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On the Hilbert Inequality With Weights

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Abstract. In this paper, it is shown that a Hilbert-type inequality with weight $\omega(n) = \pi - \frac{\theta}{\sqrt{2n+1}}$ can be established where $\theta = \frac{17}{20}$. As application, a quite sharp result of the Hardy-Littlewood inequality is obtained and some further extensions are obtained.

Keywords: *Hilbert inequality with weights, Hardy-Littlewood inequality, infimum, weight functions*

AMS subject classification: 41, 26D

1. Introduction

The Hilbert inequality may be written in the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=0}^{\infty} a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} b_n^2\right)^{\frac{1}{2}} \tag{1}$$

where (a_n) and (b_n) are sequences of real numbers such that $0 < \sum_{n=0}^{\infty} a_n^2 < +\infty$ and $0 < \sum_{n=0}^{\infty} b_n^2 < +\infty$. It is well known that the constant factor π herein is best possible, i.e. π cannot be decreased any more. But we can move the factors in $\pi = \sqrt{\pi}\sqrt{\pi}$ under the summation sign on an average and write a Hilbert-type inequality with weights of the form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} \leq \left(\sum_{n=0}^{\infty} \omega(n) a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} \omega(n) b_n^2\right)^{\frac{1}{2}} \tag{2}$$

where the weight function ω is defined by

$$\omega(n) = \pi - \frac{\theta(n)}{\sqrt{2n+1}}.$$

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Recently, a few papers (see [4, 5]) dealt with the weight function ω. Namely, in [4] it was shown that $\theta(n) > \frac{4n+1}{3(n+1)(2n+1)} > 0 \quad (n \in \mathbb{N}_0)$. Clearly, this inequality is related to n, and $\frac{4n+1}{3(n+1)(2n+1)} \to 0$ as $n \to \infty$. In addition, the expression of $\theta(n)$ is relatively complicated. Further, in [5] it was shown that $\omega(n) < \pi - \frac{\alpha}{\sqrt{n+1}}$ where $\alpha = 0.5292496^{+}$.

The purpose of the present paper is to simplify and to refine the results of [4, 5]. The method and theory employed by us are different from those in [4, 5]. To be specific, we use the expansion of functions into power series and the approximation theory. Similarly, our results can be extended to a Hilbert-type integral inequality with weights. Applying the results to the Hardy-Littlewood inequality, a sharp result there is obtained.

For convenience, we define the function θ by

$$\theta(x) = u(x) + v(x)\xi \qquad (x \geq 0) \tag{3}$$

where ξ is a constant satisfying the condition $0 < \xi < 1$ and the functions u and v are defined by

$$u(x) = 2\sqrt{2x+1}\ \arctan\sqrt{\frac{3}{2x+1}} - \frac{2x+1}{x+1} - \frac{\sqrt{3}(2x+1)}{6(x+2)} \tag{4}$$

$$v(x) = -\frac{\sqrt{3}(2x+1)(x+5)}{108(x+2)^2}, \tag{5}$$

respectively.

2. Lemmas and their proofs

In order to prove our assertions we need the following lemmas.

Lemma 1. *Let u be the function defined by (4). Then $u(x) > \frac{5\sqrt{3}}{3} - 2$ for $x \geq 8$.*

Proof. Taking the derivative of u we obtain after some simplifications

$$u'(x) = \frac{2}{\sqrt{2x+1}}\ \arctan\sqrt{\frac{3}{2x+1}} - \frac{\sqrt{3}}{x+2} - \frac{1}{(x+1)^2} - \frac{\sqrt{3}}{2(x+2)^2}.$$

