_1 \to V_2 \oplus F_2 \oplus F_1. \end{aligned}$$

Obviously, $\eta_2(z)$ is contractive. Let β_1 and β_2 be the functional model of Sz.Nagy and Foias for the simple unitary systems having respectively $\eta_1(z)$ and $\eta_2(z)$ as transfer functions. Putting

$$\begin{aligned} \nabla_1(e^{it}) &= \big(I - \eta_1(e^{it})^*\eta_1(e^{it})\big)^{\frac{1}{2}} \\ \nabla_2(e^{it}) &= \big(I - \eta_2(e^{it})^*\eta_2(e^{it})\big)^{\frac{1}{2}} \end{aligned}$$

we have the following results:

$$X^C_{\beta_1} = X^C_{\bar{\alpha}_1} = X_{\alpha_{10}}.$$

$$\nabla_2^2(e^{it}) = I - \eta_2(e^{it})^*\eta_2(e^{it}) = \begin{pmatrix} \bar{\Delta}_2^2(e^{it}) & 0 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} X_{\beta_2} &= \Big[H^2(V_2 \oplus F_2) \oplus H^2(F_1) \oplus \overline{\nabla_2 L_2(V_1 \oplus F_1)}\Big] \\ &\quad \ominus \left\{ \left(\eta_2\begin{pmatrix} h \\ h' \end{pmatrix} \oplus \nabla_2\begin{pmatrix} h \\ h' \end{pmatrix}\right) \middle| \, h \in H^2(V_1) \text{ and } h' \in H^2(F_1) \right\} \\ &= \Big[H^2(V_2 \oplus F_2) \oplus H^2(F_1) \oplus \overline{\bar{\Delta}_2 L_2(V_1)} \oplus \{0\}\Big] \\ &\quad \ominus \left\{ \bar{\theta}_2 h \oplus h' \oplus \bar{\Delta}_2 h \oplus 0 \, \middle| \, h \in H^2(V_1) \text{ and } h' \in H^2(F_1) \right\} \\ &= \Big[H^2(V_2 \oplus F_2) \oplus \{0\} \oplus \overline{\bar{\Delta}_2 L_2(V_1)} \oplus \{0\}\Big] \\ &\quad \ominus \left\{ \bar{\theta}_2 h \oplus 0 \oplus \bar{\Delta}_2 h \oplus 0 \, \middle| \, h \in H^2(V_1) \right\}. \end{aligned}$$

So we can identify X_{β_2} with $X_{\bar{\alpha}_2}$ and hence $A_{\beta_2} \equiv \bar{A}_2$.

Besides, for every $(v_1, r_1) \in V_1 \oplus F_1$ we have

$$\begin{aligned} B_{\beta_2}(v_1, r_1) &= \big(e^{-it}(\eta_2(e^{it}) - \eta_2(0))(v_1, r_1)\big) \oplus e^{-it}\nabla_2(e^{it})(v_1, r_1) \\ &= \begin{pmatrix} e^{-it}(\bar{\theta}_2(e^{it}) - \bar{\theta}_2(0))v_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} e^{-it}\bar{\Delta}_2(e^{it})v_1 \\ 0 \end{pmatrix}. \end{aligned}$$

Therefore $B_{\beta_2} \equiv \bar{B}_2$. From the fact that $A_{\beta_2} \equiv \bar{A}_2$ and $B_{\beta_2} \equiv \bar{B}_2$ we have the identity $X^C_{\beta_2} = X^C_{\bar{\alpha}_2} = X_{\alpha_{20}}$.

