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Pseudodifferential Operators with Analytic Symbols and Estimates for Eigenfunctions of Schrödinger Operators

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Abstract. We study the behavior of eigenfunctions of the Schrödinger operator $-\Delta + v$ with potential having power, exponential or super-exponential growth at infinity and discontinuities on manifolds in $\mathbb{R}^n$. We use a connection between the domain of analyticity of the main symbol $(|\xi|^2 + v(x))^{-1}$ of the parametrix $-\Delta + v$ at infinity or near singularities of v and the behavior of eigenfunctions at infinity or near singularities of potentials. Our approach is based on a general calculus of pseudodifferential operators with analytic symbols.

Keywords: *Pseudodifferential operators, analytic symbols, Schrödinger operators*

AMS subject classification: 35J15, 35J10, 35S05

1. Introduction

We study the behavior of eigenfunctions of Schrödinger operators $-\Delta+v$ with potentials having power, exponential or super-exponential growth at infinity and which is discontinuous on manifolds in $\mathbb{R}^n$. We use a connection between the domain of analyticity of the main symbol $(|\xi|^2+v(x))^{-1}$ of the parametrix $-\Delta+v$ and the behavior of the eigenfunctions at infinity or near singularities of potentials. Our approach is based on a general calculus of pseudodifferential operators with analytic symbols.

It should be noted that exponential estimates for eigenfunctions of Agmon type [8] for Schrödinger operators with decreasing potentials can be obtained from the general estimates of the present paper.

If $A = -\Delta+|x|^2$ is the Harmonic oscillator, then the well-known estimates (see, for instance, [3]) for eigenfunctions ϕ

$$\phi(x) = O\big(\exp\big(-(\tfrac{1}{2}-\varepsilon)|x|^2\big)\big) \qquad (\varepsilon > 0)$$

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is obtained in our paper from the fact that the symbol $(|\xi|^2+|x|^2)^{-1}$ of the parametrix of A has an analytic extension with respect to ξ in the tube domain $\mathbb{R}^n_\xi + i\{\eta\in\mathbb{R}^n : |\eta|<|x|\}$. Moreover, we prove that the eigenfunctions of the Schrödinger operator with potential $|x|^{2k}$ $(k\in\mathbb{N})$ are entire functions on $\mathbb{C}^n$ and we give estimates for the order and the type of these entire functions.

We also consider the Schrödinger operator $A=-\triangle+q^2(x)$ with general potentials $q^2(x)$ where $q(x)$ can be $\exp|x|^{2k}, \exp(\exp|x|^{2k}), (1-|x|^2)^{-r}+|x|^k$ $(r>1, k>0)$ and so on. In particular, as a corollary of main results of the paper we obtain that the eigenfunctions $\phi(x)$ of the Schrödinger operator

$$-\triangle+\left[|1-|x|^2|^{-r}+|x|^k\right]^2 \qquad (k>0, r>1)$$

have the behavior

$$\phi(x)=\begin{cases} O\big(\exp\big(-\frac{K}{|1-|x|^2|^{r-1}}\big)\big) & \text{as } x\to S^{n-1}, 0<K<\frac{1}{2(r-1)}\\ O\big(\exp\big(-K|x|^{k+1}\big) & \text{as } x\to\infty, 0<K<\frac{1}{k+1}\end{cases}$$

near the singularities of the potential.

The problem under consideration is connected with a class of pseudodifferential operators with double symbols $p(x,y,\xi)$ which have an analytic extensions in tube domains $\mathbb{R}^n+W_\alpha(x)$ with bases depending on x, and can have a growth at infinity faster than any polynomial. Moreover, the symbols $p(x,y,\xi)$ are discontinuous with respect to (x,y) on some manifolds in $\mathbb{R}^n$. The work is based on a general calculus of pseudodfferential operators with double symbols $p(x,y,\xi)$ in the form given by Levendorskii (see, for instance, [7, 8]). This calculus is an extension of the well-known calculus of Beals and Hörmander (see, for instance, [2]) and Feigin [3]. But we need a calculus of pseudodifferential operators with analytic symbols $p(x,y,\xi)$ with respect to ξ in tube domains with base depending on (x,y). Such calculus is presented in the current paper. Applying this calculus we obtain results on boundedness and Fredholmness of pseudodifferential operators with double symbols in Sobolev spaces with exponential weight $\exp a(x)$ compatible with the domain of analyticity of the symbol. We also obtain estimates for solutions of differential equations at infinity or near singularities of the coefficients, and the results on the behavior of eigenfunctions of Schrödinger operators are obtained as a corollary of these estimates.

A connection between the domain of analycity of the parametrix of pseudodifferential operators and exponential estimates of solutions of pseudodifferential equations at infinity was first given in the short paper [6], devoted to the study of scalar pseudodifferential operators with symbols $a(x,\xi)$ analytic with respect to the variable ξ in the tube domains $\mathbb{R}^n+iW$, where W is a bounded convex domain in $\mathbb{R}^n$ independent of x. In the paper [7] an algebra of pseudodifferential operators with operator-valued symbols analytic with respect to ξ in tube domains $\mathbb{R}^n + iW$ was considered and weighted estimates with weights $\exp a(x), a(x) = O(|x|)$ as $x \to \infty$, were obtained for solutions of partial differential equations with operator-valued coefficients.

Recently, methods based on pseudodifferential operators with symbols $a(x,\xi)$ analytically extended with respect to ξ in tube domains with bases independent on x have been used in problems of semiclassical analysis, tunneling effects and so on (see, for instance, [9 - 11]) and the references therein).

2. Auxiliary results

In this section we will formulate auxiliary results on the general calculus of double pseudodifferential operators with symbols which can have singularities on manifolds in $\mathbb{R}^n$ following [7, 8]. For $\alpha = (\alpha_1, ..., \alpha_n)$ we use the standard notations

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad D^\alpha = (-i\partial)^\alpha$$
$$p^{(\alpha)}_{(\beta,\gamma)}(x,y,\xi) = D_x^\beta D_y^\gamma \partial_\xi^\alpha p(x,y,\xi), \quad \langle x \rangle = (1+|x|^2)^{1/2}.$$

Definition 1. Let Ω be an open set in $\mathbb{R}^n$. We say that a pair of positive continuous functions $\Phi(x,\xi)$ and $\varphi(x,\xi)$ defined on $\Omega \times \mathbb{R}^n$ is a pair of *weight functions* if there are numbers $C > 0$ and $c > 0$ such that

$$\left.\begin{array}{l}
\varphi(x,\xi) \le \Phi(x,\xi) \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^n \\
|x-y| \le c\,\varphi(x,0) \text{ for all } x,y \in \Omega \\
c\,\Phi(x,\xi)^{-1}\varphi(x,\xi)^{-1} \le \Phi(x,0)^{-1}\varphi(x,0)^{-1} \le C \text{ for all } (x,\xi) \in \Omega \times \mathbb{R}^n \\
|x-y| \le c\,\varphi(x,0) \Longrightarrow \varphi(x,\xi)^{-1}\varphi(y,\xi) \le C, \Phi(x,\xi)^{-1}\Phi(y,\xi) \le c \\
|\xi-\eta| \le c\,\Phi(x,\xi) \Longrightarrow \varphi(x,\xi)^{-1}\varphi(x,\eta) \le C \\
\Phi(x,\xi)^{-1}\Phi(x,\eta) \le C \\
\Phi(x,\xi)\Phi(x,\eta)^{-1} + \varphi(x,\xi)\varphi(y,\eta)^{-1} \le C(1+\varphi(x,\eta)|\xi-\eta|)^\kappa \quad (\kappa \in \mathbb{R})
\end{array}\right\}. \tag{1}$$

It follows from condition 1 that the growth of weight functions with respect to ξ can not be faster than polynomial.

Example 2. Let $0 \le q \in C^\infty(\Omega)$ with $q(x) \to \infty$ as $x \to \partial\Omega$ and

$$|\partial^\alpha q| \le C_\alpha q(x)^{1+\delta_0|\alpha|} \qquad (\delta_0 < 1).$$

Then

$$\Phi(x,\xi) = \left(1+|\xi|^2 + q(x)^2\right)^{\frac{1}{2}} \qquad \text{and} \qquad \varphi(x) = q(x)^{-\delta} \quad (\delta_0 \le \delta < 1)$$

are a pair of weight functions.

Example 3. Let q be the same as in Example 2. Then

$$\Phi(x,\xi)=\big(1+|\xi|^2+q(x)^2\big)^{\rho} \qquad \text{and} \qquad \varphi(x,\xi)=\big(1+|\xi|^2+q(x)^2\big)^{-\delta}$$

where $\delta_0\leq\delta<\rho\leq 1$ are a pair of weight functions.

