Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 2, 431–464

A Unified Approach to Nonlinear Integro-Differential Inverse Problems of Parabolic Type

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Abstract. We give a unified approach to a class of nonlinear parabolic inverse problems involving kernels of convolution type. Our main tools are optimal regularity results, in Sobolev and Hölder spaces, for parabolic equations and analytic semigroup theory. We apply the main abstract results (Theorems 2.1 - 2.2) to a model of population dynamics, to the theory of combustion of a material with memory and, finally, to a parabolic equation with elliptic part of order $2m$, which for $m = 1$ is the heat equation with memory and with non-linearity containg derivatives up to order $2m - 1$.

Keywords: *Analytic semigroup theory, optimal regularity results, nonlinear parabolic inverse problems*

AMS subject classification: Primary 35R30, 45K05, secondary 45N05, 80A20

1. Introduction and basic notation

The aim of this paper is to give an abstract unified approach which leads to generalizations of some results already known and to face new inverse problems not yet investigated. Our abstract theory allows a wide range of applications to the study of heat propagation in materials with memory, mathematical models arising in biology, theory of combustion of a material with memory, parabolic models in viscoelasticity, and many other fields in science and technology (see, for some examples, Section 3). We can apply our results also to parabolic inverse problems concerning spatial differential operators of order $2m$.

More precisely, we deal with a direct fully nonlinear problem of parabolic

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ISSN 0232-2064 / $ 2.50

type when we consider the problem

$$\left.\begin{aligned} &u'(t) = G(u(t)) + \int_0^t h(t-s)F(u(t),u(s))\,ds + f(t) \quad (t \in [0,T]) \\ &u(0) = u_0 \end{aligned}\right\} \tag{1.1}$$

in which the term h is a given function, G and F are suitable nonlinear functions on the domains D and $D \times D$, respectively, where D is a Banach space continuously embedded into a reference Banach space X. Their properties and regularity will be specified in the sequel; the main assumption is that $G'(u_0)$ (the Fréchet derivative of G in $u_0 \in D$) is supposed to be the generator of an analytic semigroup.

Problem (1.1) is well studied in literature (see, for example, [18]). The convolution kernel h, depending only on t, has several meanings, according to the problem we are considering. For example, h represents the thermal memory in problems of heat propagation, or the mechanism of spread of infections in population dynamics models. In practice, h can be seldom considered a known term, because it is not directly measurable. So we have to face the problem to reconstruct also h together with u.

In the following we investigate the inverse problem related to a Banach space X to determine two functions

$$u : [0,T] \to X$$
$$h : [0,T] \to \mathbb{R}$$

satisfying system (1.1) with the additional condition

$$\Phi(u(t)) = g(t) \qquad (t \in [0,T]) \tag{1.2}$$

where Φ is a linear and bounded functional acting on X and g is a given function. As a rule, in applications Φ has an integral representation, and physically it represents the additional measurements on u to determine the function h.

The functional setting is referred to Hölder and Sobolev spaces of fractional order denoted by $W^{\beta,p}(0,T;X)$ where $\beta \in (0,1) \setminus \{\frac{1}{p}\}$ with $1 < p \le \infty$ and X is a Banach space. Recalling that $W^{\beta,\infty}(0,T;X)$ coincides with the space of Hölder continuous functions $C^\beta([0,T];X)$, we give a unified approach to Sobolev ($1 < p < \infty$) and Hölder ($p = \infty$) spaces, using as fundamental tools optimal regularity results for parabolic equations in those spaces, analytic semigroup theory and fixed point arguments. We recall that maximal regularity results establish the existence of linear and topological isomorphisms between suitable spaces of solutions and data (see [17]). The use of Sobolev spaces allows to lower the order of regularity required to the data in order to get a solution. Moreover, concerning optimal regularity, fractional Sobolev spaces allow a relatively easy theory, if compared with spaces of integer order (see, for example, [7]).

The plan of the paper is the following:

Section 1 contains basic notations. In Section 2 we have put the main abstract results of the paper, namely Theorems 2.1 and 2.2. Theorem 2.1 establishes the existence of a local solution of problem (1.1) - (1.2) in $W^{\beta,p}(0,\tau;D)\cap W^{1+\beta,p}(0,\tau;X)$ for fixed $\beta\in(0,1)$ and $p\in[1,+\infty]$ and some $\tau>0$, under minimal conditions on the data. We obtain such result employing a certain optimal regularity result (Theorem 4.1) valid for abstract linear parabolic problems. Theorem 2.2 is a result of global uniqueness. The following Corollary 2.1 consists in a simple consequence of Theorems 2.1 and 2.2. This is the result we employ the most in the applications. In Section 3 we have put three possible applications of the results of the previous section. The two first applications concern population dynamics and combustion of materials with memory, respectively. The third application treats fully nonlinear parabolic problems of higher order in the space variables. Section 4 contains a series of technical lemmata, which are useful in the sequel. In Section 5 we prove the results stated in the second section. Finally, the Appendix contains a detailed proof of a new (in our knowledge) optimal regularity result in the framework of fractional order Sobolev spaces. An alternative proof can be deduced also from the results of [6].

In our knowledge the first paper applying maximal regularity techniques to intergro-differential parabolic inverse problems is [16]. But therein the authors considered only linear problems in spaces of functions which are Hölder continuous in the time variable. A different functional setting for similar problems was adopted in [15].

Fully nonlinear parabolic problems are considered also in [10]. The intersection between the class of problems that the authors considered and the class of problems which is the subject of the present paper is given by problems like (1.1) - (1.2) with F depending only on $u(s)$. The functional setting is a little different and the authors in [10] find only solutions such that $u\in W^{\beta,p}(0,\tau;X)$, with β suitably small.

Other authors consider different situations and obtain results which are not comparable with ours. For example, using Laplace transform methods important results are due to J. Janno and L. von Wolfersdorf (see, for example, [13, 22, 23] and the bibliography therein). The case of h depending also on some space variables (linear equations) is treated in [3, 4], the case of a linear problem with Φ possibly nonlinear and depending also on h is treated in [9].

Now we give some basic notation that we shall use in the sequel.

If X and Y are normed spaces, we indicate with $\mathcal{L}(X,Y)$ the normed space of bounded linear operators from X to Y. If $X=Y$, we simply write $\mathcal{L}(X)$. The dual space of X is denoted by X'. If A is a linear operator in a Banach space X, we denote with $\rho(A)$ the resolvent set of A. If $p\in[1,+\infty)$, we set $p'=\frac{p}{p-1}$. We indicate with $\mathbb{N}$ the set of positive integers, i.e. naturals. The symbol $*$ is used to denote the convolution with respect to time. If $\tau\in\mathbb{R}^+$, the set of positive reals, we set

$$\Delta_\tau=\big\{(t,s)\in\mathbb{R}^2 : 0<s<t\le\tau\big\}.$$

Given a function F depending on (x,y), we indicate with d_1F and d_2F its partial derivatives with respect to x and y, respectively. If $k\in\mathbb{N}$, we indicate with $\mathcal{C}^k(\mathbb{R})$ the set of real-valued functions on $\mathbb{R}$, which are continuous together with their derivatives of order less or equal to k. If Ω is a bounded open subset of $\mathbb{R}^n$, we mention the Besov spaces $B^\theta_{p,q}(\Omega)$. For their definition and properties see, for example, [2, 21].