Let $\beta = \beta_2\beta_1$. From (2.4) we have

$$\begin{aligned} X^C_\beta = \vee \Bigg\{ &\left(\frac{\eta_1(e^{it}) - \eta_1(z)}{e^{it} - z} u, \frac{\nabla_1(e^{it})}{e^{it} - z} u \right) \\ &\oplus \left(\frac{\eta_2(e^{it}) - \eta_2(z)}{e^{it} - z} \eta_1(z) u, \frac{\nabla_2(e^{it})}{e^{it} - z} \eta_1(z) u \right) \Bigg| \, z \in \mathcal{D} \text{ and } u \in U \Bigg\}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{\eta_1(e^{it}) - \eta_1(z)}{e^{it} - z} u &= \left(\frac{\theta_1(e^{it}) - \theta_1(z)}{e^{it} - z} u, \frac{\varphi_1(e^{it}) - \varphi_1(z)}{e^{it} - z} u \right) \\ \frac{\nabla_1(e^{it})}{e^{it} - z} u &= \frac{\bar{\Delta}_1(e^{it})}{e^{it} - z} u \\ \frac{\eta_2(e^{it}) - \eta_2(z)}{e^{it} - z} \eta_1(z) u &= \left(\tfrac{\bar{\theta}_2(e^{it}) - \bar{\theta}_2(z)}{e^{it} - z} \theta_1(z) u \ 0 \right) \\ &= \left(\frac{\theta_2(e^{it}) - \theta_2(z)}{e^{it} - z} \theta_1(z) u, \frac{\varphi_2(e^{it}) - \varphi_2(z)}{e^{it} - z} \theta_1(z) u, 0 \right) \\ \frac{\nabla_2(e^{it})}{e^{it} - z} \eta_1(z) u &= \left(\frac{\bar{\Delta}_2(e^{it})}{e^{it} - z} \theta_1(z) u, 0 \right). \end{aligned}$$

Let α_{10} and α_{20} be two optimal minimal passive scattering systems with transfer functions $\theta_1(z)$ and $\theta_2(z)$, respectively, and let $\alpha = \alpha_{20}\alpha_{10}$. From the results above and Lemma 3.1 we have the identity $X^C_\beta \equiv X^C_\alpha$. Denote

$$\left.\begin{aligned} \bar{\Delta}_{k*}(e^{it}) &= (I - \bar{\theta}_k(e^{it})\bar{\theta}_k(e^{it})^*)^{\frac{1}{2}} \\ \nabla_*(e^{it}) &= (I - \eta(e^{it})\eta(e^{it})^*)^{\frac{1}{2}} \end{aligned}\right\} \tag{3.9}$$

where $\eta(z) = \eta_2(z)\eta_1(z)$. We have the following result.

Theorem 3.3. *Let $\alpha_k = (X_k, U_k, V_k, A_k, B_k, C_k, D_k)$ be optimal controllable passive scattering systems having $\varphi_k(z) : U_k \to F_k$ as best minorant outer functions corresponding to transfer functions $\theta_k(z)$ $(k = 1, 2)$ and satisfying $U_2 = V_1$. Then the system $\alpha = \alpha_2\alpha_1$ is controllable if and only if the operator*

$$\begin{aligned} \Gamma^- : \overline{\nabla_* L_2^-(V_2 \oplus F_2 \oplus F_1)} &\to \overline{\bar{\Delta}_{1*} L_2^-(V_1 \oplus F_1)} \oplus \overline{\bar{\Delta}_{2*} L_2^-(V_2 \oplus F_2)} \\ \nabla_*(h, h') &\mapsto \bar{\Delta}_{1*}(\bar{\theta}_2^* h, h') \oplus \bar{\Delta}_{2*} h \\ &\quad \big(h \in L_2^-(V_2 \oplus F_2), h' \in L_2^-(F_1)\big) \end{aligned}$$

can be continuously extended to a unitary operator.