Let us expand u' into power series of $\frac{1}{2x+1}$ and drop the negative remainder which consists of all terms with powers higher than 5. In such a way we may find via algebraic calculations

$$u'(x) < (2\sqrt{3}-4)t^2 + \left(8 - \frac{12\sqrt{3}}{5}\right)t^3 + A(t)\,t^4 < -\frac{1}{2}t^2 + 4\,t^3 + A(t)\,t^4 \tag{6}$$

where $t = \frac{1}{2x+1}$ and $A(t) = -(12 + \frac{54\sqrt{3}}{7}) + (16 + 234\sqrt{3})\,t$. In fact, when $0 < \alpha < 1$, using the inequality $\arctan\alpha < \alpha - \frac{1}{3}\alpha^3 + \frac{1}{5}\alpha^5 - \frac{1}{7}\alpha^7 + \frac{1}{9}\alpha^9$ we get

$$2\sqrt{t}\,\arctan\sqrt{3t} < 2\sqrt{3}\,t - 2\sqrt{3}\,t^2 + \frac{18\sqrt{3}}{5}\,t^3 - \frac{54\sqrt{3}}{7}\,t^4 + 18\sqrt{3}\,t^5$$

and

$$\begin{aligned}
-\frac{\sqrt{3}}{x+2} &= -\frac{2\sqrt{3}\,t}{1+3t} < -2\sqrt{3}\,t + 6\sqrt{3}\,t^2 - 18\sqrt{3}\,t^3 + 54\sqrt{3}\,t^4\\
-\frac{1}{(x+1)^2} &= -\frac{4\,t^2}{(1+t)^2} < -4\,t^2 + 8\,t^3 - 12\,t^4 + 16\,t^5\\
-\frac{\sqrt{3}}{2(x+2)^2} &= -\frac{2\sqrt{3}\,t^2}{(1+3t)^2} < -2\sqrt{3}\,t^2 + 12\sqrt{3}\,t^3 - 54\sqrt{3}\,t^4 + 216\sqrt{3}\,t^5.
\end{aligned}$$

Adding these inequalities, we get inequality (6). Notice that for $A(t)$ contained in (6) we have $A(t) < -25+422\,t$. Evidently, $A(t) < 0$ when $t \in (0, \frac{1}{17})$. Hence inequality (6) can be reduced to $u'(x) < (-\frac{1}{2} + 4t)t^2 < 0$ where $t = \frac{1}{2x+1}$ and $x \geq 8$. It follows that $u(x)$ is monotone decreasing in the interval $[8, +\infty)$ whence we have $\inf_{x\geq 8} u(x) = u(\infty) = \frac{5\sqrt{3}}{3} - 2$ and the lemma is proved ∎

Lemma 2. *Let* v *be the function defined by* (5)*. Then* $v(x) \geq -\frac{\sqrt{3}}{48}$ *for* $x \geq 0$.

Proof. Taking the derivative, after simplifications we get $v'(x) = \frac{\sqrt{3}(x-4)}{36(x+2)^3}$. Evidently, $v(4)$ is a minimun of v in $[0, +\infty)$. This implies that the lemma is true ∎

Lemma 3. *Let* θ *be the function defined by* (3)*. Then* $\theta(n) > \frac{17}{20}$ *for all* $n \in \mathbb{N}_0$.

Proof. For $n \geq 8$ we have with the use of Lemmas 1 and 2

$$\theta(n) = u(n) + v(n)\,\xi > u(n) + v(n) > \Big(\frac{5\sqrt{3}}{3} - 2\Big) - \frac{\sqrt{3}}{48} > \frac{17}{20}$$

where ξ is a constant satisfying $0 < \xi < 1$. It remains to prove only that $u(n) > \frac{5\sqrt{3}}{3} - 2$ when $0 \leq n \leq 7$. By direct computations we attain from (4)

$$\begin{array}{llll}
u(0) = 0.9500 & u(1) = 0.9320 & u(2) = 0.9198 & u(3) = 0.9130\\
u(4) = 0.9085 & u(5) = 0.9054 & u(6) = 0.9031 & u(7) = 0.9013.
\end{array}$$

This way $\theta(n) > \frac{17}{20}$ for all $n \geq 0$ and the lemma is proved ∎

3. Main results

Now let us came to our main results.