2. Main abstract results

Let D and X be Banach spaces with norms $\|\cdot\|_D$ and $\|\cdot\|$, respectively, $D\subseteq X$ and continuous embedded, and $A\in\mathcal{L}(D,X)$. We shall mainly think of A as an unbounded linear operator in X. Assume the following:

(h1) There exist $R_0>0$ and $M>0$ such that $\Sigma:=\big\{\lambda\in\mathbb{C} : |\lambda|\ge R_0 \text{ and } |\mathrm{Arg}\lambda|\le\frac{\pi}{2}\big\}\subseteq\rho(A)$. Moreover, for every $\lambda\in\Sigma$, $\|(\lambda-A)^{-1}\|_{\mathcal{L}(X)}\le M|\lambda|^{-1}$.

It is well known (see, for example, [17: Chapter 2]) that, if condition (h1) is satisfied, then A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\ge0}$, possibly not strongly continuous in 0.

Let now $\alpha\in(0,1)$ and $p\in(1,+\infty]$. We set

$$D_A(\alpha,p)=\Big\{x\in X : t\to v(t)=\|t^{1-\alpha-\frac{1}{p}}AS(t)x\|\in L^p(0,1)\Big\} \tag{2.1}$$

and, if $x\in D_A(\alpha,p)$,

$$[x]_{D_A(\alpha,p)}=\|v\|_{L^p(0,1)} \tag{2.2}$$
$$\|x\|_{D_A(\alpha,p)}=\|x\|+[x]_{D_A(\alpha,p)}. \tag{2.3}$$

$D_A(\alpha,p)$ is a Banach space with the norm $\|\cdot\|_{D_A(\alpha,p)}$ and coincides up to equivalent norms with the real interpolation space $(X,D)_{\alpha,p}$ (see [17: Proposition 2.2.2]).

We introduce now for $-\infty < a < b < +\infty, \beta \in (0,1)$ and $p \in [1,+\infty]$ the spaces $W^{\beta,p}(a,b;X)$. Namely, we set

$$W^{\beta,p}(a,b;X) = \left\{ f \in L^p(a,b;X) \;\middle|\; \begin{array}{ll} \int_a^b \big(\int_a^t \frac{\|f(t)-f(s)\|^p}{(t-s)^{1+\beta p}} ds\big) dt < \infty & \text{if } p < +\infty \\ \sup\limits_{a \le s < t \le b} \frac{\|f(t)-f(s)\|}{(t-s)^\beta} < \infty & \text{if } p = +\infty \end{array} \right\}. \tag{2.4}$$

Observe that the space $W^{\beta,\infty}(a,b;X)$ coincides with the space $C^\beta([a,b];X)$ of Hölder-continuous functions. We set also, for $k \in \mathbb{N}$,

$$W^{k+\beta,p}(a,b;X) = \Big\{ f \in W^{k,p}(a,b;X) :\; f^{(j)} \in W^{\beta,p}(a,b;X) \;\; (1 \le j \le k) \Big\}. \tag{2.5}$$

Of course, here the derivatives are intended in the sense of vector-valued distributions. If $\beta > \frac{1}{p}$, then by the Sobolev embedding theorem (see [2: Theorem 7.57]), every element of $W^{\beta,p}(a,b;X)$ can be identified with a continuous function. Before giving a more precise result, we introduce the following notation: for $1 < p \le +\infty$, $0 < \beta < 1$ and $f \in W^{\beta,p}(a,b;X)$ we set

$$[f]_{W^{\beta,p}(a,b;X)} = \begin{cases} \big(\int_a^b \big(\int_a^t \frac{\|f(t)-f(s)\|^p}{(t-s)^{1+\beta p}} ds\big) dt\big)^{\frac{1}{p}} & \text{if } 1 \le p < +\infty \\ \sup_{a \le s < t \le b} \frac{\|f(t)-f(s)\|}{(t-s)^\beta} & \text{if } p = +\infty. \end{cases} \tag{2.6}$$

We want to study the problem

$$\left. \begin{aligned} u'(t) &= G(u(t)) + \int_0^t h(t-s) F(u(t),u(s))\, ds + f(t) \quad (t \in [0,\tau]) \\ u(0) &= u_0 \\ \Phi(u(t)) &= g(t) \quad (t \in (0,\tau)) \end{aligned} \right\} \tag{2.7}$$

under the following conditions:

(k1) X and D are Banach spaces, D is continuously embedded into X.

(k2) $F \in C^2(D \times D; X)$ and F'' are uniformly Lipschitz continuous from $D \times D$ to $\mathcal{L}\big(D \times D; \mathcal{L}(D \times D; X)\big)$ in every bounded subset of $D \times D$.

(k3) $G \in C^2(D;X)$ and G'' are uniformly Lipschitz continuous from D to $\mathcal{L}(D;\, \mathcal{L}(D;X))$ in every bounded subset of D.

(k4) $u_0 \in D$.

(k5) $A = G'(u_0)$ considered as an unbounded operator in X satisfies assumption (h1).

(k6) $\Phi \in X'$.

(k7) $f \in W^{1+\beta,p}(0,T;X)$ for some $\beta \in (0,1)$ and $p \in (1,+\infty]$.

We are interested in the existence and uniqueness of a solution (u,h) of problem (2.7) belonging to

$$\big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$$

for some $\tau > 0$, with $p \in (1,+\infty]$ and $\beta \in (0,1) \setminus \{\frac{1}{p}\}$.

The main abstract results are the followings two theorems.

Theorem 2.1. *Assume that conditions* (k1) - (k7) *are satisfied and let* $p \in (1,+\infty]$, $\beta \in (0,1) \setminus \{\frac{1}{p}\}$ *and* $T > 0$. *Assume, moreover, the following:*

(i) $G(u_0) + f(0) \in D_A(1+\beta-\frac{1}{p}, p)$ *if* $\beta < \frac{1}{p}$ *and* $G(u_0) + f(0) \in D$ *if* $\beta > \frac{1}{p}$.

(ii) $g \in W^{2+\beta,p}(0,T)$.

(iii) $\Phi(u_0) = g(0)$ *and* $\Phi(G(u_0) + f(0)) = g'(0)$.

(iv) $\chi := \Phi(F(u_0,u_0)) \neq 0$.