Proof. Since an optimal controllable passive scattering system is minimal, so we can assume that $\alpha_1 = \alpha_{10}$ and $\alpha_2 = \alpha_{20}$, where α_{10} and α_{20} are modeled as (3.4). Then we have

$$X_\alpha = X_{\alpha_{10}} \oplus X_{\alpha_{20}} = X^C_{\bar{\alpha}_1} \oplus X^C_{\bar{\alpha}_2}.$$

The system α is controllable if and only if $X_\alpha = X^C_\alpha = X^C_{\bar{\alpha}_1} \oplus X^C_{\bar{\alpha}_2}$. This condition is equivalent to $X^C_\beta = X^C_{\beta_1} \oplus X^C_{\beta_2}$. Since β_1 and β_2 are simple unitary systems, this equality occurs if and only if the factorization $\eta(z) = \eta_2(z)\eta_1(z)$ is (-)regular, i.e. the operator

$$\begin{aligned} \Gamma^- : \overline{\nabla_* L_2^-(V_2 \oplus F_2 \oplus F_1)} &\to \overline{\nabla_{1*} L_2^-(V_1 \oplus F_1)} \oplus \overline{\nabla_{2*} L_2^-(V_2 \oplus F_2 \oplus F_1)} \\ \nabla_*(h, h') &\mapsto \nabla_{1*}\eta_2^*(h, h') \oplus \nabla_{2*}(h, h') \\ &\quad \big(h \in L_2^-(V_2 \oplus F_2), h' \in L_2^-(F_1)\big) \end{aligned}$$

can be continuously extended to a unitary operator. Note that

$$\overline{\nabla_{1*} L_2^-(V_1 \oplus F_1)} = \overline{\bar{\Delta}_{1*} L_2^-(V_1 \oplus F_1)}.$$

By simple computations we have

$$\overline{\nabla_{2*} L_2^-(V_2 \oplus F_2 \oplus F_1)} = \overline{\bar{\Delta}_{2*} L_2^-(V_2 \oplus F_2)} \oplus \{0\}.$$

From these equalities the theorem is proved ∎

Now we can state a criterion for conserving the optimal minimal property of a cascade connection.

Theorem 3.4. *Let α_k be optimal minimal passive scattering systems having $\varphi_k(z) : U_k \to F_k$ as best minorant outer functions corresponding to its transfer functions $\theta_k(z) : U_k \to V_k$ $(k = 1, 2)$ and satisfying $U_2 = V_1$. In order that $\alpha = \alpha_2\alpha_1$ is an optimal minimal passive scattering system, it is necessary and sufficient that the following two conditions hold:*

a) $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ *is the best minorant outer function corresponding to $\theta(z) = \theta_\alpha(z) = \theta_2(z)\theta_1(z)$.*

b) *The operator*

$$\begin{aligned}
\Gamma^- : \overline{\bar\Delta_* L_2^-(V_2 \oplus F_2 \oplus F_1)} &\to \overline{\bar\Delta_{1*} L_2^-(V_1 \oplus F_1)} \oplus \overline{\bar\Delta_{2*} L_2^-(V_2 \oplus F_2)}\\
\bar\Delta_*(h, h') &\mapsto \bar\Delta_{1*}(\bar\theta_2^* h, h') \oplus \bar\Delta_{2*} h \quad \big(h \in L_2^-(V_2 \oplus F_2), h' \in L_2^-(F_1)\big)\\
\bar\Delta_{k*}(e^{it}) &= \big(I - \bar\theta_k(e^{it})\bar\theta_k(e^{it})^*\big)^{\frac{1}{2}}\\
&= \big(I - \theta_k(e^{it})\theta_k(e^{it})^* - \varphi_k(e^{it})\varphi_k(e^{it})^*\big)^{\frac{1}{2}}\\
\bar\Delta_*(e^{it}) &= \big(I - \theta(e^{it})\theta(e^{it})^* - \varphi(e^{it})\varphi(e^{it})^*\big)^{\frac{1}{2}}
\end{aligned}$$

where $\varphi(z)$ is the best minorant outer function corresponding to $\theta(z)$ can be continuously extended to a unitary operator.