We list examples of sets Ω and functions q that satisfy the estimates given above:

1) $\Omega=\mathbb{R}^n, q(x)=\exp|x|^2$
2) $\Omega=\mathbb{R}^n\backslash\{0\}, q(x)=\langle x\rangle^l+|x|^{-m}\quad (m\geq 1, l\ in \mathbb{R})$
3) $\Omega=\mathbb{R}^n\backslash\{0\}, q(x)=\exp\exp(\langle x\rangle^l+|x|^{-m})\quad (lm>0)$
4) $\Omega=\{x:\ |x|<1\}, q(x)=(1-|x|^2)^{-m}\quad (m\geq 1)$
5) $\Omega=\mathbb{R}^n\backslash\mathbb{S}^{n-1}, q(x)=|1-|x|^2|^{-m}+\langle x\rangle^l\quad (m\geq 1, l\geq 0)$
6) $q(x)=d^{-\gamma}(x)\ \ (\gamma>1)$ where $d\in C^\infty(\Omega)$ is a regularized distance from the point $x\in\Omega$ to the boundary $\partial\Omega$. This means

$$\begin{aligned} c_1\operatorname{dist}(x,\partial\Omega)\leq d(x)&\leq c_2\operatorname{dist}(x,\partial\Omega)\\ |\partial_x^\alpha d(x)|&\leq c_\alpha d(x)^{1+|\alpha|}. \end{aligned}$$

Definition 4. Denote by $O(\Phi,\varphi)$ the class of positive C^∞ functions $\lambda(x,\xi)$ defined on $\Omega\times\mathbb{R}^n$ such that

$$\begin{aligned} &|\lambda(x,\xi)^{-1}\lambda(y,\xi)|\leq C\ \text{ if } |x-y|\leq c\,\varphi(x,0)\\ &|\lambda(x,\eta)^{-1}\lambda(x,\xi)|\leq C\big(1+\varphi(x,\eta)|\xi-\eta|\big)^{\kappa}\quad (\kappa\in\mathbb{R}). \end{aligned}$$

Example 5. Let Φ and φ be the same as in Example 2. Then

$$\lambda(x,\xi)=\big(1+|\xi|^2+q(x)^2\big)^{m/2}\qquad (m\in\mathbb{R})$$

is in the class $O(\Phi,\varphi)$.

Let $p(x,y,\xi)$ be a smooth complex-valued function defined on $\Omega\times\Omega\times\mathbb{R}^n$. We say that a pair $(x,y)\in\Omega\times\Omega$ is in $\operatorname{supp}_{xy}p$ if there is $\xi\in\mathbb{R}^n$ such that $(x,y,\xi)\in\operatorname{supp}p$.

Definition 6. Let $\lambda\in O(\Phi,\varphi)$. Denote by $S(\lambda,\Phi,\varphi)$ the space of complex-valued functions $p(x,y,\xi)$ defined on $\Omega\times\Omega\times\mathbb{R}^n$ such that:

a) If $(x,y)\in\operatorname{supp}_{xy}p$, then $|x-y|\leq c(p)\varphi(x,0)$ for some $c(p)>0$.

b) For all multi-indices α,β,γ

$$n^\alpha_{\beta,\gamma}(p)=\sup_{\Omega_x\times\Omega_y\times\mathbb{R}^n_\xi}\lambda^{-1}(x,\xi)\Phi^{|\alpha|}(x,\xi)\varphi^{|\beta|+|\gamma|}(x,\xi)\big|p^{(\alpha)}_{(\beta,\gamma)}(x,y,\xi)\big|<\infty.$$

The constants $n^{\alpha}_{\beta,\gamma}(p)$ define the Fréchet space topology in $S(\lambda,\Phi,\varphi)$.

We associate with a function $p(x,y,\xi) \in S(\lambda,\Phi,\varphi)$ a pseudodifferential operator with double symbol

$$Op(p)u(x) = \tfrac{1}{(2\pi)^n} \int d\xi \int e^{i(x-y,\xi)} p(x,y,\xi)u(y)\,dy \quad \big(u \in C_0^\infty(\Omega)\big) \tag{2}$$

where the integrals are taken over the entire space (this is always supposed if the limits of integration are not indicated). The integral in (2) exists only as repeated.

By $OPS(\lambda,\Phi,\varphi)$ we denote the class of pseudodifferential operators with double symbol. Note that the usual pseudodifferential operators $p(x,D)$ are not included in this class. Such inclusion is possible if $\Omega = \mathbb{R}^n$, $p(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $p(x,\xi)$ has a growth at infinity not faster than polynomial.

Let $p(x,\xi)$ be a polynomial with respect to the variable ξ satisfying the estimates

$$\sup_{\Omega\times\mathbb{R}^n} \lambda^{-1}(x,\xi)\Phi^{|\alpha|}(x,\xi)\varphi^{|\beta|}(x,\xi)\big|p^{(\alpha)}_{(\beta)}(x,\xi)\big| < \infty \tag{3}$$

and let

$$\chi(x,y) = \theta\Big(c^{-1}|x-y|\big(\varphi(x,0)^{-1} + \varphi(y,0)^{-1}\big)\Big)$$

where $c > 0$ is sufficiently small, $\theta \in C_0^\infty(\mathbb{R})$ and $\theta(x) = 1$ in a neighborhood of the origin. Then the differential operator $p(x,D)$ can be written as a pseudodifferential operator with double symbol

$$p(x,D) = Op\big(p(x,\xi)\chi(x,y)\big). \tag{4}$$

It is easy to check that $p(x,\xi)\chi(x,y) \in OPS(\lambda,\Phi,\varphi)$.

Proposition 7.

(a) *Let $\lambda \in O(\Phi,\varphi)$ and $A = Op(a) \in OPS(\lambda,\varphi,\Phi)$. Then for all $N \in \mathbb{N}$*

$$A = Op\Big\{\chi(x,y)\sum_{|\alpha|<N} \frac{1}{\alpha!} a^{(\alpha)}_{(0,\alpha)}(x,x,\xi) + a_N(x,y,\xi)\Big\}$$

where $a_N \in S(\lambda\pi^{-N},\Phi,\varphi)$ and $\pi(x,\xi) = \Phi(x,\xi)\varphi(x,\xi)$.

(b) *Let $a_1(x,y,\xi) \in S(\lambda_1,\Phi,\varphi)$ and $a_2(x,y,\xi) \in S(\lambda_2,\Phi,\varphi)$. Then*

$$B = Op(a_1)Op(a_2) \in OPS(\lambda_1\lambda_2,\Phi,\varphi)$$

and $B = Op(b)$ where

$$b(x,y,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} B_\alpha(x,y,\xi) + r_N(x,y,\xi)$$

for an arbitrary $N \in \mathbb{N}$, with

$$B_\alpha(x,y,\xi) = D_z^\alpha\big(a_1^{(\alpha)}(x,z,\xi)a_2(z,y,\xi)\big)\big|_{z=x} \quad \text{and} \quad r_N \in S\big(\lambda_1\lambda_2\pi^{-N},\Phi,\varphi\big).$$

Moreover, for each α,β,γ there exist constants $C_{\alpha\beta\gamma} > 0$ and $M \in \mathbb{N}$ such that

$$n_{\beta\gamma}^\alpha(r_N) \le C_{\alpha\beta\gamma} \max_{\substack{N\le|\alpha'|\le M, |\alpha''|\le M \\ N\le|\beta'|+|\beta''|+|\gamma'|+|\gamma''|\le M}} n_{\beta'\gamma'}^{\alpha'}(a_1)n_{\beta''\gamma''}^{\alpha''}(a_2).$$

Definition 8. We set

$$H(\lambda,\Phi,\varphi) = \operatorname{span}\big\{Au : \, u \in L_2(\Omega), A \in S(\lambda,\Phi,\varphi)\big\}.$$

The space $H(\lambda,\Phi,\varphi)$ endowed with the finest topology in which each operator $A \in OPS(\lambda,\Phi,\varphi)$ is continuous from $H(\lambda,\Phi,\varphi)$ into $L_2(\Omega)$.