Theorem 1. *If* $0<\sum_{n=0}^{\infty} a_n^2<\infty$ *and* $0<\sum_{n=0}^{\infty} b_n^2<+\infty$, *then*

$$\sum_{m,n=0}^{\infty}\frac{a_m b_n}{m+n+1}<\left\{\sum_{n=0}^{\infty}\Big(\pi-\frac{\theta}{\sqrt{2n+1}}\Big)a_n^2\right\}^{\frac12}\left\{\sum_{n=0}^{\infty}\Big(\pi-\frac{\theta}{\sqrt{2n+1}}\Big)b_n^2\right\}^{\frac12} \tag{7}$$

where $\theta=\frac{17}{20}$.

Proof. We apply Cauchy's inequality to estimate the left-hand side of (7) as follows:

$$\begin{aligned}
&\sum_{m,n=0}^{\infty}\frac{a_m b_n}{m+n+1}\\
&\quad=\sum_{m,n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^{\frac12}}\Big(\frac{2m+1}{2n+1}\Big)^{\frac14}\frac{b_n}{(m+n+1)^{\frac12}}\Big(\frac{2n+1}{2m+1}\Big)^{\frac14}\\
&\quad\le\left\{\sum_{m,n=0}^{\infty}\frac{a_m^2}{m+n+1}\Big(\frac{2m+1}{2n+1}\Big)^{\frac12}\right\}^{\frac12}\left\{\sum_{m,n=0}^{\infty}\frac{b_n^2}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac12}\right\}^{1/2}\\
&\quad=\left\{\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac12}\right)a_n^2\right\}^{\frac12}\\
&\qquad\times\left\{\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}\frac{1}{m+n+1}\Big(\frac{2n+1}{2m+1}\Big)^{\frac12}\right)b_n^2\right\}^{\frac12}\\
&\quad=\left\{\sum_{n=0}^{\infty}\omega(n)a_n^2\right\}^{\frac12}\left\{\sum_{n=0}^{\infty}\omega(n)b_n^2\right\}^{\frac12}
\end{aligned}$$

where $\omega(n)=\sum_{m=0}^{\infty}\frac{1}{m+n+1}\big(\frac{2n+1}{2m+1}\big)^{1/2}$. Let us define the function F by

$$F(t)=\frac{1}{t+n+1}\Big(\frac{2n+1}{2t+1}\Big)^{\frac12}.$$

Applying the Euler-Maclaurin summation fomula to $\omega(n)$ we get

$$\omega(n)=F(0)+\sum_{m=1}^{\infty}F(m)=F(0)+\int_1^{\infty}F(t)\,dt+\frac12F(1)+R(n) \tag{8}$$

where $R(n)$ is the remainder. See [2, 3] for various expressions of it. Here we give the remainder in the form $R(n)=-\frac{\xi}{12}F'(1)\quad(0<\xi<1)$. By computation we obtain the relation $\sqrt{2n+1}\,R(n) = v(n)\xi$ where v is the function defined by (5), and $\int_1^\infty F(t)\,dt = \pi - 2\arctan\sqrt{3/(2n+1)}$. In view of (4) we may write (8) in form

$$\omega(n) = \pi - \frac{u(n) + v(n)\,\xi}{\sqrt{2n+1}} = \pi - \frac{\theta(n)}{\sqrt{2n+1}}$$

where θ is the function defined by (3). Basing on Lemma 3 we get $\omega(n) < \pi - \frac{\theta}{\sqrt{2n+1}}$ where $\theta = \frac{17}{20}$ and the proof of the theorem is completed ∎

Remark. Theorem 1 is obviously an improvement on the result of [5] because $\frac{\theta}{\sqrt{2n+1}} = \frac{\theta}{\sqrt{2}(n+\frac{1}{2})^{\frac{1}{2}}} > \frac{\theta}{\sqrt{2}(n+1)^{\frac{1}{2}}} > \frac{\alpha}{(n+1)^{\frac{1}{2}}}$ where $\theta = \frac{17}{20}$ and $\alpha = 0.5292496^{+}$.