Proof. *Necessary condition.* We can suppose that $\alpha_1 = \alpha_{10}$ and $\alpha_2 = \alpha_{20}$. Since $\alpha = \alpha_2\alpha_1$ is optimal, then by Theorem 3.1 it is necessary that $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ is the best minorant outer function of $I - \theta(z)^*\theta(z)$. In this case we have $\nabla_* = \bar\Delta_*$ with ∇_* defined by (3.9). Indeed, we have

$$\begin{aligned}
\nabla_*^2 &= I - \eta\eta^*\\
&= I - \begin{pmatrix} \bar\theta_2 & 0 \\ 0 & 1 \end{pmatrix} \bar\theta_1 \bar\theta_1^* \begin{pmatrix} \bar\theta_2^* & 0 \\ 0 & 1 \end{pmatrix}\\
&= \begin{pmatrix} I - \theta\theta^* & -\theta\theta_1^*\varphi_2^* & -\theta\varphi_1^* \\ -\varphi_2\theta_1\theta^* & I - \varphi_2\theta_1\theta_1^*\varphi_2^* & -\varphi_2\theta_1\varphi_1^* \\ -\varphi_1\theta^* & -\varphi_1\theta_1^*\varphi_2^* & I - \varphi_1\varphi_1^* \end{pmatrix}\\
&= I - \begin{pmatrix} \theta \\ \varphi_2\theta_1 \\ \varphi_1 \end{pmatrix} (\theta^* \ \theta_1^*\varphi_2^* \ \varphi_1^*)\\
&= \bar\Delta_*^2.
\end{aligned}$$

Since α is controllable, then by Theorem 3.3 and the equality $\nabla_* = \bar\Delta_*$ condition b) holds.

Sufficient condition. Condition a) leads to $\alpha = \alpha_2\alpha_1$ is optimal. Since $\binom{\varphi_2(z)\theta_1(z)}{\varphi_1(z)}$ is the best minorant outer function of $I - \theta(z)^*\theta(z)$, we have

$$\overline{\nabla_* L_2^-(V_2 \oplus F_2 \oplus F_1)} = \overline{\bar{\Delta}_* L_2^-(V_2 \oplus F_2 \oplus F_1)}.$$

Then by condition b) and Theorem 3.3 it follows that α is controllable. Since the passive scattering system α is optimal and controllable it is also minimal and the proof is completed ∎

Corollary 3.1. *Let α_k be optimal minimal passive scattering systems having $\varphi_k(z) = 0$ as best minorant outer functions corresponding to transfer functions $\theta_k(z) : U_k \to V_k \quad (k = 1, 2)$ and let $U_2 = V_1$. Then the system $\alpha = \alpha_2\alpha_1$ is optimal minimal if and only if the following two conditions hold:*

a) *$\varphi(z) = 0$ is the best minorant outer function corresponding to $\theta(z)$ $(= \theta_\alpha(z))$.*

b) *The factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is $(-)$regular.*

Proof. This result is evident since from $\varphi_k(z) = 0$ there follows that $F_k = \{0\}$, $\bar{\Delta}_{k*} = (I - \theta_k\theta_k^*)^{\frac{1}{2}} = \Delta_{k*}$ and $\bar{\Delta}_* = (I - \theta\theta^*)^{\frac{1}{2}} = \Delta_*$. Then condition b) coincides with the (-)regular notion of the factorization $\theta(z) = \theta_2(z)\theta_1(z)$ ∎

By duality, the notion of (+)regular factorization [6] was also introduced. The factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is said to be (+)*regular* if the operator

$$\Gamma^+ : \Delta h \to \Delta_1 h \oplus \Delta_2\theta_1 h \qquad (h \in H^2(U))$$

can be continuously extended to a unitary operator from $\overline{\Delta H^2(U)}$ onto $\overline{\Delta_1 H^2(U_1)} \oplus \overline{\Delta_2 H^2(U_2)}$. It is easy to see that the factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is (-)regular if and only if the factorization $\tilde{\theta}(z) = \tilde{\theta}_1(z)\tilde{\theta}_2(z)$ is (+)regular where $\tilde{\theta}(z) = \theta(\bar{z})^*$ and $\tilde{\theta}_k(z) = \theta_k(\bar{z})^* \ (k = 1, 2)$. Applying this result and Theorem 3.4 to the dual systems α_{1*}, α_{2*} and α_*, we have the following result.