Proposition 9. *Let $\lambda,\mu \in O(\Phi,\varphi)$. The following statements hold:*

a) *The embedding $\mathcal{D}(\Omega) \subset H(\lambda,\Phi,\varphi) \subset \mathcal{D}'(\Omega)$ is continuous and $\mathcal{D}(\Omega)$ is dense in $H(\lambda,\Phi,\varphi)$.*

b) *The embedding $H(\mu,\Phi,\varphi) \subset H(\lambda,\Phi,\varphi)$ is continuous if $\lambda(x,\xi) \le C\mu(x,\xi)$, and this embedding is compact if*

$$\lim_{\varepsilon\to 0} \inf_{(x,\xi)\in m_\varepsilon} \mu(x,\xi)^{-1}\lambda(x,\xi) = 0$$

where

$$m(\varepsilon) = \big\{(x,\xi) \in \Omega\times\mathbb{R}^n : |x|+|\xi| > \varepsilon^{-1}\big\}\cup\big\{(x,\xi) \in \Omega\times\mathbb{R}^n : \operatorname{dist}(x,\partial\Omega) < \varepsilon\big\}.$$

c) *There are operators $\mathcal{L}^\lambda \in OPS(\lambda^{-1},\Phi,\varphi)$ and $\mathcal{M}^\lambda \in OPS(\lambda,\Phi,\varphi)$ such that $\mathcal{L}^\lambda : L_2(\Omega) \to H(\lambda,\Phi,\varphi)$ and $\mathcal{M}^\lambda : H(\lambda,\Phi,\varphi) \to L_2(\Omega)$ are topological isomorphisms.*

d) *$H^*(\lambda,\Phi,\varphi) = H(\lambda^{-1},\Phi,\varphi)$.*

A norm on $H(\lambda,\Phi,\varphi)$ can be defined as

$$\|u\|_{H(\lambda,\Phi,\varphi)} = \|\mathcal{L}^\lambda u\|_{L_2(\Omega)}.$$

Denote

$$H_{-\infty}(\Phi,\varphi) = \bigcup_{\lambda\in O(\Phi,\varphi)} H(\lambda,\Phi,\varphi) \quad \text{and} \quad H_\infty(\Phi,\varphi) = \bigcap_{\lambda\in O(\Phi,\varphi)} H(\lambda,\Phi,\varphi)$$

with the topology of the inductive and projective limits, respectively.

Let us consider an important example of functional space $H(\lambda, \Phi, \varphi)$ in which the norm is defined in an explicit way. Let $q(x)$ be the function in Example 2, $\lambda(x,\xi) = \left(1 + |\xi|^2 + q(x)^2\right)^{1/2}$, $\Phi(x,\xi) = \lambda(x,\xi)$ and $\varphi(x,\xi) = q(x)^{-\delta}$ $(\delta_0 \le \delta < 1)$. Then $\mu = \lambda^m$ is in $O(\Phi, \varphi)$ for all $m \in \mathbb{N}$. An equivalent norm in $H(\lambda^m, \varphi, \Phi)$ is

$$\|u\|_{H(\lambda^m,\Phi,\varphi)} = \left(\sum_{|\alpha| \le m} \left\| q^{m-|\alpha|} \partial^\alpha u \right\|^2_{L_2(\Omega)} \right)^{\frac{1}{2}}.$$

If m is positive but non integer, the norm in $H(\lambda^m, \Phi, \varphi)$ can be defined by means of interpolation, and for negative m by means of duality.

Let φ and Φ be the same weight functions as in Example 2. Then $H_\infty(\Phi, \varphi)$ is a countably-normed space with topology given by the semi-norms

$$\|u\|_m = \left(\sum_{|\alpha| \le m} \int_\Omega q(x)^{2(m-|\alpha|)} |\partial^\alpha u(x)|^2 dx \right)^{\frac{1}{2}} \qquad (m \in \mathbb{N}_0).$$

$H_{-\infty}(\Phi, \varphi)$ is the dual space of distributions under $H_\infty(\Phi, \varphi)$.

Proposition 10. *If $A = Op(a) \in OPS(\mu, \Phi, \varphi)$, then $A : H(\lambda\mu) \to H(\lambda)$ is a continuous operator, and there exist constants $C > 0$ and $N > 0$ such that*

$$\|A\|_{\mathcal{B}(H(\lambda\mu,\Phi,\varphi),H(\lambda,\Phi,\varphi)))} \le C \sum_{|\alpha|+|\beta|+|\gamma| \le N} n^\alpha_{\beta,\gamma}(a)$$

where herein and in what follows $\mathcal{B}(H_1, H_2)$ is the space of bounded linear operators acting from H_1 into H_2.

Example 11. Let $q(x)$ be the function from Example 2,

$$p(x,D) = \sum_{|\alpha| \le m} a_\alpha q^{m-|\alpha|} D^\alpha \qquad (k \ge 1)$$

where $a_\alpha \in C^\infty(\Omega)$ satisfies the estimates

$$|\partial^\beta a_\alpha(x)| \le c_{\alpha\beta} q(x)^{\delta|\beta|} \qquad (x \in \Omega, \delta \le \delta_0). \tag{5}$$

The differential operator $p(x,D)$ can be represented by means of construction (4) as a pseudodifferential operator with double symbol $p(x,y,\xi) = p(x,\xi)\chi(x,y) \in S(\lambda^m, \Phi, \varphi)$ where Φ and φ are the weight functions in Example 2 and $\lambda(x,\xi) = \left(1 + |\xi|^2 + q(x)^2\right)^{1/2}$. The differential operator $p(x,D)$ is bounded from $H(\lambda^s, \Phi, \varphi)$ into $H(\lambda^{s-m}, \Phi, \varphi)$.

Below we will formulate conditions for operators in $OPS(\lambda, \Phi, \varphi)$ to be compact or to be Fredholm (see [7, 8]).

Proposition 12. *Let* $\lambda, \mu \in O(\Phi, \varphi), a \in S(\mu, \Phi, \varphi)$ *and*

$$\lim_{\varepsilon \to 0} \inf_{(x,\xi) \in m(\varepsilon)} \pi(x, \xi) = \infty.$$

Then:

a) *If for all* α, β, γ

$$\lim_{\varepsilon \to 0} \sup_{(x,\xi) \in m(\varepsilon)} \left(\mu^{-1} \Phi^{|\alpha|} \varphi^{|\beta|+|\gamma|}\right)(x, \xi) \left| a_{(\beta,\gamma)}^{(\alpha)}(x, x, \xi) \right| = 0,$$

then $Op(a) : H(\lambda\mu, \Phi, \varphi) \to H(\lambda, \Phi, \varphi)$ *is a compact operator.*

b) *If*

$$\lim_{\varepsilon \to 0} \inf_{y \in \Omega, (x,\xi) \in m(\varepsilon)} \mu^{-1}(x, \xi) |a(x, x, \xi)| > 0, \tag{6}$$

then $Op(a) : H(\lambda\mu, \Phi, \varphi) \to H(\lambda, \Phi, \varphi)$ *is a Fredholm operator, and* $\ker Op(a)$ *and* $\operatorname{coker} Op(a)$ *are in* $H_\infty(\Phi, \varphi)$. *Moreover, if condition* (6) *holds, then the implication*

$$\left. \begin{array}{r} u \in H_{-\infty}(\Phi, \varphi) \\ Op(a)u \in H_\infty(\Phi, \varphi) \end{array} \right\} \implies u \in H_\infty(\Phi, \varphi)$$

holds.

Let $A = Op(a) \in S(\mu, \varphi, \Phi)$. Then A can be considered as an unbounded operator in $L_2(R^n)$ with domain $H(\mu, \varphi, \Phi)$.

Proposition 13 (see [7: p. 68]). *Let the conditions of Proposition* 12 *hold. Then:*

1) *A is closed.*

2) *If* $\mu(x, \xi) \to \infty$ *when* $|x| + |\xi| + \operatorname{dist}^{-1}(x, \partial\Omega) \to \infty$, *then the spectrum of A is either* $\mathbb{C}$ *or a discrete set. In the last case every point of the spectrum is an eigenvalue, the corresponding eigenspaces are finite-dimensional, and all eigenfunctions of A are in* $H_\infty(\Phi, \varphi)$.

3) *If A is a symmetric operator on* $H_\infty(\Phi, \varphi)$, *then A has a self-adjoint extension with a discrete spectrum.*

In what follows we need in a representation of pseudodifferential operators in the class $OPS(\lambda, \Phi, \varphi)$ as an oscillatory integral. We will suppose that the condition

$$\pi(x, \xi) = \Phi(x, \xi)\varphi(x, \xi) \geq C\langle\xi\rangle^\gamma \qquad (\gamma > 0) \tag{7}$$

holds.