Corollary 1. *If* $0 < \sum_{n=0}^{\infty} a_n^2 < +\infty$, *then*

$$\sum_{m,n=0}^{\infty} \frac{a_m a_n}{m+n+1} < \sum_{n=0}^{\infty} \Big(\pi - \frac{\theta}{\sqrt{2n+1}}\Big) a_n^2 \tag{9}$$

where $\theta = \frac{17}{20}$.

Clearly, this is an immediate consequence of (7).

Theorem 2. *Let* $f, g \in L^2[0, +\infty)$. *Then*

$$\int_0^\infty\!\!\int_0^\infty \frac{f(x)g(y)}{x+y+1}\,dxdy$$
$$\leq \pi\Big\{\int_0^\infty \Big(1 - \frac{1}{2\sqrt{2x+1}}\Big) f^2(x)\,dx\Big\}^{\frac{1}{2}} \Big\{\int_0^\infty \Big(1 - \frac{1}{2\sqrt{2x+1}}\Big) g^2(x)\,dx\Big\}^{\frac{1}{2}}. \tag{10}$$

Equality herein holds if and only if $f = 0$ *or* $g = 0$.

Proof. Similar to the proof of Theorem 1 we obtain

$$\int_0^\infty\!\!\int_0^\infty \frac{f(x)g(y)}{x+y+1}\,dxdy \leq \Big\{\int_0^\infty p(x) f^2(x)\,dx\Big\}^{\frac{1}{2}} \Big\{\int_0^\infty p(x) g^2(x)\,dx\Big\}^{\frac{1}{2}}$$

where the weight function p is defined by

$$P(x) = \int_0^\infty \frac{1}{x+y+1}\Big(\frac{2x+1}{2y+1}\Big)^{\frac{1}{2}} dy = \pi - 2\arctan\frac{1}{\sqrt{2x+1}} = \pi - \frac{\alpha(x)}{\sqrt{2x+1}}$$

where $\alpha(x) = 2\sqrt{2x+1}\,\arctan\frac{1}{\sqrt{2x+1}}$. It is easy to prove that the function α is monotonely increasing in the interval $[0, +\infty)$. In fact,

$$\alpha'(x) = \frac{2}{\sqrt{2x+1}}\arctan\frac{1}{\sqrt{2x+1}} - \frac{1}{x+1}.$$

Notice that $\arctan t > t - \frac{1}{3}t^3$ when $0 < t < 1$. Hence

$$\begin{aligned}\alpha'(x) &> \frac{2}{2x+1} - \frac{2}{3(2x+1)^2} - \frac{1}{x+1}\\ &= \frac{1}{(2x+1)(x+1)} - \frac{2}{3(2x+1)^2}\\ &= \frac{1}{2x+1}\Big(\frac{1}{x+1} - \frac{1}{3x+\frac{3}{2}}\Big)\\ &> 0\end{aligned}$$

and $\alpha'(0) = \frac{\pi}{2} - 1 > 0$. So our assertion is proved. Hence $\inf_{x\geq 0}\alpha(x) = \alpha(0) = \frac{\pi}{2}$ and $p(x) \leq \pi - \frac{\alpha(0)}{\sqrt{2x+1}} = \pi\big(1 - \frac{1}{2\sqrt{2x+1}}\big)$. It follows that (10) is valid and the theorem is proved ∎

Corollary 2. *If $f \in L^2[0,+\infty)$, then*

$$\int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y+1}\,dxdy \leq \pi\int_0^\infty \Big(1 - \frac{1}{2\sqrt{2x+1}}\Big)f^2(x)\,dx. \tag{11}$$

Equality herein holds if and only if $f = 0$.

4. Applications

Let $f \in L^2(0,1)$ and $f(x) \neq 0$ for all x. If $a_n = \int_0^1 x^n f(x)\,dx \quad (n \in \mathbb{N}_0)$, then we get the Hardy-Littlewood inequality (cf. [1]) in the form

$$\sum_{n=0}^\infty a_n^2 < \pi\int_0^1 f^2(x)\,dx \tag{12}$$

where π is the best constant that keeps (12) valid. The following improvement of (12) will be obtained by means of Corollary 1.

