Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 1, 249–256

Global Existence for some Integro-Differential Equations with Delay Subject to Non-Local Conditions

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Abstract. By making use of the Leray-Schauder fixed point theorem we prove the global existence of solutions to some integro-differential equations with delay subject to non-local conditions, and this problem is considered in an arbitrary Banach space.

Keywords: *Nonlocal conditions, mild solutions, semigroups, Schaefer's fixed point theorem, integro-differential equations*

AMS subject classification: 35A05

1. Introduction

We are concerned in this paper with the study of global existence of solutions to some semilinear evolution integro-differential equation subject to non-local conditions. In fact, following the steps of Ntouyas and Tsamatos [6], we shall prove the global existence of solutions to the initial value problem

$$\left.\begin{aligned} &x'(t) + Ax(t) \\ &\quad = F\left(t, x(\sigma_1(t)), \int_0^t g\left(t, s, x(\sigma_2(s)), \int_0^s k\big(s, \tau, x(\sigma_3(\tau))\big) d\tau\right) ds\right) \\ &x(0) + h\big(t_1, ..., t_p, x(\cdot)\big) = x_0 \end{aligned}\right\} \qquad (1)$$

for $0 \leq t \leq T$, where

$0 < t_1 < ... < t_p \leq T$
$(X, \|\cdot\|)$ is a Banach space
$\{S(t)\}_{t\geq 0}$ is a linear semigroup in this space
$-A$ is the infinitesimal generator of this semigroup

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ISSN 0232-2064 / $ 2.50

I is the closed interval $[0, T]$ and
$F : I \times X \times X \to X$
$g : I \times I \times X \times X \to X$
$k : I \times I \times X \to X$
$h \in C(I, X) \to X$

are given functions. We assume, furthermore, that $\sigma_i \in C(I, I)$ and $\sigma_i(t) \le t$ for all $t \in I$ $(i = 1, 2, 3)$. We point out that the expression $h\big(t_1, ..., t_p, x(\cdot)\big)$ means that the function x is valued only on the set $\{t_1, ..., t_p\}$.

In fact, several physical problems have motivated specialists to consider non-local conditions $(1)_2$, which allow measurements at various points of the initial interval I including 0 rather than at a sole point as in the classical Cauchy initial value problem. We remember that problems dealing with non-local conditions were considered by L. Byszewski [1, 2] who proved the existence and uniqueness of mild, strong and classical solutions to some initial value problem of the form

$$\left.\begin{array}{l} x'(t) = Ax(t) + f(t, x(t)) \quad (t \in I) \\ x(0) + g\big(t_1, ..., t_p, x(\cdot)\big) = x_0 \end{array}\right\}.$$

2. Global existence

Let $\{S(t)\}_{t \ge 0}$ be a compact semigroup with infinitesimal generator $-A$ satisfying the estimate

$$\|S(t)\| \le Me^{\omega t} \qquad (t \ge 0) \tag{2}$$

for some constants $M > 0$ and $\omega \in \mathbb{R}_+$. We recall that any solution $t \to x(t)$ to the functional equation

$$x(t) = S(t)\big(x_0 - h(t_1, ..., t_p, x(\cdot))\big) + \int_0^t S(t-s)F\left(s, x(\sigma_1(s)), \int_0^s g\left(s, \theta, x(\sigma_2(\theta)), \int_0^\theta k\big(\theta, \tau, x(\sigma_3(\tau))\big)d\tau\right)d\theta\right)ds$$

is called a *mild solution* of problem (1).

We are now in position to state our result about the global existence of solutions to problem (1).

Theorem 1. *Let the function $F : I \times X \times \to X$ satisfy the following conditions:*

(H1) *For each $t \in I$ the function $F(t, \cdot, \cdot)$ belongs to $C(X \times X, X)$, and for each $(x, y) \in X \times X$ the function $F(\cdot, x, y)$ is strongly measurable.*

(H2) *There exists continuous functions* $p, q : I \to [0, \infty)$ *and a number* $\alpha \geq 1$ *such that*

$$\|F(t, x, y)\| \leq p(t)\|x\|^{\alpha} + q(t)\|y\|$$

for all $x, y \in X$ *and for all* $t \in I$.