Proposition 14. *Let*

$$L(\partial_y) = g(x,\xi)^{-1}\big(1 - R(x,\xi)^{-1}\Delta_y\big)$$

where $R(x,\xi) = \Phi(x,\xi)\varphi(x,\xi)^{-1}, g(x,\xi) = \big(1 + R(x,\xi)^{-1}|\xi|^2\big), p(x,y,\xi) \in S(\lambda,\Phi,\varphi)$, $u \in C_0^\infty(\Omega), \lambda(x,\xi) \le c\langle\xi\rangle^m$ *and condition* (7) *holds. Then*

$$\begin{aligned} Op(p)u(x) &= \int d\xi \int e^{-i(x-y,\xi)} p(x,y,\xi)u(y)\,dy \\ &= \tfrac{1}{(2\pi)^n} \iint e^{-i(x-y,\xi)} L(\partial_y)^N \big(p(x,y,\xi)u(y)\big) dy d\xi \\ &= \lim_{\varepsilon\to 0} \tfrac{1}{(2\pi)^n} \iint e^{-\varepsilon\xi^2} e^{-i(x-y,\xi)} p(x,y,\xi)u(y)\,dyd\xi \end{aligned}$$

where $2\gamma N \ge m + n + 1$.

Proof. Applying the Leibnitz-Hörmander formula for the differentiation of a product and estimates (7) we obtain

$$\begin{aligned} &\big|L^N(\partial_y)\{p(x,y,\xi)u(y)\}\big| \\ &\qquad \le d_N(x)\Bigg(\sum_{|\alpha|\le 2N} n^0_{0,\alpha}(p)\Bigg)\Bigg(\sum_{|\alpha|\le 2N} |u^{(\alpha)}(y)|\Bigg)(1+|\xi|^2)^{\frac{m}{2}-\gamma N} \end{aligned}$$

where $N \in \mathbb{N}$. This implies that $Op(p)u(x)$ can be written as an oscillatory integral

$$Op(p)u(x) = \tfrac{1}{(2\pi)^n} \iint e^{-i(x-y,\xi)} L(\partial_y)^N p(x,y,\xi)u(y)\,dyd\xi$$

where $2\gamma N \ge m + n + 1$. Let

$$J_\varepsilon(x) = \tfrac{1}{(2\pi)^n} \iint_{\Omega\times\mathbb{R}^n} e^{-\varepsilon\xi^2} e^{-i(x-y,\xi)} p(x,y,\xi)u(y)\,dyd\xi.$$

By integrating by parts herein we obtain

$$J_\varepsilon(x) = \tfrac{1}{(2\pi)^n} \iint_{\Omega\times\mathbb{R}^n} e^{-\varepsilon\xi^2} e^{-i(x-y,\xi)} L(\partial_y)^N p(x,y,\xi)u(y)\,dyd\xi$$

where the last integral is absolutely convergent uniformly with respect to $\varepsilon > 0$ if $2\gamma N > n + m + 1$. Thus if $2\gamma N > n + m + 1$, then we can pass to the limit in $J_\varepsilon(x)$ and obtain

$$Op(p)u(x) = \lim_{\varepsilon\to 0} J_\varepsilon(x) = \tfrac{1}{(2\pi)^n} \iint_{\Omega\times\mathbb{R}^n} e^{-i(x-y,\xi)} L(\partial_y)^N p(x,y,\xi)u(y)\,dyd\xi$$

what completes the proof. ∎

3. Pseudodifferential operators with analytic symbols in exponential weighted classes

Let $\alpha : \Omega \to \mathbb{R}_+$ be a positive continuous function. We set

$$W_\alpha(x) = \big\{\eta \in \mathbb{R}^n : |\eta| < \alpha(x)\big\}.$$

Definition 15. We say that $p \in S_{W_\alpha}(\lambda, \Phi, \varphi)$ if $p \in S(\lambda, \Phi, \varphi)$ and p satisfies the following condition: for any fixed point $(x, y) \in \Omega \times \Omega$ the function $p(x, y, \xi)$ has an analytic extension with respect to the variable ξ in the tube domain $\mathbb{R}^n_\xi + iW_\alpha(x)$ and, for all multi-indices α, β, γ,

$$\sup_{\substack{(x,y)\in\Omega\times\Omega \\ \zeta\in\mathbb{R}^n_\xi + iW_\alpha(x)}} \lambda(x,\xi)^{-1}\Phi^{|\alpha|}(x,\xi)\varphi^{|\beta|+|\gamma|}(x,\xi)\big|p^{(\alpha)}_{(\beta,\gamma)}(x,y,\zeta)\big| \le n^\alpha_{\beta,\gamma}(p) \tag{8}$$

where $\zeta = \xi + i\eta$. The best constants $n^\alpha_{\beta,\gamma}(p)$ in this estimates define the Fréchet space topology in $S_{W_\alpha}(\lambda, \Phi, \varphi)$.

The following proposition concerning the algebra $S_{W_\alpha}(\lambda, \Phi, \varphi))$ is similar to Propositions 7 and 9.

Proposition 16.

a) *Let $\lambda \in O(\Phi, \varphi)$ and $A = Op(a) \in OPS_{W_\alpha}(\lambda, \varphi, \Phi)$. Then for all $N > 0$*

$$A = Op\bigg\{\chi(x,y)\sum_{|\alpha|<N}\frac{1}{\alpha!}a^{(\alpha)}_{(0,\alpha)}(x,x,\xi) + a_N(x,y,\xi)\bigg\}$$

where $a_N \in S_{W_\alpha}(\lambda\pi^{-N}, \Phi, \varphi)$.

(b) *Let $a_1(x,y,\xi) \in S_{W_\alpha}(\lambda_1, \Phi, \varphi)$ and $a_2(x,y,\xi) \in S_{W_\alpha}(\lambda_2, \Phi, \varphi)$. Then $B = Op(a_1)Op(a_2) \in OPS_{W_\alpha}(\lambda_1\lambda_2, \Phi, \varphi)$, $B = Op(b)$ with*

$$b(x,y,\xi) = \sum_{|\beta|<N}\frac{1}{\beta!}B_\beta(x,y,\xi) + r_N(x,y,\xi) \qquad (N \in \mathbb{N})$$

where

$$B_\beta(x,y,\xi) = D^\beta_z\big(a^{(\beta)}_1(x,z,\xi)a_2(z,y,\xi)\big)\big|_{z=x} \quad \textit{and} \quad r_N \in S_{W_\alpha}(\lambda_1\lambda_2\pi^{-N}, \Phi, \varphi)$$

and for each α, β, γ there exist constants $C_{\alpha\beta\gamma} > 0$ and $M \in \mathbb{N}$ such that

$$n^\alpha_{\beta\gamma}(r_N) \le C_{\alpha\beta\gamma} \max_{\substack{N\le|\alpha'|\le M, |\alpha''|\le M \\ N\le|\beta'|+|\beta''|+|\gamma'|+|\gamma''|\le M}} n^{\alpha'}_{\beta'\gamma'}(a_1)\, n^{\alpha''}_{\beta''\gamma''}(a_2).$$

The proof of this propositions is similar to that of [7: Propositions 7 and 9]. Hence it is omitted.

Definition 17. Let Φ and φ be weight functions. We say that a weight $b(x) = \exp a(x) \in \Lambda(\Phi, \varphi)$ if $a \in C^\infty(\Omega)$ and for all multi-indices β

$$|\partial_x^\beta \nabla a(x)| \leq n_\beta(a)\Phi(x,0)\varphi^{-|\beta|}(x,0). \tag{9}$$

The best constants $n_\beta(a)$ herein define a topology in $\Lambda(\Phi, \varphi)$.

Let $a \in C^1(\Omega)$ and $x, y \in \Omega$. Then we set

$$g_a(x,y) = \int_0^1 (\nabla a)\big(x - t(x-y)\big)\,dt.$$

Proposition 18. *Suppose $p(x,y,\xi) \in S_{W_\alpha}(\lambda, \Phi, \varphi)$, $b(x) = \exp a(x) \in \Lambda(\Phi, \varphi)$ and $g_a(x,y) \in W_\alpha(x)$ if $(x,y) \in \operatorname{supp}_{xy} p$. Then the symbol $p\big(x,y,\xi \pm ig_a(x,y)\big) \in S(\lambda, \Phi, \varphi)$.*

Proof. We have

$$|\partial_x^\beta \partial_y^\gamma g_a(x,y)| = \left| \int_0^1 \partial_x^\beta \partial_y^\gamma \nabla a\big(x - t(x-y)\big)\,dt \right| \leq \sup_{t \in [0,1]} \left|\partial_x^\beta \partial_y^\gamma \nabla a\big(x - t(x-y)\big)\right|.$$

By Definition 15,

$$\left|\partial_x^\beta \partial_y^\gamma \nabla a(x - t(x-y))\,dt\right| \leq C_{\beta+\gamma} \frac{\Phi\big(x - t(x-y), 0\big)}{\varphi^{|\beta|+|\gamma|}\big(x - t(x-y), 0\big)},$$

and by Definition 1 we have

$$\begin{aligned} &\Phi\big(x - t(x-y), 0\big) \leq c\Phi(x,0) \\ &\varphi(x,0) \geq c^{-1}\varphi\big(x - t(x-y), 0\big) \end{aligned} \qquad \big(t \in [0,1], (x,y) \in \operatorname{supp}_{xy} p\big).$$

Thus

$$\left|\partial_x^\beta \partial_y^\gamma g_a(x,y)\right| \leq C_{\beta,\gamma} \frac{\Phi(x,0)}{\varphi^{|\beta|+|\gamma|}(x,0)} \leq C'_{\beta,\gamma} \frac{\Phi(x,\xi)}{\varphi^{|\beta|+|\gamma|}(x,\xi)}$$

if $(x,y) \in \operatorname{supp}_{xy} p$. The estimate

$$\left|\partial_x^\beta \partial_y^\gamma \partial_\xi^\alpha p\big(x,y,\xi \pm ig_a(x,y)\big)\right| \leq C_{\alpha\beta\gamma}(p,a)\lambda(x,\xi)\Phi(x,\xi)^{-|\alpha|}\varphi(x,\xi)^{-|\beta|-|\gamma|}$$

follows from both above given estimates. ∎

The following theorem is a key result for investigation of pseudodifferential operators of the class $S_{W_\alpha}(\lambda, \varphi, \Phi)$ in exponential weighted Sobolev spaces.