Theorem 3. *Under the assumptions just described we have*

$$\Big(\sum_{n=0}^\infty a_n^2\Big)^2 < \Big\{\sum_{n=0}^\infty \Big(\pi - \frac{\theta}{\sqrt{2n+1}}\Big)a_n^2\Big\}\int_0^1 f^2(x)\,dx \tag{13}$$

where $\theta = \frac{17}{20}$.

Proof. By our assumption, $a_n^2 = \int_0^1 a_n x^n f(x)\,dx$. Using the Cauchy-Schwarz inequality and Corollary 1 we get

$$\begin{aligned}\left(\sum_{n=0}^{\infty} a_n^2\right)^2 &= \left(\sum_{n=0}^{\infty}\int_0^1 a_n x^n f(x)dx\right)^2 \\ &= \left(\int_0^1\left(\sum_{n=0}^{\infty} a_n x^n\right) f(x)dx\right)^2 \\ &\le \int_0^1\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 dx \int_0^1 f^2(x)dx \qquad (14) \\ &= \left\{\sum_{m,n=0}^{\infty}\frac{a_m a_n}{m+n+1}\right\}\int_0^1 f^2(x)dx \\ &\le \left\{\sum_{n=0}^{\infty}\left(\pi - \frac{\theta}{\sqrt{2n+1}}\right)a_n^2\right\}\int_0^1 f^2(x)dx\end{aligned}$$

where $\theta = \frac{17}{20}$. Since $f(x) \neq 0$ for all x, $a_n \neq 0$ for all $n \geq 0$. Therefore it is impossible to take equality in (14). It follows that (13) is valid ∎

Remark. If in (13) we replace θ by zero, then (12) follows. Clearly, this is a refinement of the Hardy-Littlewood inequality.

Theorem 4. *Let $g \in L^2(0,1)$ with $g(t) \neq 0$ for all t and define f by*

$$f(x) = \int_0^1 t^x g(t)dt \qquad (x \geq 0).$$

Then

$$\left(\int_0^{\infty} f^2(x)dx\right)^2 < \pi\left(\int_0^{\infty}\left(1 - \frac{1}{\sqrt{2x+1}}\right)f^2(x)dx\right)\int_0^1 g^2(t)dt. \qquad (15)$$

Proof. We may write f^2 in the form $f^2(x) = \int_0^1 f(x)t^x g(t)dt$. Applying the Cauchy-Schwarz inequality and using Corollary 2 we get

$$\begin{aligned}\left(\int_0^\infty f^2(x)dx\right)^2 &= \left\{\int_0^\infty \left(\int_0^1 f(x)t^x g(t)dt\right)dx\right\}^2 \\ &= \left\{\int_0^1 \left(\int_0^\infty f(x)t^x dx\right)g(t)dt\right\}^2 \\ &\le \int_0^1 \left(\int_0^\infty f(x)t^x dx\right)^2 dt \int_0^1 g^2(t)dt \\ &= \int_0^1 \left(\int_0^\infty f(x)t^x dx\right)\left(\int_0^\infty f(y)t^y dy\right)dt \int_0^1 g^2(t)dt \\ &= \int_0^1 \left(\int_0^\infty\int_0^\infty f(x)f(y)t^{x+y}dxdy\right)dt \int_0^1 g^2(t)dt \\ &= \int_0^\infty\int_0^\infty \frac{f(x)f(y)}{x+y+1}dxdy \int_0^1 g^2(t)dt \\ &\le \pi\int_0^\infty \left(1-\frac{1}{2\sqrt{2x+1}}\right)f^2(x)dx \int_0^1 g^2(t)dt. \end{aligned} \tag{16}$$

Since $g(t) \neq 0$ for all t, whence $f(x) \neq 0$ for all x. It is impossible to take equality in (16). Hence (15) is valid ∎

Remark. We point out that if $\frac{1}{2\sqrt{2x+1}}$ contained in (15) is replaced by zero, then we obtain immediately a new inequality of the form $\int_0^\infty f^2(x)dx < \pi\int_0^1 g^2(t)dt$. Obviously, this is an extension of the Hardy-Littlewood inequality (12).

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