(v) *If* $\beta > \frac{1}{p}$, *then* $A[G(u_0) + f(0)] + f'(0) + \mathcal{H}F(u_0,u_0) \in D_A(\beta - \frac{1}{p}, p)$ *where* $\mathcal{H}$ *is defined as*

$$\mathcal{H} = \chi(0)^{-1}\big\{g''(0) - \Phi\big[A(G(u_0) + f(0)) + f'(0)\big]\big\}. \tag{2.8}$$

(vi) $F(u_0,u_0) \in \overline{D}$.

Consider the functions v_0 *and* h_0 *defined in* (5.12) *and* (5.14), *respectively, and for* $R > 0$ *and* $\tau \in (0,T]$ *set*

$$B(\tau,R) = \left\{ (v,h) \in W^{\beta,p}(0,\tau;D) \times W^{\beta,p}(0,\tau) \;\middle|\; \begin{array}{l} \max\left\{ \begin{array}{c} \|v - v_0\|_{W^{\beta,p}(0,\tau;D)} \\ \|h - h_0\|_{W^{\beta,p}(0,\tau)} \end{array} \right\} \leq R \\ h(0) = \mathcal{H} \text{ if } \beta > \frac{1}{p} \end{array} \right\} \tag{2.9}$$

with the norms $\|\cdot\|_{W^{\beta,p}(0,\tau;D)}$ *and* $\|\cdot\|_{W^{\beta,p}(0,\tau)}$ *defined in* (4.21).

Then for every $R > 0$ *there exists* $\tau(R) \in (0,T]$ *such that for every* $\tau \in (0,\tau(R)]$ *problem* (2.7) *has a unique solution*

$$(u,h) \in \big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$$

such that if $v := \partial_t u$, *then* $(v,h) \in B(\tau,R)$.

Theorem 2.2. *Assume that the assumptions of Theorem* 2.1 *are satisfied. Let* $T > 0$ *and let*

$$(u_1, h_1) \in \big(W^{2+\beta,p}(0,T;X) \cap W^{1+\beta,p}(0,T;D)\big) \times W^{\beta,p}(0,T)$$

be a solution of problem (2.7) *with* $\tau = T$. *Set* $v_1 = \partial_t u_1$ *and assume, moreover, the following:*

- **a)** *For all* $t \in [0,T)$, *the operator* $A_{v_1,h_1}(t)$ *defined in* (2.7) *satisfies condition* (h1), *with* R_0 *and* M *which can depend on* t.
- **b)** *For all* $t \in [0,T)$, $\Phi(F(u_1(t), u_0)) \neq 0$.
- **c)** *For all* $t \in [0,T)$, $F(u_1(t), u_0) \in \overline{D}$.

Then (u_1, h_1) *is the unique solution of problem* (2.7) *belonging to*

$$\big(W^{2+\beta,p}(0,T;X) \cap W^{1+\beta,p}(0,T;X)\big) \times W^{\beta,p}(0,T).$$

Corollary 2.1. *Assume that the assumptions of Theorem* 2.1 *are satisfied and that, moreover,* $F(u, u_0) \in \overline{D}$ *for every* $u \in D$. *Then there exists a* $\tau > 0$ *such that problem* (2.7) *has a unique solution* $(u,h) \in \big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$.

3. Applications

We are now in the position to solve some problems of particular interest, applying the abstract results of Section 1. We have decided to give the details, for sake of simplicity, in three different fields of science even though many other applications can be mentioned.

Problem P_1: *Population dynamics.* We consider the well known Lotka-Volterra model with diffusion. For other results concerning systems of this form, see also [8]. The non-linearity in the integral term appearing in the first equation in the sequel is analogous to that of the Kermack-McKendrick system arising in the theory of spread of infections. The semilinear case with particular diffusion terms has been studied in [9]. Here we consider a more general case.

Let Ω be an open bounded set in $\mathbb{R}^n$, lying on one side of $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2. The problem is to determine two functions

$$u : [0,T] \times \Omega \to \mathbb{R}$$
$$h : [0,T] \to \mathbb{R}$$

satisfying the integro-differential system

$$\left.\begin{aligned} \partial_t u(t,x) &= d(u(t,x))\Delta u(t,x) + bu(t,x)\int_0^t h(t-s)u(s,x)\,ds \\ &\quad + f(u(t,x)) \quad \big((t,x)\in[0,T]\times\Omega\big) \\ u(0,x) &= u_0(x) \qquad (x\in\Omega) \\ u(t,x') &= 0 \qquad \big((t,x')\in[0,T]\times\partial\Omega\big) \\ \int_\Omega \phi(x)u(t,x)\,dx &= g(t) \qquad (t\in[0,T]) \end{aligned}\right\} \tag{3.1}$$

where the following conditions are fulfilled:

(h$_{11}$) $d, f \in \mathcal{C}^3(\mathbb{R})$, d real-valued and positive.
(h$_{12}$) $b \in \mathbb{R}\setminus\{0\}$.
(h$_{13}$) $\phi \in L^{p'}(\Omega)$ for a certain $p\in(1,+\infty)$ with $p > \frac{n}{2}$.
(h$_{14}$) $g \in W^{2+\beta,p}(0,T)$ for some $\beta\in(0,1)$ with $\beta\notin\{\frac{1}{p}, \frac{3}{2p}-1, \frac{3}{2p}\}$.

We have the following

Theorem 3.1. *Consider problem* (3.1) *under conditions* (h$_{11}$) - (h$_{14}$). *Assume, moreover, the following:*

(h$_{15}$) $u_0 \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$.
(h$_{16}$) *If* $G(u_0) := d(u_0)\Delta u_0 + f(u_0)$, *then*

$$G(u_0) \in \begin{cases} B_{p,p}^{2(1+\beta-\frac{1}{p})}(\Omega) & \text{if } \beta < \frac{3}{2p}-1 \\ \big\{u \in B_{p,p}^{2(1+\beta-\frac{1}{p})}(\Omega) : \, u_{|\partial\Omega} = 0\big\} & \text{if } \frac{3}{2p}-1 < \beta < \frac{1}{p} \\ W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) & \text{if } \beta > \frac{1}{p}. \end{cases}$$

(h$_{17}$) $\int_\Omega \phi(x)u_0(x)\,dx = g(0)$, $\int_\Omega \phi(x)\big[d(u_0(x))\Delta u_0(x) + f(u_0(x))\big]dx = g'(0)$.
(h$_{18}$) $\chi := \int_\Omega \phi(x)u_0(x)^2 dx \neq 0$.
(h$_{19}$) $d'(u_0)G(u_0)\Delta u_0 + d(u_0)\Delta G(u_0) + f'(u_0)G(u_0) + \mathcal{H}bu_0^2$

$$\in \begin{cases} B_{p,p}^{2(\beta-\frac{1}{p})}(\Omega) & \text{if } \frac{1}{p} < \beta < \frac{3}{2p} \\ \big\{u\in B_{p,p}^{2(\beta-\frac{1}{p})}(\Omega) : \, u_{|\partial\Omega} = 0\big\} & \text{if } \frac{3}{2p} < \beta \end{cases}$$

with

$$\mathcal{H} = \chi^{-1}\Big[g''(0) - \int_\Omega \phi\big\{d(u_0)\Delta G(u_0) + \big[d'(u_0)\Delta u_0 + f'(u_0)\big]G(u_0)\big\}dx\Big].$$