Theorem 3.5. *Let α_k be *-optimal minimal passive scattering systems having $\varphi_{*k}(z) : F_k' \to V_k$ as *-best minorant outer functions corresponding to transfer functions $\theta_k(z) : U_k \to V_k \quad (k = 1, 2)$ and let $U_2 = V_1$. Then the system $\alpha = \alpha_2\alpha_1$ is *-optimal minimal if and only if the following two conditions hold:*

a) *$\binom{\varphi_{*2}(z)}{\theta_2(z)\varphi_{*1}(z)}$ is the *-best minorant outer function corresponding to $\theta(z) = \theta_\alpha(z) = \theta_2(z)\theta_1(z)$.*

b) *The operator*

$$\Gamma^+ : \overline{\barwedge H^2(U_1 \oplus F_1' \oplus F_2')} \to \overline{\barwedge_1 H^2(U_1 \oplus F_1')} \oplus \overline{\barwedge_2 H^2(U_2 \oplus F_2')}$$
$$\barwedge(h, h') \mapsto \barwedge_1 h \oplus \barwedge_2(\bar{\theta}_{*1}h, h') \quad \big(h \in H^2(U_1 \oplus F_1'), h' \in H^2(F_2')\big)$$

can be continuously extended to a unitary operator where

$$\bar{\wedge}(e^{it}) = \Big(I - \theta(e^{it})^*\theta(e^{it}) - \varphi_*(e^{it})^*\varphi_*(e^{it})\Big)^{\frac{1}{2}}$$

$$\bar{\wedge}_k(e^{it}) = \Big(I - \theta_k(e^{it})^*\theta_k(e^{it}) - \varphi_{*k}(e^{it})^*\varphi_{*k}(e^{it})\Big)^{\frac{1}{2}}$$

$$\varphi_*(z) = \begin{pmatrix} \varphi_{*2}(z) \\ \theta_2(z)\varphi_{*1}(z) \end{pmatrix}, \ \bar{\theta}_{*1}(z) = \big(\theta_1(z)\ \varphi_{*1}(z)\big).$$

Corollary 3.2. *Let α_k be *-optimal minimal passive scattering systems having $\varphi_{*k}(z) = 0$ as *-best minorant outer functions corresponding to transfer functions $\theta_k(z) : U_k \to V_k \quad (k = 1, 2)$ and let $U_2 = V_1$. Then the system $\alpha = \alpha_2\alpha_1$ is *-optimal minimal if and only if the following conditions hold:*

a) *$\varphi_*(z) = 0$ is the *-best minorant outer function corresponding to $\theta(z) = \theta_\alpha(z)$.*

b) *The factorization $\theta(z) = \theta_2(z)\theta_1(z)$ is $(+)$regular.*

4. Relations between optimal and *-optimal systems

In this section we will call a linear system $\alpha = (X, U, V, A, B, C, D)$ a *left dilation* of the linear system $\alpha' = (X', U, V, A', B', C', D')$ if there exists a subspace G_* such that

$$X = G_* \oplus X'$$

and

$$A^*G_* \subset G_*, \quad B^*G_* = \{0\}, \quad A' = A|_{X'}, \quad B' = B, \quad C' = C|_{X'}, \quad D' = D.$$

Similarly, a linear system $\alpha = (X, U, V, A, B, C, D)$ is a *right dilation* of the system $\alpha' = (X', U, V, A', B', C', D')$ if there exists a subspace G such that

$$X = X' \oplus G$$

and

$$AG \subset G, \quad CG = \{0\}, \quad A'^* = A^*|_{X'}, \quad B'^* = B^*|_{X'}, \quad C'^* = C^*, \quad D'^* = D^*. \blacksquare$$

It is not difficult to verify that if α is a left or right dilation of α', then the transfer functions of α and α' are equal in a certain neighbourhood of 0. Besides, if α is a left dilation of α', then the dual system α_* is a right dilation of α'_*.