Theorem 19. *Let* $p(x,y,\xi) \in S_{W_\alpha}(\lambda,\varphi,\Phi)$, $b(x) \in \Lambda(\Phi,\varphi)$ *and* $g_a(x,y) \in W_\alpha(x)$ *if* $(x,y) \in \operatorname{supp}_{xy} p$. *Then the operator* $bOp(p)b^{-1} \in OPS(\lambda,\varphi,\Phi)$, *and*

$$bOp(p)b^{-1}u(x) = \tfrac{1}{(2\pi)^n} \int d\xi \int e^{-i(x-y,\xi)} p(x,y,\xi - ig_a(x,y))u(y)\,dy$$

where $u \in C_0^\infty(\Omega)$ *and* $p\big(x,y,\xi - ig_a(x,y)\big) \in S(\lambda,\Phi,\varphi)$.

Proof. The statement $p\big(x,y,\xi - ig_a(x,y)\big) \in S(\lambda,\varphi,\Phi)$ has been proved in Proposition 18. According to the Lagrange formula,

$$a(x) - a(y) = \big(x-y, g_a(x,y)\big).$$

Applying Proposition 14 we obtain the representation

$$bOp(p)b^{-1}u(x) = \lim_{\varepsilon\to 0} \tfrac{1}{(2\pi)^n} \iint e^{-i(x-y,\xi+ig_a(x,y))-\varepsilon\xi^2} p(x,y,\xi)u(y)\,dyd\xi.$$

For an arbitrary $\varepsilon > 0$ the integrand herein has a finite support with respect to y, and exponentially decreases with respect to ξ as $\xi \to \infty$. Hence, the integral can be written as

$$bOp(p)b^{-1}u(x) = \lim_{\varepsilon\to 0} \int_\Omega J(\varepsilon,x,y)u(y)\,dy$$

where

$$J(\varepsilon,x,y) = \int_{\mathbb{R}^n} p(x,y,\xi) e^{[-\varepsilon\xi^2 - i(x-y,\xi+ig_a(x,y))]} d\xi.$$

After the change of variables $\tau = \xi + ig_a(x,y)$ we obtain

$$J(\varepsilon,x,y) = \int_{\mathbb{R}^n + ig_a(x,y)} p\big(x,y,\tau - ig_a(x,y)\big) e^{[-\varepsilon(\tau - ig_a(x,y))^2 - i(x-y,\tau)]} d\tau. \quad (10)$$

Note that this change of variables is possible because $x,y \in \Omega$, $g_a(x,y) \in W_\alpha(x)$ on $\operatorname{supp}_{xy} p$ and $p(x,y,\zeta)$ is defined in the domain $\Omega \times \Omega \times (\mathbb{R}^n + iW_\alpha(x))$. Since $p(x,y,\zeta)$ is an analytic function with respect to ζ in $\mathbb{R}^n + iW_\alpha(x)$ and $g_a(x,y) \in W_\alpha(x)$, there exists a tube domain

$$T_{x,y,\varepsilon} = \mathbb{R}^n + i\big\{\eta \in \mathbb{R}^n : |(\eta, g_a(x,y))| < \varepsilon\big\}$$

where ε is a sufficiently small number so that $p\big(x,y,\tau - ig_a(x,y)\big)$ is an analytic function with respect to τ on $T_{x,y,\varepsilon}$. Thus the integrand in (10) is an analytic function with respect to τ in the layer $T_{x,y,\varepsilon}$ containing $\mathbb{R}^n$. Moreover, it is exponentially decreasing if $\operatorname{Re}\tau \to \infty$ uniformly with respect to $\operatorname{Im}\tau \in$

$\operatorname{Im} T_{x,y,\varepsilon}$. Thus we can use the Cauchy-Poincare theorem [14] and replace integration in (10) along the plane $\mathbb{R}^n - ig_a(x,y)$ by integration along $\mathbb{R}^n$ and write

$$J(\varepsilon, x, y) = \int_{\mathbb{R}^n} p\big(x, y, \tau - ig_a(x,y)\big) e^{[-\varepsilon(\tau - ig_a(x,y))^2 - i(x-y,\tau)]} d\tau.$$

Thus

$$bOp(p)b^{-1}u(x) =$$
$$\lim_{\varepsilon\to 0} \tfrac{1}{(2\pi)^n} \iint_{\Omega\times\mathbb{R}^n} e^{-i(x-y,\xi)-\varepsilon(\xi - ig_a(x,y))^2} p\big(x, y, \xi - ig_a(x,y)\big) u(y)\, dy d\xi.$$

A slight modification of Proposition 14 allow us to pass to the limit in that integral and to obtain

$$bOp(p)b^{-1}u(x) = \tfrac{1}{(2\pi)^n} \int d\xi \int e^{-i(x-y,\xi)} p\big(x, y, \xi - ig_a(x,y)\big) u(y)\, dy.$$

Thus the theorem is proved ∎

Corollary 20. *Let the conditions of Theorem* 19 *be fulfilled. Then for an arbitrary* $N > 0$

$$bOp(p)b^{-1} - Op\Big(\chi(x,y) \sum_{|\alpha|<N} \tfrac{1}{\alpha!} (p_b)^{(\alpha)}_{(0,\alpha)}\big(x, x, \xi - i\nabla a(x)\big)\Big) \in OpS(\lambda\pi^{-N}, \Phi, \varphi)$$

where

$$\chi(x,y) = \theta\Big(c^{-1}|x-y|\big(\varphi(x,0)^{-1} + \varphi(y,0)^{-1}\big)\Big)$$

for $\theta \in C_0^\infty(\mathbb{R})$ *with* $\theta(x) = 1$ *for all* x *in a neighborhood of the origin and* $c > 0$ *is a sufficiently small constant. In particular, for* $N = 1$ *we obtain*

$$bOp(p)b^{-1} - Op\Big(\chi(x,y) p\big(x, x, \xi - i\nabla a(x)\big)\Big) \in OPS(\lambda\pi^{-1}, \Phi, \varphi).$$

Corollary 20 follows from Proposition 7.

Corollary 21. *Let*

$$P = p(x, D) = \sum_{|\beta|\le m} p_\beta(x) D^\beta$$

be a differential operator with symbol $p(x,\xi) = \sum_{|\beta|\le m} p_\beta(x)\xi^\beta$ *satisfying estimates* (3)*. Then*

$$P = Op(p_c(x, y, \xi)) \in OPS_{W_\alpha}(\lambda, \Phi, \varphi)$$

for an arbitrary function α, where

$$p_c(x,y,\xi) = p(x,\xi)\theta\Big(c^{-1}(x-y)\big(\varphi(x,0)+\varphi(y,0)\big)^{-1}\Big)$$

for the above given $\theta \in C_0^\infty(\mathbb{R}^n)$.

If $b = \exp a \in \Lambda(\Phi,\varphi)$, then $bPb^{-1} \in OPS(\lambda,\Phi,\varphi)$ and

$$bPb^{-1} = Op\big(p_c\big(x,y,\xi - ig_a(x,y)\big)\big) = p\big(x, D - i\nabla a(x)\big) + T$$

where $T \in OPS(\lambda\pi^{-1},\Phi,\varphi)$ and

$$p\big(x, D - i\nabla a(x)\big)u(x) = \tfrac{1}{(2\pi)^n}\int d\xi \int p\big(x,\xi - i\nabla a(x)\big)u(y)e^{i(x-y,\xi)}dy$$

is a differential operator.