Then for some $\tau \in (0,T]$ problem (3.1) *has a unique solution*

$$(u,h) \in \big(W^{2+\beta,p}(0,\tau;L^p(\Omega)) \cap W^{1+\beta,p}(0,\tau;W^{2,p}(\Omega))\big) \times W^{\beta,p}(0,\tau).$$

Proof. We introduce a proper functional setting. For this, first put

$$X = L^p(\Omega) \tag{3.2}$$
$$D = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega). \tag{3.3}$$

Observe that, owing to condition (h_{13}), $D \subseteq C(\overline{\Omega})$. Further, set

$$F : D \times D \to X, \qquad F(u_1,u_2) = bu_1u_2 \quad (u_1,u_2 \in D). \tag{3.4}$$

It can be easily seen that F satisfies condition (k2). Finally, set

$$G : D \to X, \qquad G(u) = d(u)\Delta u + f(u) \quad (u \in D). \tag{3.5}$$

Then G satisfies condition (k3). Condition (k4) is exactly assumption (h_{15}). We have also, for every $v \in D$,

$$G'(u_0)v = d(u_0)\Delta v + [d'(u_0)\Delta u_0 + f'(u_0)]v$$

so that condition (k5) is satisfied by [20: Section 3.8]. Condition (k6) follows from assumption (h_{13}), setting

$$\Phi(u) = \int_\Omega \phi(x)u(x)\,dx. \tag{3.6}$$

Concerning assumptions (i) - (v) in Theorem 2.1 it suffices to say that, if $\theta \in (0,1) \setminus \{\frac{1}{2p}\}$, we have

$$D_A(\theta,p) = \begin{cases} B^{2\theta}_{p,p}(\Omega) & \text{if } \theta < \frac{1}{2p} \\ \big\{u \in B^{2\theta}_{p,p}(\Omega) : u_{|\partial\Omega} = 0\big\} & \text{if } \frac{1}{2p} < \theta \end{cases} \tag{3.7}$$

(see [11]). Finally, assumption (vi) is obviously satisfied, as D is dense in X ∎

Problem $\mathbf{P}_2$: *Combustion of a material with memory.* The semilinear version of the model was studied in [4] in spaces of Hölder-continuous functions. In this application we consider the quasilinear version in Sobolev spaces. We denote by u the temperature, by v the density of the combustible and, as usual, by h the memory kernel. The system governing the evolution of (u, v, h) is given by

$$\left.\begin{aligned}
\partial_t u(t,x) &= D_{1,1}(u(t,x), v(t,x))\Delta u(t,x) \\
&\quad + \int_0^t h(t-s) D_{1,2}(u(s,x), v(s,x))\Delta u(s,x)\, ds \\
&\quad + f_1(u(t,x), v(t,x)) \quad \big((t,x) \in [0,T] \times \Omega\big) \\
\partial_t v(t,x) &= D_{2,1}(u(t,x), v(t,x))\Delta v(t,x) \\
&\quad + f_2(u(t,x), v(t,x)) \quad \big((t,x) \in [0,T] \times \Omega\big) \\
u(0,x) &= u_0(x) \qquad (x \in \Omega) \\
v(0,x) &= v_0(x) \qquad (x \in \Omega) \\
u(t,x') &= v(t,x') = 0 \qquad \big((t,x') \in [0,T] \times \partial\Omega\big) \\
\int_\Omega \phi(x) u(t,x)\, dx &= g(t) \qquad (t \in [0,T]).
\end{aligned}\right\} \tag{3.8}$$

The problem is to determine three functions

$$\begin{aligned}
u &: [0,T] \times \Omega \to \mathbb{R} \\
v &: [0,T] \times \Omega \to \mathbb{R} \\
h &: [0,T] \to \mathbb{R}
\end{aligned}$$

satisfying the above integro-differential system.

The functions f_1 and f_2 are given by the Arhenius kinetics (see [4]), while the term $\int_\Omega \phi(x) u(t,x)\, dx = g(t)$ represents the additional measurements on the temperature to identify the convolution kernel h which is not directly measurable.

In the following we shall assume the following:

($\mathbf{h}_{21}$) $D_{1,1}, D_{1,2}, D_{2,1} \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R})$, $D_{1,1}(u,v) > 0$ and $D_{2,1}(u,v) > 0$ for all $(u,v) \in \mathbb{R}^2$.

($\mathbf{h}_{22}$) $f_1, f_2 \in \mathcal{C}^3(\mathbb{R}^2, \mathbb{R})$.

($\mathbf{h}_{23}$) Ω is an open bounded subset in $\mathbb{R}^n$, lying on one side of $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^2.

($\mathbf{h}_{24}$) $p \in (\frac{n}{2} \vee 1, +\infty)$.

($\mathbf{h}_{25}$) $\phi \in L^{p'}(\Omega)$.

($\mathbf{h}_{26}$) $u_0, v_0 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

We introduce now a proper functional setting. For this we put first

$$X = L^p(\Omega) \times L^p(\Omega) \tag{3.9}$$
$$D = \big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\big) \times \big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\big). \tag{3.10}$$

Further, if $U = (u, v)$ and $V = (z, w)$ are elements of D, we define

$$G(U) = \big(D_{1,1}(u, v)\Delta u + f_1(u, v), D_{2,1}(u, v)\Delta v + f_2(u, v)\big) \tag{3.11}$$
$$F(U, V) = (D_{1,2}(V)\Delta z, 0) \tag{3.12}$$
$$U_0 = (u_0, v_0) \tag{3.13}$$

and

$$\begin{aligned} A(V) &= G'(U_0)(V) \\ &= \Big(D_{1,1}(U_0)\Delta z + \Delta u_0 \frac{\partial D_{1,1}}{\partial x}(U_0)z \\ &\qquad + \Delta u_0 \frac{\partial D_{1,1}}{\partial y}(U_0)w + \frac{\partial f_1}{\partial x}(U_0)z + \frac{\partial f_1}{\partial y}(U_0)w, \\ &\qquad D_{2,1}(U_0)\Delta w + \Delta v_0 \frac{\partial D_{2,1}}{\partial x}(U_0)z \\ &\qquad + \Delta v_0 \frac{\partial D_{2,1}}{\partial y}(U_0)w + \frac{\partial f_2}{\partial x}(U_0)z + \frac{\partial f_2}{\partial y}(U_0)w\Big). \end{aligned} \tag{3.14}$$