(H3) $g : I \times I \times X \times X \to X$ *and* $k : I \times I \times X \to X$ *are continuous functions such that*

$$\begin{aligned} \|g(t,s,x,y)\| &\leq m_1(t,s)\,\|x\|^{\alpha-1}\Psi(\|x\|) + m_2(s)\,\|y\| \\ \|k(t,s,z)\| &\leq m_3(t,s)\,\|z\|^{\alpha-1}\Psi(\|z\|) \end{aligned}$$

for all $x, y, z \in X$ *and all* $t, s \in I$, *where* $\Psi : [0, \infty) \to (0, \infty)$ *is a continuous non-decreasing function,* $m_1 : I \times I \to [0, \infty)$ *is continuous and differentiable with respect to the first variable almost everywhere,* $m_2 : I \to [0, \infty)$ *and* $m_3 : I \times I \to [0, \infty)$ *are given continuous functions.*

(H4) $\{S(t)\}_{t \geq 0}$ *is a compact semigroup satisfying* (2).

(H5) *There is a constant* $H > 0$ *such that* $\|h(t_1, ..., t_p, x(\cdot))\| \leq H$ *for all* $x \in X$.

Then, if

$$\int_0^T \widetilde{Q}(t)\,dt < \int_a^{\infty} \frac{dz}{\Psi(z) + z^{\alpha} + z}$$

where

$$\widetilde{Q}(t) = \max\left\{\omega, Mp(t), Mq(t), \tfrac{1}{\alpha}\left(m_1(t,t) + \int_0^t \left|m_2(t)m_3(t,\tau) + \frac{\partial m_1(t,\tau)}{\partial t}\right| d\tau\right)\right\}$$

and where $a = M(\|x_0\| + H)$, *problem* (1) *has at least one mild solution on* I.

In order to prove the above theorem we need the following fixed point result due to Schaefer [3, 7].

Lemma 2. *Let* Ω *be a convex subset of a normed linear space* V *containing its zero element* 0. *If* $\mathcal{A} : \Omega \to \Omega$ *is a completely continuous operator, then either* $\mathcal{A}$ *has a fixed point or the subset*

$$\left\{x \in \Omega : x = \lambda \mathcal{A}x \ \ \textit{for some}\ \lambda \in (0,1)\right\}$$

is unbounded.

Proof of Theorem 1. Consider the initial value problem

$$\left.\begin{array}{l} x'(t) + \lambda Ax(t) = \lambda F\Big(t, x(\sigma_1(t)), \displaystyle\int_0^t g\Big(t, s, x(\sigma_2(s)), \int_0^s k\big(s, \tau, x(\sigma_3(\tau))\big)d\tau\Big)ds\Big) \\ x(0) + h(t_1, ..., t_p, x(\cdot)) = x_0 \end{array}\right\}$$

for $0 \leq t \leq T$ and some $\lambda \in (0,1)$, whose corresponding mild solution x satisfies the integral equation

$$\begin{aligned} x(t) &= S_\lambda(t)\big(x_0 - h(t_1, ..., t_p, x(\cdot))\big) \\ &\quad + \int_0^t S_\lambda(t-s)F\Big(s, x(\sigma_1(s)), \int_0^s g\Big(s, \theta, x(\sigma_2(\theta)), \int_0^\theta k\big(\theta, \tau, x(\sigma_3(\tau))\big)d\tau\Big)d\theta\Big)ds \end{aligned}$$

where $\{S_\lambda(t)\}_{t\geq 0}$ is the semigroup with infinitesimal generator $-\lambda A$. Again, we have the estimate

$$\|S_\lambda(t)\| \leq Me^{\omega t} \qquad (t \geq 0).$$

According to the given assumptions, it is clear that

$$\begin{aligned} \|x(t)\| \leq Me^{\omega t}(\|x_0\| + H) &+ Me^{\omega t}\int_0^t e^{-\omega s}\Big\{p(s)\|x(\sigma_1(s))\|^\alpha \\ &+ q(s)\int_0^s \Big(m_1(s,\theta)\|x(\sigma_2(\theta))\|^{\alpha-1}\Psi(\|x(\sigma_2(\theta))\|) \\ &+ m_2(\theta)\int_0^\theta m_3(\theta,\tau)\|x(\sigma_3(\tau))\|^{\alpha-1}\Psi(\|x(\sigma_3(\tau))\|d\tau\Big)d\theta\Big\}ds. \end{aligned}$$