Example 22. Let $P = p(x,D)$ be the differential operator from Example 11. If $b = \exp a$ with $a(x) = Kq(x)^{1-\delta_0}$, then

$$|\partial^\alpha(\nabla a(x))| \le C_2 q(x)^{1+|\alpha|\delta_0} \le C_2\frac{\Phi(x,0)}{\varphi^{|\alpha|}(x,0)}.$$

Thus $b \in \Lambda(\Phi,\varphi)$ and

$$bPb^{-1} = p\Big(x, D - i(1-\delta_0)Kq(x)^{-\delta_0}\nabla q(x)\Big) + T$$

where $T \in OPS(\lambda\pi^{-1},\Phi,\varphi)$.

Let $b \in \Lambda(\Phi,\varphi)$ and $\lambda \in O(\Phi,\varphi)$. We say that the function $u \in H^b(\lambda,\Phi,\varphi)$ if

$$\|u\|_{H^b(\lambda,\Phi,\varphi)} = \|bu\|_{H(\lambda,\Phi,\varphi)} < \infty.$$

Let us denote

$$H^b_\infty(\Phi,\varphi) = \bigcap_{\lambda\in O(\Phi,\varphi)} H^b(\lambda,\Phi,\varphi) \quad \text{and} \quad H^b_{-\infty}(\Phi,\varphi) = \bigcup_{\lambda\in O(\Phi,\varphi)} H^b(\lambda,\Phi,\varphi).$$

Since the multiplication by the weight b is an isomorphism of $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$, we have the dense imbeddings

$$\mathcal{D}(\Omega) \subset H^b_\infty(\Phi,\varphi) \subset H^b_{-\infty}(\Phi,\varphi) \subset \mathcal{D}'(\Omega). \tag{11}$$

Proposition 23. *Let $P = Op(p(x,y,\xi))$ where $p(x,y,\xi) \in S_{W_\alpha}(\lambda,\Phi,\varphi)$ and $b = \exp a \in \Lambda(\Phi,\varphi)$ with $g_a(x,y) \in W_\alpha(x)$ if $(x,y) \in \operatorname{supp}_{xy} p$. Then $P : H^b(\lambda\mu,\Phi,\varphi) \to H^b(\mu,\Phi,\varphi)$ is a bounded operator, and there exist constants $C > 0$ and $M \in \mathbb{N}$ such that*

$$\|P\|_{\mathcal{B}(H^b(\lambda\mu,\Phi,\varphi),H^b(\lambda,\Phi,\varphi))} \le C\sum_{|\alpha|+|\beta|+|\gamma|\le M} n^\alpha_{\beta\gamma}(p)\sum_{|\alpha|\le M} n_\alpha(a)$$

where $n^\alpha_{\beta\gamma}(p)$ are the constants in (8) and $n_\alpha(a)$ are the constants in (9).

The proof follows immediately from Theorem 19.

Theorem 24. *Let* $\lambda, \mu \in O(\Phi,\varphi), p(x,y,\xi) \in S_{W_\alpha}(\mu,\Phi,\varphi)$ *and* $b(x) = \exp a(x) \in \Lambda(\Phi,\varphi)$ *with* $g_a(x,y) \in W_\alpha(x)$ *if* $(x,y) \in \operatorname{supp}_{xy} p$. *Then*

$$P = Op(p) : \, H^b(\lambda\mu,\Phi,\varphi) \to H^b(\lambda,\Phi,\varphi)$$

is a Fredholm operator if

$$\lim_{\varepsilon\to 0} \inf_{(x,\xi)\in m(\varepsilon)} \mu^{-1}(x,\xi)\big|p\big(x,x,\xi - i(\nabla a)(x)\big)\big| > 0. \tag{11}$$

If this condition is fulfilled, then $\ker P$ *and* $\operatorname{coker} P$ *belong* $H^b_\infty(\Phi,\varphi)$. *Moreover, the implication*

$$\left.\begin{aligned} u \in H^b_{-\infty}(\Phi,\varphi) \\ Pu \in H^b_\infty(\Phi,\varphi) \end{aligned}\right\} \quad\Longrightarrow\quad u \in H^b_\infty(\Phi,\varphi) \tag{12}$$

holds.

Proof. It is evident that $Op(p) : \, H^b(\lambda\mu,\Phi,\varphi) \to H^b(\mu,\Phi,\varphi)$ is a Fredholm operator if and only if $bOp(p)b^{-1} : \, H(\lambda\mu,\Phi,\varphi) \to H(\mu,\Phi,\varphi)$ is so. According to Theorem 19, $bOp(p)b^{-1} \in OPS(\lambda,\Phi,\varphi)$. Moreover, by Corollary 20 the operator

$$bOp(p)b^{-1} - Op\Big(\theta(x,y)p\big(x,x,\xi - i\nabla a(x)\big)\Big) \in S(\lambda\pi^{-1},\Phi,\varphi)$$

and therefore, according to Proposition 12/a), this operator is compact from $H(\lambda\mu)$ into $H(\mu)$. Applying Proposition 12/b) we obtain that P is a Fredholm operator and the implication stated holds.

Corollary 25. *The differential operator* $p(x,D) : \, H^b(\lambda\mu,\Phi,\varphi) \to H^b(\mu,\Phi,\varphi)$ *is a Fredholm operator if*

$$\lim_{\varepsilon\to 0} \inf_{(x,\xi)\in m(\varepsilon)} \lambda(x,\xi)^{-1}\big|p(x,\xi - i\nabla a(x)\big| > 0. \tag{13}$$

If this condition holds, then

$$\ker a(x,D) \subset H_\infty(\Phi,\varphi).$$

Example 26. Let $(-\Delta + q^{2k})^m \quad (m \in \mathbb{N}, k > 1)$ where q is the same as in Example 2. Let

$$\Phi(x,\xi) = \lambda(x,\xi) = (1 + |\xi|^2 + q(x)^{2k})^{\frac{1}{2}},$$

$\varphi(x,\xi) = q(x)^{-\delta_0}$, $\lambda(x,\xi) = \big(1 + |\xi|^2 + q(x)^{2k}\big)^{m/2}$ and $b(x) = \exp Kq_0(x)^{k-\delta_0}$ $(\in \Lambda(\Phi,\varphi))$. Then condition (13) is fulfilled if

$$|KM(k-\delta_0)| < 1 \qquad \text{with } M = \sup_{x\in\Omega} \frac{|\nabla q|}{q^{1+\delta_0}}.$$

4. Exponential estimates of solutions of differential equations

In this section we suppose that $\Omega = \mathbb{R}^n \backslash \mathcal{M}$ where $\mathcal{M}$ is a compact closed manifold in $\mathbb{R}^n$ of dimension $n-1$, Φ and φ are weight functions in Ω and $\mu \in O(\Phi, \varphi)$. We consider the partial differential equation

$$P = p(x,D)u(x) = f(x) \qquad \big(x \in \Omega, p(x,\xi) \in S(\mu, \Phi, \varphi)\big) \tag{14}$$

with coefficients which can have discontinuities on $\mathcal{M}$ and growth at infinity.

Let the weight $b \in \Lambda(\Phi, \varphi)$ be such that $\lim_{x\to\infty} b(x) = \infty$ and $\lim_{x\to M} b(x) = \infty$. If condition (12) of Theorem 24 is fulfilled, then implication (13) holds, but below we are interested in a more strong result, namely, the implication

$$\left.\begin{array}{r} u \in H^{b^{-1}}_{-\infty}(\Phi,\varphi) \\ Pu \in H^{b}_{\infty}(\Phi,\varphi) \end{array}\right\} \quad \Longrightarrow \quad u \in H^{b}_{\infty}(\Phi,\varphi).$$

The main result of this paper is the following theorem on exponential decay of solutions of equation (14) near singularities of coefficients and at infinity.

Theorem 27. *Let $P = p(x,D) \in S(\lambda, \Phi, \varphi)$ be a partial differential operator and*

$$\lim_{\varepsilon\to 0} \inf_{(x,\xi)\in m(\varepsilon), \eta \in W_\alpha} \lambda^{-1}(x,\xi)|p(x,\xi + i\eta)| > 0. \tag{15}$$

Let $b(x) = \exp a(x) \in \Lambda(\Phi,\varphi)$ and $\nabla a(x) \in W_\alpha(x)$ if $x \in \Omega$. Then:

1) *$P : H_{b^t}(\mu,\Phi,\varphi) \to H_{b^t}(\lambda^{-1}\mu,\Phi,\varphi)$ is a Fredholm operator for all $t \in [-1,1]$ with index independent of t.*

2) *The implication*

$$\left.\begin{array}{r} u \in H^{b^{-1}}(\lambda\mu,\Phi,\varphi) \\ Pu \in H^{b}(\mu,\Phi,\varphi) \end{array}\right\} \quad \Longrightarrow \quad u \in H^{b}(\lambda\mu,\Phi,\varphi)$$

holds.