Now we can give the following

Theorem 3.2. *Consider problem* (3.8) *under assumptions* (h_{21}) - (h_{26}). *Further, let* X, D, G, F, U_0, A *have the meaning declared in* (3.9) – (3.14), *respectively, and let also* $\beta \in (0, 1) \setminus \{\frac{1}{p}, \frac{3}{2p}, \frac{3}{2p} - 1\}$. *Concerning the data* $U_0 = (u_0, v_0)$, ϕ *and* g *we suppose the following:*

- **(i)** *If* $\beta < \frac{1}{p}$, *then* $G(U_0) \in B_{p,p}^{2(1+\beta-\frac{1}{p})}(\Omega)^2$, *and* $G(U_0)$ *vanishes in* $\partial\Omega$ *if* $\beta > \frac{3}{2p} - 1$.
- **(ii)** *If* $\beta > \frac{1}{p}$, *then* $G(U_0) \in D$.
- **(iii)** $g \in W^{2+\beta,p}(0, T)$.
- **(iv)** $\int_\Omega \phi(x)u_0(x)\, dx = g(0)$.
- **(v)** $\int_\Omega \phi(x)\big[D_{1,1}(u_0(x), v_0(x))\Delta u_0(x) + f_1(u_0(x), v_0(x))\big]dx = g'(0)$.
- **(vi)** $\chi := \int_\Omega \phi(x)D_{1,2}(u_0(x), v_0(x))\Delta u_0(x)\, dx \neq 0$.
- **(vii)** *If* $\beta > \frac{1}{p}$, *let the set* $\mathcal{H}$ *be defined as in* (2.8). *Then* $AG(U_0) + \mathcal{H}F(U_0, U_0) \in \big[B_{p,p}^{2(\beta-\frac{1}{p})}(\Omega)\big]^2$ *and the expression vanishes in* $\partial\Omega$ *if* $\beta > \frac{3}{2p}$.

Then for some $\tau \in (0,T]$ *problem* (3.8) *has a unique solution*

$$((u,v),h) = (U,h) \in \big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau).$$

Proof. The proof follows the same arguments as in the proof of Theorem 3.1 ∎

Problem P$_3$: *A fully nonlinear inverse problem with spatial derivatives of order* $2m$ (the case $m=1$ represents the heat equation with memory). The problem is to determine two functions

$$\begin{aligned} u &: [0,\tau] \times \Omega \to \mathbb{R} \\ h &: [0,T] \to \mathbb{R} \end{aligned}$$

satisfying the system

$$\left.\begin{aligned} \partial_t u(t,x) &= d(A(x,\partial_x)u(t,x)) + \int_0^t h(t-s)e(A(x,\partial_x)u(s,x))\,ds \\ &\quad + f(t,x) \quad (t \in [0,\tau], x \in \Omega) \\ B_j(x,\partial_x)u(t,x) &= 0 \qquad (j \in \{1,...,m\}, t \in [0,\tau], x \in \partial\Omega) \\ u(0,x) &= u_0(x) \qquad (x \in \Omega) \\ \int_{\overline{\Omega}} u(t,x)\mu(dx) &= g(t) \qquad (t \in [0,\tau]) \end{aligned}\right\} \tag{3.15}$$

under the following assumptions:

(h$_{31}$) $d, e \in \mathcal{C}^3(\mathbb{R})$, $d'(u) > 0$ for all $u \in \mathbb{R}$.

(h$_{32}$) $\Omega \subset \mathbb{R}^n$ is an open bounded subset, lying on one side of $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^{2m} $(m \in \mathbb{N})$.

(h$_{33}$) $A(x,\partial_x)$ is a strongly elliptic operator of order $2m$ with coefficients in $C(\overline{\Omega})$.

(h$_{34}$) $B_j(x,\partial_x)$ $(j = 1,\dots,m)$ is a linear differential operator of order $m_j < 2m$ with coefficients in $C^{2m-m_j}(\partial\Omega)$.

(h$_{35}$) $\{B_j(x,\partial_x)\}_{1\le j\le m}$ is a normal system of boundary operators in the sense of [20: Definition 3.7.1].

(h$_{36}$) The operator $A(x,\partial_x)$ with vanishing boundary conditions $B_j(x,\partial_x)$ $(1 \le j \le m)$ has $\mathrm{Arg}\lambda = \theta$ as a ray of minimal growth of the resolvent in the sense of [20: Definition 3.8.1].

(h$_{37}$) μ is a Borel measure in $\overline{\Omega}$.

We set

$$X = C(\overline{\Omega}) \tag{3.16}$$

and

$$D = \left\{ u \in \bigcap_{1<p<+\infty} W^{2m,p}(\Omega) \,\middle|\, \begin{array}{l} A(x,\partial_x)u \in X \text{ and } B_j(x,\partial_x)u \equiv 0 \\ \text{in } \partial\Omega \text{ for all } j = 1, ..., m \end{array} \right\}. \tag{3.17}$$

If u and v are elements of D, we define

$$G(u) = d(A(x,\partial_x)u) \tag{3.18}$$

$$F(u) = e(A(x,\partial_x)u). \tag{3.19}$$

Observe that

$$G'(u)v = d'(A(x,\partial_x)u)A(x,\partial_x)v.$$

Now we can give the following

Theorem 3.3. *Consider problem* (3.15) *under assumptions* (h_{31}) - (h_{37}). *Further, let* X, D, G, F *be as in* (3.16) – (3.19), *respectively, and let* $\beta \in (0,1)$ *such that* $2m\beta \notin \mathbb{N}$. *Concerning the data* u_0, f, μ, g *we suppose the following:*

(i) $u_0 \in D$.

(ii) $f \in C^{1+\beta}(0,T;X)$.

(iii) $G(u_0) + f(0) \in D$.

(iv) $g \in C^{2+\beta}(0,T)$.

(v) $\int_\Omega u_0(x)\mu(dx) = g(0)$.

(vi) $\int_\Omega [G(u_0)(x) + f(0,x)]\mu(dx) = g'(0)$.

(vii) $\chi := \int_\Omega F(u_0)(x)\mu(dx) \neq 0$.

(viii) *Put* $A = G'(u_0)$ *and let* $\mathcal{H}$ *be defined as in* (2.8). *Then* $A[G(u_0) + f(0)] + f'(0) + \mathcal{H}F(u_0) \in C^{2m\beta}(\overline{\Omega})$. *Moreover, for all* $j \in \{1, ..., m\}$ *such that* $m_j < 2m\beta$, $B_j(x,\partial_x)\big\{\big[A[G(u_0) + f(0)] + f'(0) + \mathcal{H}F(u_0)\big]\big\}$ *vanishes in* $\partial\Omega$.