Denoting by $e^{\omega t}u(t)$ the right-hand side of the above inequality, we obtain at once

$$u(0) = M(\|x_0\| + H) \qquad \text{and} \qquad \|x(t)\| \leq e^{\omega t}u(t) \;\; (0 \leq t \leq T)$$

and

$$\begin{aligned} u'(t) = Me^{-\omega t}&\Big\{p(t)\|x(\sigma_1(t))\|^\alpha \\ &+ q(t)\int_0^t \Big(m_1(t,\theta)\|x(\sigma_2(\theta))\|^{\alpha-1}\Psi(\|x(\sigma_2(\theta))\|) \\ &+ m_2(\theta)\int_0^\theta m_3(\theta,\tau)\|x(\sigma_3(\tau))\|^{\alpha-1}\Psi(\|x(\sigma_3(\tau))\|)d\tau\Big)d\theta\Big\}. \end{aligned}$$

From the fact that u is increasing and $\sigma_i(t) \le t$ for $i = 1, 2, 3$ it follows that

$$\begin{aligned} u'(t) = Me^{-\omega t}\Big\{& p(t)e^{\alpha\omega t}u^{\alpha}(t) \\ &+ q(t)\int_0^t \Big(m_1(t,\theta)e^{(\alpha-1)\omega\theta}u^{\alpha-1}(\theta)\Psi(e^{\omega\theta}u(\theta)) \\ &+ m_2(\theta)\int_0^{\theta} m_3(\theta,\tau)e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))d\tau\Big)d\theta\Big\} \\ \le Me^{-\omega t}Q(t)\Big\{& e^{\alpha\omega t}u^{\alpha}(t) + \int_0^t \Big(m_1(t,\theta)e^{(\alpha-1)\omega\theta}u^{\alpha-1}(\theta)\Psi(e^{\omega\theta}u(\theta)) \\ &+ m_2(\theta)\int_0^{\theta} m_3(\theta,\tau)e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))d\tau\Big)d\theta\Big\} \end{aligned}$$

where $Q(t) = \max\{p(t), q(t)\}$. Setting

$$\begin{aligned} v^{\alpha}(t) = {} & e^{\alpha\omega t}u^{\alpha}(t) \\ &+ \int_0^t \Big(m_1(t,\theta)e^{(\alpha-1)\omega\theta}u^{\alpha-1}(\theta)\Psi(e^{\omega\theta}u(\theta)) \\ &+ m_2(\theta)\int_0^{\theta} m_3(\theta,\tau)e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))\,d\tau\Big)d\theta \end{aligned}$$

we obtain

$$v(0) = u(0) = M(\|x_0\| + H) \qquad \text{and} \qquad v(t) \ge e^{\omega t}u(t) \ \ (t \in I).$$

Moreover, differentiating $v^{\alpha}(t)$ we get

$$\begin{aligned} \alpha v^{\alpha-1}(t)v'(t) = {} & \alpha\omega e^{\alpha\omega t}u^{\alpha}(t) + \alpha e^{\alpha\omega t}u^{\alpha-1}(t)u'(t) \\ &+ m_1(t,t)e^{(\alpha-1)\omega t}u^{\alpha-1}(t)\Psi(e^{\omega t}u(t)) \\ &+ m_2(t)\int_0^t m_3(t,\tau)e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))d\tau \\ &+ \int_0^t \frac{\partial m_1(t,\tau)}{\partial t}e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))\,d\tau \end{aligned}$$