Proof. The operator b^tPb^{-t} can be written as a pseudodifferential operator with double symbol of the form

$$b^tPb^{-t} = Op\Big(p\big(x, \xi - itg_a(x,y)\big)\theta\big(c^{-1}(x-y)(\varphi(x)^{-1} + \varphi(y)^{-1})\big)\Big).$$

It follows from condition (16) and Theorem 24 that

$$b^tPb^{-t} : H_{b^t}(\mu,\Phi,\varphi) \to H_{b^t}(\lambda^{-1}\mu,\Phi,\varphi)$$

is a Fredholm operator for all $t \in [-1,1]$. By Proposition 10, the function

$$t \to b^t P b^{-t} : \, H_{b^t}(\mu, \Phi, \varphi) \to H_{b^t}(\lambda^{-1}\mu, \Phi, \varphi)$$

is continuous, so that the index $b^t P b^{-t}$ does not depend on $t \in [-1,1]$. It follows from (11) that $H^b(\mu, \Phi, \varphi) \subset H^{b^{-1}}(\mu, \Phi, \varphi)$, and this embedding is dense. Then the kernels

$$\begin{aligned} &\ker A : \, H^b(\mu, \Phi, \varphi) \to H^b(\lambda^{-1}\mu, \Phi, \varphi) \\ &\ker A : \, H^{b^{-1}}(\mu, \Phi, \varphi) \to H^{b^{-1}}(\lambda^{-1}\mu, \Phi, \varphi) \end{aligned}$$

coincide [5: p. 308]. Moreover, if the equation $Au = f$ with $f \in H^b(\lambda^{-1}\mu, \Phi, \varphi)$ is solvable in $H^{b^{-1}}(\mu, \Phi, \varphi)$, then $u \in H^b(\mu, \Phi, \varphi)$ $\blacksquare$

Corollary 28. *Let the conditions of Theorem* 27 *be fulfilled. Then the implication*

$$\left.\begin{aligned} u \in H^{b^{-1}}_{-\infty}(\Phi, \varphi) \\ Pu \in H^b_\infty(\Phi, \varphi) \end{aligned}\right\} \quad \Longrightarrow \quad u \in H^b_\infty(\Phi, \varphi)$$

holds.

Proof. If $u \in H^{b^{-1}}_{-\infty}(\Phi, \varphi)$, then there exists $\mu \in O(\Phi, \varphi)$ such that $u \in H^{b^{-1}}(\lambda\mu)$. By Theorem 27, $u \in H^b_\infty(\lambda\mu)$. Consequently, $u \in H^b_\infty(\Phi, \varphi)$ $\blacksquare$

5. Estimates of eigenfunctions of the Schrödinger operator

5.1 Decreasing potentials. Let us consider the Schrödinger operator with potential v

$$A = -\Delta + v$$

where v is a real-valued C^∞-function satisfying the estimates $|\partial_x^\alpha v(x)| \leq c_\alpha \langle x \rangle^{-|\alpha|}$ and $\lim_{x\to\infty} v(x) = 0$. The operator A considered as unbounded in $L_2(\mathbb{R}^n)$ is selfadjont with domain $H^2(\mathbb{R}^n)$ and it has a discrete spectrum distributed on $(-\infty, 0)$. Let u_j be an eigenfunction with eigenvalue $-\mu_j^2, \mu_j > 0$. The operator A is included in the usual algebra of pseudodifferential operators with weight functions $\Phi(x,\xi) = \langle \xi \rangle, \varphi(x,\xi) = \langle x \rangle$ and $\lambda(x,\xi) = \langle \xi \rangle^2$.

Let $W_{\mu_j} = \{\eta \in \mathbb{R}^n : \, |\eta| < \mu_j\}$. Then

$$\lim_{R\to\infty} \inf_{|x| \geq R, \zeta \in \mathbb{R}^n + iW_{\mu_j}} \left|\zeta^2 + \mu_j^2 + v(x)\right| \langle \xi \rangle^{-2} > 0.$$

Thus the weight $a(x) = e^{\kappa\langle x\rangle}$ $(0 < \kappa < \mu_j)$ and the operator $A + \mu_j^2$ satisfy all conditions of Theorem 27. Thus if u_j is an eigenfunction of A with eigenvalue $-\mu_j^2$, then $|\partial_x^\alpha u_j(x)| \le C_\alpha e^{-\kappa\langle x\rangle}$ $(0 < \kappa < \mu_j)$ for all α.

5.2 Potentials of power growth at infinity. 5.2.2 Estimates for eigenfunctions. Let the potential $v(x) \ge 0$ and

$$v(x) = \langle x\rangle^{2k}(1 + r(x)) \qquad (k > 0)$$

where r satisfies the estimates

$$|\partial_x^\alpha r(x)| \le c_\alpha \langle x\rangle^{-|\alpha|} \qquad \text{and} \qquad \lim_{x\to\infty} r(x) = 0. \tag{16}$$

The operator A is included in the usual algebra of pseudodifferential operators with weight functions

$$\begin{aligned}\Phi(x,\xi) &= (1 + |\xi|^2 + |x|^{2k})^{\frac{1}{2}}\\ \varphi(x,\xi) &= (1 + |\xi|^2 + |x|^{2k})^{\frac{1}{2k}}\\ \lambda(x,\xi) &= \Phi(x,\xi)^2.\end{aligned}$$

The operator A considered as unbounded with domain $H(\lambda, \Phi, \varphi)$ is self-adjoint and has a discrete spectrum. Let us introduce a weight $b(x) = \exp \kappa\langle x\rangle^{k+1}$. It is easy to see that the weight $a \in \Lambda(\Phi, \varphi)$. Let $W_\alpha = \{\eta \in \mathbb{R}^n : |\eta| < |x|^k\}$. Then

$$\lim_{R\to\infty} \inf_{|x|\ge R, \zeta\in\mathbb{R}^n + iW_{\alpha\mu}} |\zeta^2 + v(x) - \mu|\lambda(x,\xi)^{-1} > 0.$$

Thus the weight $\exp \kappa\langle x\rangle^{k+1}$ $(0 < \kappa < \frac{1}{k+1})$ and the operator $A - \mu$ $(\mu \in \mathbb{R})$ satisfy all conditions of Theorem 27. Thus if u_μ is an eigenfunction of A with eigenvalue $\mu \in \mathbb{R}$, then $u_\mu \in H^\infty(\Phi, \varphi, e^{\kappa\langle x\rangle^{k+1}})$ where $0 < \kappa < \frac{1}{k+1}$. By the embedding theorem, $u \in C^\infty(\mathbb{R}^n)$ and, for all α and $m \in \mathbb{N}$,

$$|\partial_x^\alpha u_\mu(x)| \le C_\alpha e^{-\kappa\langle x\rangle^{k+1}} \qquad (0 < \kappa < \tfrac{1}{k+1}).$$

For $k = 2$ we obtain the estimate of eigenfunctions of the perturbed Harmonic oscillator $-\Delta + |x|^2(1 + r(x))$ where r satisfies conditions (16). The estimate for eigenfunctions

$$|\partial_x^\alpha u_\mu(x)| \le C_\alpha e^{-(\frac{1}{2}-\varepsilon)\langle x\rangle^2}$$

holds where $\varepsilon > 0$ is arbitrary. This estimate is well-known for the Harmonic oscillator (see [3]).