(ix) *If* $\min_{1\le j\le m} m_j = 0$, *then* $F(u_0)$ *vanishes in* $\partial\Omega$.

Then for some $\tau \in (0,T]$ *problem* (3.15) *has a unique solution*

$$(U,h) \in \big(C^{2+\beta}(0,\tau;X) \cap C^{1+\beta}(0,\tau;D)\big) \times C^\beta(0,\tau).$$

Proof. The proof can be obtained from Corollary 2.1 using the following known facts:

a) The operator A satisfies assumption (h_1) (see [19]).

b) If $\alpha \in (0,1)$ and $2m\alpha \notin \mathbb{N}$, one has (see [1])

$$D_A(\alpha,\infty) = \left\{ u \in C^{2m\alpha}(\overline{\Omega}) \,\middle|\, \begin{array}{l} B_j(x,\partial_x)u \equiv 0 \text{ in } \partial\Omega \;\forall\, j \in \{1, ..., m\} \\ \text{such that } m_j < 2m\alpha \end{array} \right\}. \tag{3.20}$$

c) The set $\overline{D}$ coincides with X if $\min_{1\le j\le m} m_j \ge 1$ and with $\{f \in X : f_{|\partial\Omega} = 0\}$ if $\min_{1\le j\le m} m_j = 0$ ∎

4. Preliminary lemmata

Here we give some preliminary lemmata and theorems that will be useful in Section 5.

Lemma 4.1. *Let X be a Banach space, and let $p > 1$, $1 > \gamma > \frac{1}{p}$ and $-\infty < a < b < +\infty$. Then, for every $u \in W^{\gamma,p}(a,b;X)$,*

$$\|u\|_{L^\infty(a,b;X)} \le \|u(a)\| + C(b-a)^{\gamma-\frac{1}{p}}[u]_{W^{\gamma,p}(a,b;X)}$$

where $C > 0$ is a constant independent of a, b and u.

Proof. We have

$$\|u\|_{L^\infty(a,b;X)} \le \|u(a)\| + \|u - u(a)\|_{L^\infty(a,b;X)}.$$

So we are reduced to treat the case $u(a) = 0$. Let $C > 0$ be such that, for every $v \in W^{\gamma,p}(0,1;X)$ with $v(0) = 0$, $\|v\|_{L^\infty(0,1;X)} \le C\,[v]_{W^{\gamma,p}(0,1;X)}$. Then, if $u \in W^{\gamma,p}(a,b;X)$ and $u(a) = 0$,

$$\begin{aligned}\|u\|_{L^\infty(a,b;X)} &= \big\|u(a+(b-a))\big\|_{L^\infty(0,1;X)}\\ &\le C\big[u(a+(b-a))\big]_{W^{\gamma,p}(0,1;X)}\\ &= C(b-a)^{\gamma-\frac{1}{p}}[u]_{W^{\gamma,p}(a,b;X)}.\end{aligned}$$

and the lemma is proved ∎

To introduce a suitable norm in $W^{\beta,p}(a,b;X)$ for $\beta < \frac{1}{p}$ we shall use the following known

Proposition 4.1. *Let $1 < p < +\infty$ and $f \in W^{\beta,p}(a,b;X)$. Then:*

a) *If $0 < \beta < \frac{1}{p}$, then $\int_a^b (t-a)^{-\beta p}\|f(t)\|^p dt < +\infty$.*

b) *If $\frac{1}{p} < \beta < 1$, then $\int_a^b (t-a)^{-\beta p}\|f(t) - f(a)\|^p dt \le C\,[f]^p_{W^{\beta,p}(a,b;X)}$ where $C > 0$ is independent of a, b and f.*

Proof. See [5: Lemma 7] ∎

Next, we put

$$\|f\|_{W^{\beta,p}(a,b;X)} = \begin{cases} \big(\int_a^b (t-a)^{-\beta p}\|f(t)\|^p dt\big)^{\frac{1}{p}} + [f]_{W^{\beta,p}(a,b;X)} & \text{if } 0 < \beta < \frac{1}{p} \\ \|f(a)\| + [f]_{W^{\beta,p}(a,b;X)} & \text{if } \frac{1}{p} < \beta < 1. \end{cases} \tag{4.1}$$

Finally, if $k \in \mathbb{N}$ and $\beta \in (0,1)$ with $\beta \ne \frac{1}{p}$, we set

$$\|f\|_{W^{k+\beta,p}(a,b;X)} = \sum_{j=0}^{k} \|f^{(j)}\|_{W^{\beta,p}(a,b;X)}. \tag{4.2}$$

Our interest in these spaces and norms depends on the following theorem the proof of which will be given in Section 6.

Theorem 4.1. *Consider the problem*

$$\left.\begin{aligned} &u'(t) = Au(t) + f(t) \quad (t \in [a,b]) \\ &u(a) = u_0 \end{aligned}\right\} \tag{4.3}$$

under assumption (h1) *on* A. *Then the following conditions are necessary and sufficient in order that the mild solution*

$$u(t) = S(t-a)u_0 + \int_a^t S(t-s)f(s)\,ds \tag{4.4}$$

of problem (4.3) *belongs to* $W^{1+\beta,p}(a,b;X) \cap W^{\beta,p}(a,b;D)$ *for* $1 < p \le +\infty$ *and* $\beta \in (0,1) \setminus \{\frac{1}{p}\}$*:*

(i) $f \in W^{\beta,p}(a,b;X)$.

(ii) *If* $0 < \beta < \frac{1}{p}$, *then* $u_0 \in D_A(1+\beta-\frac{1}{p}, p)$.

(iii) *If* $\frac{1}{p} < \beta < 1$, *then* $Au_0 + f(a) \in D_A(\beta - \frac{1}{p}, p)$.

Moreover, if $b - a \le T_0$ *and* $\beta < \frac{1}{p}$, *there exists a constant* $C(T_0) > 0$ *independent of* a, b *and* f, u_0 *such that*

$$\begin{aligned} &\|u\|_{W^{1+\beta,p}(a,b;X)} + \|u\|_{W^{\beta,p}(a,b;D)} \\ &\qquad \le C(T_0)\Big[\|f\|_{W^{\beta,p}(a,b;X)} + \|u_0\|_{D_A(1+\beta-\frac{1}{p},p)}\Big]. \end{aligned} \tag{4.5}$$

Finally, if $b-a \le T_0$ *and* $\beta > \frac{1}{p}$, *there exists a constant* $C(T_0) > 0$ *independent of* a, b *and* f, u_0 *such that*

$$\begin{aligned} &\|u\|_{W^{1+\beta,p}(a,b;X)} + \|u\|_{W^{\beta,p}(a,b;D)} \\ &\qquad \le C(T_0)\Big[\|f\|_{W^{\beta,p}(a,b;X)} + \|u_0\|_D + \|Au_0 + f(a)\|_{D_A(\beta-\frac{1}{p},p)}\Big]. \end{aligned} \tag{4.6}$$

Referring for the proof of the theorem to Section 6, we go on with a series of technical lemmata.