$$
\begin{aligned}
&\leq \alpha\omega e^{\alpha\omega t}u^{\alpha}(t)+\alpha e^{\alpha\omega t}u^{\alpha-1}(t)u'(t)\\
&\quad + m_1(t,t)e^{(\alpha-1)\omega t}u^{\alpha-1}(t)\Psi(e^{\omega t}u(t))\\
&\quad + \int_0^t \left| m_2(t)m_3(t,\tau)+\frac{\partial m_1(t,\tau)}{\partial t}\right| e^{(\alpha-1)\omega\tau}u^{\alpha-1}(\tau)\Psi(e^{\omega\tau}u(\tau))\,d\tau\\
&\leq \alpha\omega e^{\alpha\omega t}u^{\alpha}(t)+\alpha e^{\alpha\omega t}u^{\alpha-1}(t)u'(t)\\
&\quad + m_1(t,t)e^{(\alpha-1)\omega t}u^{\alpha-1}(t)\Psi(e^{\omega t}u(t))\\
&\quad + \big(e^{(\alpha-1)\omega t}u^{\alpha-1}(t)\Psi(e^{\omega t}u(t))\big)\int_0^t \left| m_2(t)m_3(t,\tau)+\frac{\partial m_1(t,\tau)}{\partial t}\right| d\tau\\
&\leq \alpha\omega v^{\alpha}(t)+\alpha e^{\omega t}v^{\alpha-1}(t)MQ(t)e^{-\omega t}v^{\alpha}(t)+m_1(t,t)v^{\alpha-1}(t)\Psi(v(t))\\
&\quad + (v^{\alpha-1}(t)\Psi(v(t)))\int_0^t \left| m_2(t)m_3(t,\tau)+\frac{\partial m_1(t,\tau)}{\partial t}\right| d\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
v' &\leq \omega v + MQ(t)v^{\alpha} + \tfrac{1}{\alpha}\left(m_1(t,t)+\int_0^t \left| m_2(t)m_3(t,\tau)+\frac{\partial m_1(t,\tau)}{\partial t}\right| d\tau\right)\Psi(v)\\
&\leq \widetilde{Q}(t)\big(\Psi(v)+v^{\alpha}+v\big).
\end{aligned}
$$

Accordingly,

$$
\int_a^{v(t)} \frac{dz}{\Psi(z)+z^{\alpha}+z} \leq \int_0^T \widetilde{Q}(t)dt < \int_a^{\infty}\frac{dz}{\Psi(z)+z^{\alpha}+z}
$$

from which we get $v(t)\leq c$, for some constant $c=c(\alpha,\omega,T,\widetilde{Q})>0$. Then $\|x(t)\|\leq c$ for all $t\in I$.

Define the operator $\mathcal{T}: V\to V$, where $V=C(I,X)$, by

$$
\begin{aligned}
(\mathcal{T}x)(t) &= S(t)\big(x_0 - h(t_1,...,t_p,x(\cdot))\big)\\
&\quad + \int_0^t S(t-s)F\left(s,x(\sigma_1(s)),\int_0^s g\left(s,\theta,x(\sigma_2(\theta)),\int_0^{\theta} k\big(\theta,\tau,x(\sigma_3(\tau))\big)d\tau\right)d\theta\right)ds.
\end{aligned}
$$

It is worth to note that if $y\in V$ is such that $\|y(t)\|\leq r$ for some $r>0$, then

$$
\begin{aligned}
&\left\| F\left(t,y(t),\int_0^t g\left(t,\theta,y(\theta),\int_0^{\theta} k(\theta,\tau,y(\tau))\,d\tau\right)d\theta\right)\right\|\\
&\quad \leq p(t)\,\|y(t)\|^{\alpha} + q(t)\int_0^t \bigg(m_1(t,\theta)\|y(\theta)\|^{\alpha-1}\Psi(\|y(\theta)\|)\\
&\qquad + m_2(\theta)\int_0^{\theta} m_3(\theta,\tau)\|y(\tau)\|^{\alpha-1}\Psi(\|y(\tau)\|)d\tau\bigg)d\theta\\
&\quad \leq r^{\alpha}p(t)+\Psi(r)r^{\alpha-1}q(t)\int_0^t\left(m_1(t,\theta)+m_2(\theta)\int_0^{\theta} m_3(\theta,\tau)\,d\tau\right)d\theta.
\end{aligned}
$$

In what follows we shall denote the last term of the above inequality by $F_r(t)$. It is clear that, for each $r > 0$, the function F_r is summable over I.

In order to prove the continuity of the operator $\mathcal{T}$ we consider a sequence $\{x_n\}_{n\geq 1} \subset V$ converging in V to some function $\hat{x} \in V$. Thus the sequence $\{x_n(t)\}_{n\geq 1}$ and $\hat{x}(t)$ must be contained in some closed ball $B(0,r) \subset X$ for all $t \in I$. Define the mapping $\Phi : I \times V \to X$ by

$$\Phi(t,x) = F\bigg(t, x(\sigma_1(t)), \int_0^t g\bigg(t,s,x(\sigma_2(s)), \int_0^s k\big(s,\tau,x(\sigma_3(\tau))\big)d\tau\bigg)ds\bigg).$$