Theorem 4.1. *In order that two optimal or *-optimal passive scattering systems α_1 and α_2 have the same transfer function $\theta_{\alpha_1}(z) = \theta_{\alpha_2}(z)$, it is necessary and sufficient that there exists an optimal or *-optimal minimal passive scattering system α' such that α_1 and α_2 are left or right dilations of α', respectively.*

Proof. Let α_1 and α_2 be optimal passive scattering systems having the same transfer function $\theta(z)$. Put $X_1' = X_{\alpha_1}^C$ and $X_2' = X_{\alpha_2}^C$. Consider the systems

$$\alpha_k' = (X_k', U, V, A_k', B_k', C_k', D_k')$$

where

$$A_k' = A_k|_{X_k'}, \quad B_k' = B_k, \quad C_k' = C_k|_{X_k'}, \quad D_k' = D_k.$$

Then α_k is a left dilation of α_k' and $A_k^n B_k = {A'}_k^n B_k'$ $(k = 1, 2)$. Let

$$\begin{aligned} T:\ & X_1' \to X_2' \\ & {A'}_1^n B_1' u \mapsto {A'}_2^n B_2' u \quad (n \in \mathbb{N}, u \in U) \\ \sum_{k=0}^{n} & {A'}_1^k B_1' u_k \mapsto \sum_{k=0}^{n} {A'}_2^k B_2' u_k. \end{aligned}$$

From the fact that α_1 and α_2 are optimal and have the same transfer function it is easy to verify that T is well defined and also a unitary operator. From the definition of T we have

$$TA_1'({A'}_1^n B_1' u) = {A'}_2^{n+1} B_2' u = A_2' T({A'}_1^n B_1' u).$$

Thus

$$TA_1' = A_2' T. \tag{4.1}$$

Moreover,

$$TB_1' = B_2'. \tag{4.2}$$

Since α_1 and α_2 have the same transfer function,

$$\theta(z) = D_1 + zC_1(I - zA_1)^{-1} B_1 = D_2 + zC_2(I - zA_2)^{-1} B_2.$$

It follows that

$$\theta(0) = D_1 = D_2 \tag{4.3}$$

and hence

$$\sum_{n=0}^{\infty} z^n C_1 A_1^n B_1 = \sum_{n=0}^{\infty} z^n C_2 A_2^n B_2.$$

Therefore $C_1A_1^nB_1 = C_2A_2^nB_2$. From the definition of α'_1 and α'_2 we may deduce that $C'_1{A'}_1^nB'_1 = C'_2{A'}_2^nB'_2$. Thus

$$C'_1 = C'_2T. \tag{4.4}$$

The operator T is unitary and satisfies (4.1) - (4.4). Thus α'_1 and α'_2 are unitarily equivalent. Moreover, since $A_k^nB_k = {A'}_k^nB'_k$ and $X'_k = X_{\alpha_k}^C$, α'_k is an optimal minimal passive scattering system.

We recall that systems

$$\alpha_1 = (X_1, U, V, A_1, B_1, C_1, D_1)$$
$$\alpha_2 = (X_2, U, V, A_2, B_2, C_2, D_2)$$

are said to be *unitarily equivalent* if there exixts a unitary oprator $T : X_1 \to X_2$ satisfying $TA_1 = A_2T$, $TB_1 = B_2$, $C_1 = C_2T$ and $D_1 = D_2$. It is obvious that if α_1 and α_2 are unitarily equivalent, then they have the same transfer function.