Lemma 4.2. *Let* Y *be a Banach space, and let* $p \in (1,+\infty], \beta \in (0,1) \setminus \{\frac{1}{p}\}$ *and* $h \in W^{\beta,p}(a,b;Y)$. *Then*

$$\|h\|_{L^1(a,b;Y)} \le \begin{cases} C(b-a)^{\beta+\frac{1}{p'}}\|h\|_{W^{\beta,p}(a,b;Y)} & \text{if } \beta < \frac{1}{p} \\ C\big[(b-a)\|h(a)\|_Y + (b-a)^{\beta+\frac{1}{p'}}[h]_{W^{\beta,p}(a,b;Y)}\big] & \text{if } \beta > \frac{1}{p} \end{cases} \tag{4.7}$$

where $C > 0$ *is independent of* a, b *and* h.

Proof. Assertion (4.7) follows considering first the case $h(a) = 0$, using the homogeneity argument of Lemma 4.1, then being reduced to the previous case by writing $h = (h - h(a)) + h(a)$ ∎

Lemma 4.3. *Let Y, W, Z be Banach spaces and let $(\cdot,\cdot) : Y \times W \to Z$ be a continuous and bilinear mapping. Further, let $k \in W^{\beta,p}(a,b,W)$ and $h \in W^{\gamma,p}(a,b;Y)$ for some $p > 1$ and $0 < \beta \le \gamma < 1$ with $\gamma > \frac{1}{p}$. Then $(h,k) \in W^{\beta,p}(a,b;Z)$. Moreover,*

$$\|(h,k)\|_{W^{\beta,p}(a,b;Z)} \le \begin{cases} C\Big[\|h(a)\|_Y + (b-a)^{\gamma-\frac{1}{p}}[h]_{W^{\gamma,p}(a,b;Y)}\Big]\|k\|_{W^{\beta,p}(a,b;W)} & \text{if } \beta < \frac{1}{p} \ + \\ C\Big[\|h(a)\|_Y\|k\|_{W^{\beta,p}(a,b;W)} + (b-a)^{\gamma-\beta} & \\ \quad \times [h]_{W^{\gamma,p}(a,b;Y)}\big(\|k(a)\|_W + (b-a)^{\beta-\frac{1}{p}}[k]_{W^{\beta,p}(a,b;W)}\big)\Big] & \text{if } \beta > \frac{1}{p} \end{cases} \tag{4.9}$$

where $C > 0$ is a constant independent of a, b and h, k.

Proof. By Lemma 4.1, h is bounded with values in Y. So $k \to (h,k)$ belongs to $\mathcal{L}\big(L^p(a,b;W), L^p(a,b;Z)\big)$. It can also easily seen that the same operator maps $W^{\gamma,p}(a,b;W)$ into $W^{\gamma,p}(a,b;Z)$. In fact, if $k \in W^{\gamma,p}(a,b;W)$, then

$$\begin{aligned} &\int_a^b \bigg(\int_a^t \frac{\|(h(t),k(t)) - (h(s),k(s))\|_Z^p}{(t-s)^{1+\gamma p}}\,ds\bigg)dt \\ &\quad \le C\bigg[\int_a^b \bigg(\int_a^t \frac{\|(h(t)-h(s)\|_Y^p}{(t-s)^{1+\gamma p}}\,ds\bigg)\|k(t)\|_W^p dt \\ &\qquad + \int_a^b \bigg(\int_a^t \|h(s)\|_Y^p \frac{\|(k(t)-k(s)\|_W^p}{(t-s)^{1+\gamma p}}\,ds\bigg)dt\bigg] \\ &\quad \le C\Big[\|h\|_{L^\infty(a,b;Y)}^p[k]_{W^{\gamma,p}(a,b;W)}^p + \|k\|_{L^\infty(a,b;W)}^p[h]_{W^{\gamma,p}(a,b;Y)}^p\Big]. \end{aligned}$$

By interpolation, adapting in a trivial way [14: Proposition 2.4], if $k \in W^{\beta,p}(a,b;W)$, then we have $(h,k) \in W^{\beta,p}(a,b;Z)$.

It remains to show estimates (4.9). This follows considering first the case $h(a) = 0$, using the homogeneity argument of Lemma 4.1, then being reduced to the previous case by writing $h = (h - h(a)) + h(a)$ ∎

Lemma 4.4. *Let Y and Z be Banach spaces and let $H : Y \to Z$ of class C^1. Assume for a certain $R > 0$, there exists $L > 0$ such that, for every $y_0, y_1 \in Y$ satisfying $\max\{\|y_0\|_Y, \|y_1\|_Y\} \le R$,*

$$\|H'(y_0) - H'(y_1)\|_{\mathcal{L}(Y,Z)} \le L\|y_0 - y_1\|_Y. \tag{4.10}$$

Further, let $p > 1, \frac{1}{p} < \gamma < 1$ and $\tau > 0$, and let $u \in W^{\gamma,p}(a,b;Y)$ be such that $\|u\|_{L^\infty(a,b;Y)} \le R$. Then $H \circ u \in W^{\gamma,p}(a,b;Z)$ and

$$[H \circ u]_{W^{\gamma,p}(a,b;Z)} \le \big(\|H'(0)\|_{\mathcal{L}(Y,Z)} + RL\big)[u]_{W^{\gamma,p}(a,b;Y)}. \tag{4.11}$$

Moreover, if $u_1, u_2 \in W^{\gamma,p}(a,b;Y)$ *are such that* $\max\left\{\|u_1\|_{L^\infty(a,b;Y)}, \|u_2\|_{L^\infty(a,b;Y)}\right\}$ $\leq R$, *then*

$$
\begin{aligned}
&\big[H\circ u_1 - H\circ u_2\big]_{W^{\gamma,p}(a,b;Z)} \\
&\qquad \leq \frac{L}{2}\big([u_1]_{W^{\gamma,p}(a,b;Y)} + [u_2]_{W^{\gamma,p}(a,b;Y)}\big)\|u_1-u_2\|_{L^\infty(a,b;Y)} \\
&\qquad\quad + \big(\|H'(0)\|_{\mathcal{L}(Y,Z)} + LR\big)[u_1-u_2]_{W^{\gamma,p}(a,b;Y)}.
\end{aligned}
\tag{4.12}
$$

Proof. If $\|y\|_Y \leq R$, one has

$$
\|H'(y)\|_{\mathcal{L}(Y,Z)} \leq \|H'(0)\|_{\mathcal{L}(Y,Z)} + LR.
$$

This implies that, if $\max\{\|y_1\|_Y, \|y_2\|_Y\} \leq R$, then

$$
\|H(y_1) - H(y_2)\|_Z \leq \big(\|H'(0)\|_{\mathcal{L}(Y,Z)} + LR\big)\|y_1 - y_2\|_Y
$$

and from this (4.10) follows immediately.