5.2.2 Analytic properties of eigenfunctions. Let us consider the differential operator

$$A_{k,l} = \big((-1)^l \Delta^l + |x|^{2k}\big)^m \qquad (m, l \in \mathbb{N}, k > 0).$$

This operator is included in the algebra of pseudodifferential operators in the class $OPS(\lambda, \Phi, \varphi)$ where

$$\begin{aligned}
\Phi(x,\xi) &= (1 + |\xi|^{2l} + |x|^{2k})^{\frac{1}{2}} \\
\varphi(x,\xi) &= (1 + |\xi|^{2l} + |x|^{2k})^{\frac{1}{2k}} \\
\lambda(x,\xi) &= (1 + |\xi|^{2l} + |x|^{2k})^{m}.
\end{aligned}$$

Then $A_{k,l}$ with domain $H(\lambda, \Phi, \varphi)$ is self-adjoint on $L_2(\mathbb{R}^n)$ and has a discrete spectrum. Let $b(x) = \exp \kappa \langle x \rangle^p$ $\;(p = \frac{k}{l} + 1)$, with $0 < \kappa < p$ if l is odd and $\mathfrak{k} > 0$ arbitrary if l is even. Then all eigenfunctions u of A are in $H_b^\infty(\Phi, \varphi)$. This implies

$$|\partial^\alpha u(x)| \le C_\alpha e^{-\kappa} \langle x \rangle^p \qquad (0 < \kappa < p)$$

for all α. Let us consider the analytic properties of eigenfunctions of $A_{k,l}$ in the case when $k, l \in \mathbb{N}$. We set $\hat{A}_{k,l} = F A_{k,l} F^{-1}$ where $\hat{u}(\xi) = (Fu)(\xi) = \int_{\mathbb{R}^n} u(x) e^{-i(x,\xi)} dx$ is the Fourier transform. It is evident that $\hat{A}_{k,l} = A_{l,k}$. Thus if u is an eigenfunction of $A_{k,l}$, then $\hat{u}$ is an eigenfunction of $\hat{A}_{k,l} = A_{l,k}$ satisfying the estimates $|\partial_\xi^\alpha \hat{u}(\xi)| \le C_\alpha e^{-a' \langle \xi \rangle^{p'}}$ $\;(p' = 1 + \frac{l}{k})$ for all α, where $0 < a' < \frac{1}{p'}$ if k is odd and $a' > 0$ arbitrary if k is even. These estimates imply (see, for instance, [14]) that u can be analytically extended as an entire function $u(x + iy)$ on $\mathbb{C}^n$ and estimates

$$|\partial_x^\alpha u(x + iy)| \le C_{\alpha'} e^{(a+\varepsilon)\langle y \rangle^p} \qquad (z = x + iy \in \mathbb{C}^n)$$

hold for all α where a satisfies the equation $(p'a')^p (pa)^{p'} = 1$ and $\varepsilon > 0$ is arbitrary.

Example 29. The eigenfunctions u of the Harmonic oscillator have an analytic extension on $\mathbb{C}^n$ as entire functions satisfying the estimates

$$|\partial_x^\alpha u(x + iy)| \le C'_\alpha e^{(\frac{1}{2}+\varepsilon)\langle y \rangle^2} \qquad (z = x + iy \in \mathbb{C}^n, \varepsilon > 0).$$

5.3 Super-exponential growth of potentials at infinity. Let $q \in C^\infty(\mathbb{R}^n)$ be a positive function satisfying the estimates

$$|\partial_x^\alpha q(x)| \le c_\alpha q(x)^{1+\delta_0|\alpha|} \qquad (\delta_0 < 1)$$

and $v(x) = q^2(x)(1 + r(x))$ where r is real-valued and satisfies estimates (16). We consider the Schrödinger operator with potential v and assume

$$\Phi(x,\xi) = \left(1 + |\xi|^2 + q(x)^2\right)^{\frac{1}{2}}$$
$$\varphi(x,\xi) = \left(1 + |\xi|^2 + q(x)\right)^{-\frac{\delta_0}{2}}$$
$$\lambda(x,\xi) = \Phi(x,\xi)^2.$$

Let $W_q(x) = \{\eta \in \mathbb{R}^n : |\eta| < q(x)\}$. Then

$$\lim_{R\to\infty} \inf_{|x|\geq R, \zeta \in \mathbb{R}^n + iW_q(x)} |\zeta^2 + v(x) - \mu| \lambda(x,\xi)^{-1} > 0 \qquad (\mu \in \mathbb{C}).$$

Let the weight $b(x) = \exp K q_0(x)^{1-\delta_0} \in \Lambda(\Phi,\varphi)$, and let $K(1-\delta_0) < 1$. Then the weight b and the operator $A - \mu \quad (\mu \in \mathbb{C})$ satisfy all conditions of Theorem 27. Thus if u_μ is an eigenfunction of A with eigenvalue $\mu \in \mathbb{C}$, then $u_\mu \in H_b^\infty(\Phi,\varphi)$ where $0 < K < \frac{1}{M(1-\delta_0)}$ with $M = \sup \frac{|\nabla q(x)|}{q(x)^{1+\delta_0}}$. By the embedding theorem,

$$|\partial_x^\alpha u_\mu(x)| \leq C_\alpha e^{-Kq(x)^{1-\delta_0}} \qquad (0 < K < \tfrac{1}{(1-\delta_0)M})$$

for all α and $m \in \mathbb{N}$.

Example 30. Let $q(x) = \exp\langle x\rangle^k \quad (k > 0)$. Thus eigenfunctions of $-\Delta + q(x)^2$ have the estimates

$$|\partial_x^\alpha u_\mu(x)| \leq C_\alpha \exp\left(K \exp((1-\delta_0)\langle x\rangle^k)\right) \qquad (K < 1)$$

where $\delta_0 = 0$ if $k \leq 1$ and $\delta_0 > 0$ is arbitrary small if $k > 1$.

5.4 Potentials with discontinuities on submanifolds. Let $\mathcal{M}$ be a closed submanifold in $\mathbb{R}^n$ of dimension $n-1$, $d_{\mathcal{M}}(x)$ be the regularized distance from the point $x \in \mathbb{R}^n$ to the manifold $\mathcal{M}$, that is $d_{\mathcal{M}} \in C^\infty(\mathbb{R}^n \setminus \mathcal{M})$ satisfies the estimates

$$c_1 \mathrm{dist}\,(x,\mathcal{M}) \leq d_{\mathcal{M}}(x) \leq c_2 \mathrm{dist}\,(x,\mathcal{M})$$
$$|\partial_x^\alpha d_{\mathcal{M}}(x)| \leq c_\alpha d_{\mathcal{M}}(x)^{1+|\alpha|} \quad (x \in \mathbb{R}^n \setminus \mathcal{M}).$$

Let

$$q(x) = d_{\mathcal{M}}(x)^{-\gamma} + q_0(x) \qquad (\gamma > 1) \tag{18}$$

where $q_0 \in C^\infty(\mathbb{R}^n)$ with $q_0(x) \geq 0$, $\lim_{x\to\infty} q_0(x) = \infty$ and

$$|\partial_x^\alpha q_0(x)| \leq c_\alpha q_0(x)^{1+\delta_0|\alpha|} \qquad (\delta_0 < 1).$$

We introduce the weight

$$b(x) = \exp K_{\mathcal{M}}\omega_{\mathcal{M}}(x)d_{\mathcal{M}}(x)^{-\gamma+1} + K_\infty\omega_\infty(x)q_0(x)^{1-\delta_0} \tag{19}$$

where $\omega_{\mathcal{M}} \in C_0^\infty(\mathbb{R}^n)$, with $\omega_{\mathcal{M}}(x) = 1$ in a small neighborhood of $\mathcal{M}$, $\omega_\infty \in C^\infty(\mathbb{R}^n)$, $\omega_\infty(x) = 1$ if $|x| \geq 2R$ and $\omega_\infty(x) = 0$ if $|x| \leq R$ where R is sufficiently large. Let $A = -\Delta + q(x)^2$ be the Schrödinger operator where q is given by (18). Then, if $\Phi(x,\xi), \varphi(x,\xi), \lambda(x,\xi)$ are given by (19), then A with domain $H(\lambda,\Phi,\varphi)$ is a self-adjoint operator on $L_2(\mathbb{R}^n)$ with a discrete spectrum and all eigenfunctions u belong to the space $H_b^\infty(\Phi,\varphi)$ where b is given by (20) and

$$0 < K_{\mathcal{M}} < \frac{1}{(\gamma-1)M_d} \qquad \text{and} \qquad 0 < K_\infty < \frac{1}{(1-\delta_0)M_{q_0}}$$

with

$$M_d = \lim_{\varepsilon\to 0} \sup_{\mathrm{dist}(x,\mathcal{M})<\varepsilon} \frac{|\nabla d_{\mathcal{M}}(x)|}{d_{\mathcal{M}}(x)^2} \qquad \text{and} \qquad M_{q_0} = \lim_{R\to\infty} \sup_{|x|>R} \frac{|\nabla q_0(x)|}{q_0(x)^{1-\delta_0}}.$$

By the embedding theorem,

$$\partial_x^\alpha u(x) = \begin{cases} O\big(\exp\big(-K_{\mathcal{M}}d_{\mathcal{M}}(x)^{-\gamma+1}\big)\big) & (x\to\mathcal{M}, 0 < K_{\mathcal{M}} < \frac{1}{(\gamma-1)M_d}) \\ O\big(\exp(-K_\infty q_0(x)^{1-\delta_0})\big) & (x\to\infty, 0 < K_\infty < \frac{1}{(1-\delta_0)M_{q_0}}) \end{cases}$$

for all α and $m \in \mathbb{N}$.

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