From hypotheses (H1) and (H3) it follows that

$$\lim_{n\to\infty} \Phi(t,x_n) = \Phi(t,\hat{x}).$$

On the other hand, we have

$$\|\Phi(t,x_n) - \Phi(t,\hat{x})\| \leq 2F_r(t)$$

and so we may conclude, by the dominated convergence theorem, that

$$\begin{aligned}\|\mathcal{T}x_n - \mathcal{T}\hat{x}\| = \sup_{t\in I}\bigg\| & \int_0^t S(t-s)\big\{\Phi(s,x_n) - \Phi(s,\hat{x})\big\}ds \\ & - S(t)\Big[h\big(t_1,\ldots,t_p,x_n(\cdot)\big) - h\big(t_1,\ldots,t_p,\hat{x}(\cdot)\big)\Big]\bigg\| \\ \to 0 &\end{aligned}$$

when $n \to \infty$. This shows that $\mathcal{T}$ is continuous.

For every number $r > 0$ we set

$$B_{r,V} = \big\{x \in V : \|x(t)\| \leq r\big\}.$$

To show that the image $\mathcal{T}(B_{r,V})$ is precompact in V, according to the Arzela-Ascoli theorem, we have to check only the precompactness in X of the set $\mathcal{T}(B_{r,V})(t)$ for each $t \in I$. Let t be fixed in $(0,T]$ and let $n \in \mathbb{N}$ be such that $\frac{1}{n} < t$. For every $x \in B_{r,V}$ we have

$$\begin{aligned}(\mathcal{T}x)(t) &= S(t)\big(x_0 - h(t_1,...,t_p,x(\cdot))\big) \\ &\quad + S(\tfrac{1}{n})\int_0^{t-\frac{1}{n}} S(t-s-\tfrac{1}{n})\Phi(s,x)\,ds + \int_{t-\frac{1}{n}}^t S(t-s)\Phi(s,x)\,ds.\end{aligned}$$

We set

$$(S_nx)(t) = S(t)\big(x_0 - h(t_1,..,t_p,x(\cdot))\big) + S(\tfrac{1}{n})\int_0^{t-\frac{1}{n}} S(t-s-\tfrac{1}{n})\Phi(s,x)\,ds$$

and

$$(T_n x)(t) = \int_{t-\frac{1}{n}}^{t} S(t-s)\Phi(s,x)\, ds.$$

For each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $x \in B_{r,V}$ we have

$$\|(T_n x)(t)\| \leq \int_{t-\frac{1}{n}}^{t} \|S(t-s)\| F_r(s)\, ds < \varepsilon$$

and thus

$$G_{0,t} := T_{n_0}(B_{r,V})(t) \subset B_{\varepsilon,X} := \big\{ z \in X : \|z\| < \varepsilon \big\}.$$

On the other hand, since the operatorS_{n_0} is compact, then the set $H_{0,t} = S_{n_0}(B_{r,V})(t)$ is precompact in X and thus it can be covered by m balls $B_\varepsilon(y_1), ..., B_\varepsilon(y_m) \subset X$. If we endow the product $X \times X$ with the norm $\|(x,y)\|_2 = \max(\|x\|, \|y\|)$, then

$$G_{0,t} \times H_{0,t} \subset \bigcup_{i=1}^{m} \big(B_{\varepsilon,X} \times B_\varepsilon(y_i) \big) \subset \bigcup_{i=1}^{m} B_\varepsilon(0, y_i)$$

where

$$B_\varepsilon(0,y_i) = \Big\{ (u,v) \in X \times X : \max(\|u\|, \|v - y_i\|) < \varepsilon \Big\}.$$

Therefore, the set $G_{0,t} \times H_{0,t}$ is precompact in $X \times X$ and thus

$$\overline{G_{0,t} \times H_{0,t}} = \overline{G_{0,t}} \times \overline{H_{0,t}}$$

is compact in $X \times X$ from which we deduce that $\overline{G_{0,t}} + \overline{H_{0,t}}$ is compact in X. Now, since

$$\mathcal{T}(B_{r,V})(t) = (S_{n_0} + T_{n_0})(B_{r,V})(t) \subset \overline{G_{0,t}} + \overline{H_{0,t}},$$

the set $\mathcal{T}(B_{r,V})(t)$ is precompact and accordingly the operator $\mathcal{T}$ is completely continuous. Gathering all the preceding results, we conclude that $\mathcal{T}$ has a fixed point in V which is exactly the expected mild solution we are seeking ∎

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Received 09.05.2001