Conversely, suppose that there exists an optimal minimal passive scattering system α' such that α_1 and α_2 are left dilations of α'. Then $\theta_{\alpha_1}(z) = \theta_{\alpha'}(z) = \theta_{\alpha_2}(z)$. Besides, we have for $k = 1, 2$

$$X_k = G_{*k} \oplus X'$$

$$A_k^*G_{*k} \subset G_{*k}, \quad B_k^*G_{*k} = \{0\}, \quad A' = A_k|_{X'}, \quad B' = B_k.$$

Therefore it follows that $A_k^nB_k = {A'}^nB'$. Since α' is optimal, so are α_1 and α_2.

By applying this proof to the dual systems α_{1*} and α_{2*}, we have the result for the case concerning the *-optimality ∎

Now we introduce the new notions of partially unitary equivalence and *-partially unitary equivalence which will be used in our next theorem.

Definition 4.1.

a) The systems

$$\alpha_1 = (X_1, U, V, A_1, B_1, C_1, D_1)$$
$$\alpha_2 = (X_2, U, V, A_2, B_2, C_2, D_2)$$

are said to be *partially unitarily equivalent* if there exist subspaces $X'_1 \subset X_1$ and $X'_2 \subset X_2$ and a unitary operator $T : X'_1 \to X'_2$ such that

$$A_kX'_k \subset X'_k,\ B_kU \subset X'_k \quad (k = 1, 2) \tag{4.5}$$

$$TA_1|_{X'_1} = A_2T, \quad TB = B_2, \quad C_1|_{X'_1} = C_2T, \quad D_1 = D_2. \tag{4.6}$$

b) If the operator $T : X_1' \to X_2'$ satisfies

$$A_k^* X_k' \subset X_k',\ C_k^* V \subset X_k' \quad (k = 1, 2)$$
$$TA_1^*|_{X_1'} = A_2^* T, \quad B_1^*|_{X_1'} = B_2^* T, \quad TC_1^* = C_2^*, \quad D_1^* = D_2^*,$$

then α_1 and α_2 are said to be *-*partially unitarily equivalent.*

In the proof of the next theorem we will see that if linear systems α_1 and α_2 are partially or *-partially unitarily equivalent, then they have the same transfer function in a certain neighbourhood of 0.

Theorem 4.2. *Let α_1 and α_2 be optimal or *-optimal passive scattering systems. Then α_1 and α_2 have the same transfer function if and only if α_1 and α_2 are partially or *-partially unitarily equivalent, respectively.*

Proof. Suppose that α_1 and α_2 are optimal passive scattering systems having the same transfer function. Put $X_1' = X_{\alpha_1}^C$ and $X_2' = X_{\alpha_2}^C$. Then the operator $T : X_1' \to X_2'$ in the proof of Theorem 4.1 will satisfy the conditions for α_1 and α_2 to be partially unitarily equivalent.

Conversely, suppose that α_1 and α_2 are partially unitarily equivalent. Then there exist subspaces $X_1' \subset X_1$ and $X_2' \subset X_2$ and a unitary operator $T : X_1' \to X_2'$ satisfying (4.5) - (4.6). For $k = 1, 2$ put

$$A_k' = A_k|_{X_k}, \quad B_k' = B_k, \quad C_k' = C_k|_{X_k'}, \quad D_k' = D_k. \tag{4.7}$$

By (4.5) and (4.7) we see that α_k is a left dilation of α_k' and hence α_k and α_k' have the same transfer function. Moreover, by (4.6) we may deduce that α_1' and α_2' are unitarily equivalent. Therefore they have the same transfer function and so do α_1 and α_2. The case of *-optimal passive scattering systems can be proved in a similar way ∎

Acknowledgement. The author wishes to acknowledge several helpful suggestions made by the referees, which greatly improved the presentation of this paper.

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Received 11.12.2000; in revised form 11.09.2001