Now we prove (4.11). We restrict ourselves to the case $p < +\infty$. We have, if $u_1, u_2 \in W^{\gamma,p}(a,b;Y)$ are such that $\max\{\|u_1\|_{L^\infty(a,b;Y)}, \|u_2\|_{L^\infty(a,b;Y)}\} \leq R$,

$$
\begin{aligned}
&\left(\int_a^b\left(\int_a^t \frac{\big\|H(u_1(t)) - H(u_2(t)) - H(u_1(s)) + H(u_2(s))\big\|_Z^p}{(t-s)^{1+\gamma p}}\,ds\right)dt\right)^{\frac{1}{p}} \\
&\quad = \left(\int_a^b\left(\int_a^t (t-s)^{-1-\gamma p}dsdt\right\|\int_0^1\Big[H'\big(u_2(t)+\theta(u_1(t)-u_2(t))\big)(u_1(t)-u_2(t)) \\
&\qquad - H'\big(u_2(s)+\theta(u_1(s)-u_2(s))\big)(u_1(s)-u_2(s))\Big]d\theta\left\|_Z^p\right)^{\frac{1}{p}} \\
&\quad \leq \int_0^1\left(\int_a^b\left(\int_a^t (t-s)^{-1-\gamma p}\Big\|H'\big(u_2(t)+\theta(u_1(t)-u_2(t))\big)(u_1(t)-u_2(t))\right.\right. \\
&\qquad \left.\left.- H'\big(u_2(s)+\theta(u_1(s)-u_2(s))\big)(u_1(s)-u_2(s))\Big\|_Z^p ds\right)dt\right)^{\frac{1}{p}} d\theta
\end{aligned}
$$

by the Minkowski inequality. Now, for every $\theta \in [0,1]$,

$$
\begin{aligned}
&\Big(\int_a^b \Big(\int_a^t (t-s)^{-1-\gamma p} \Big\| H'\big(u_2(t)+\theta(u_1(t)-u_2(t))\big)(u_1(t)-u_2(t)) \\
&\qquad - H'\big(u_2(s)+\theta(u_1(s)-u_2(s))\big)(u_1(s)-u_2(s))\Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\quad \le \Big(\int_a^b \Big(\int_a^t (t-s)^{-1-\gamma p} \Big\| [H'\big(u_2(t)+\theta(u_1(t)-u_2(t))\big) \\
&\qquad - H'\big(u_2(s)+\theta(u_1(s)-u_2(s))\big)(u_1(t)-u_2(t))\Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_a^b \Big(\int_a^t (t-s)^{-1-\gamma p} \Big\| H'\big(u_2(s)+\theta(u_1(s)-u_2(s))\big) \\
&\qquad \times \big(u_1(t)-u_2(t)-u_1(s)+u_2(s)\big)\Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\quad \le L\|u_1-u_2\|_{L^\infty(a,b;Y)} \Big(\int_a^b \Big(\int_a^t (t-s)^{-1-\gamma p} \|(1-\theta)(u_2(t)-u_2(s)) \\
&\qquad + \theta(u_1(t)-u_1(s))\|_Y^p ds\Big) dt\Big)^{\frac{1}{p}} + \big(\|H'(0)\|_{\mathcal{L}(Y,Z)} + LR\big)[u_1-u_2]_{W^{\gamma,p}(a,b;Y)}.
\end{aligned}
$$

Integrating in θ, we get the conclusion ∎

Lemma 4.5. *Let X be a Banach space, $p>1$, and let $\beta,\gamma \in (0,1)$ both different from $\frac{1}{p}$. Further, let $v \in W^{\beta,p}(a,b;X)$. Then*

$$
\begin{aligned}
&\Big\| \int_a^{\cdot} v(s)\, ds \Big\|_{W^{\gamma,p}(a,b;X)} \\
&\qquad \le \begin{cases} C(b-a)^{1+\beta-\gamma}\|v\|_{W^{\beta,p}(a,b;X)} & \text{if } \beta < \frac{1}{p} \\ C(b-a)^{1+\frac{1}{p}-\gamma}\big[\|v(a)\| + (b-a)^{\beta-\frac{1}{p}}[v]_{W^{\beta,p}(a,b;X)}\big] & \text{if } \beta > \frac{1}{p} \end{cases}
\end{aligned}
$$

where $C>0$ is a constant independent of a,b and v.

Proof. Clearly, $\int_a^{\cdot} v(s)\, ds \in W^{1+\beta,p}(a,b;X) \subseteq W^{\gamma,p}(a,b;X)$. The estimates asserted can be obtained employing again the homogeneity arguments of Lemma 4.1 ∎

We shall need the following generalization of [3: Theorem 3.1]:

Lemma 4.6. *Let Y,W,Z be Banach spaces and $(\cdot,\cdot) : Y \times W \to Z$ a continuous bilinear mapping. Further, for $\tau > 0$ let $h \in L^1(0,\tau;Y)$ and $k \in L^1(\Delta_\tau;W)$ and set*

$$
z(t) = \int_0^t \big(h(t-s), k(t,s)\big) ds \tag{4.13}
$$

for $t \in (0,\tau)$. Finally, let $p > 1$ and $0 < \beta < 1$ with $\beta \neq \frac{1}{p}$, and in the case $\beta > \frac{1}{p}$ assume $k \in C(\overline{\Delta_\tau}; W)$ and $k(t,0) = 0$ for every $t \in [0,\tau]$. Then

$$\|z\|_{W^{\beta,p}(0,\tau;Z)} \le C\|h\|_{L^1(0,\tau;Y)} \sup_{0<\sigma<\tau} \|k(\cdot+\sigma,\cdot)\|_{W^{\beta,p}(0,\tau-\sigma;W)}$$

where $C > 0$ is a constant independent of h, k, τ.

Proof. We consider here only the case $p < +\infty$ and start by proving the assertion assuming $k(t,0) \equiv 0$ in the case $\beta > \frac{1}{p}$. We have

$$\begin{aligned}
&\Big(\int_0^\tau \Big(\int_0^t (t-s)^{-1-\beta p} \\
&\qquad \times \Big\| \int_0^t \big(h(\sigma), k(t,t-\sigma)\big)d\sigma - \int_0^s \big(h(\sigma), k(s,s-\sigma)\big)d\sigma \Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\quad \le \Big(\int_0^\tau \Big(\int_0^t (t-s)^{-1-\beta p} \Big\| \int_0^s \big(h(\sigma), k(t,t-\sigma) - k(s,s-\sigma)\big)d\sigma \Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\qquad + \Big(\int_0^\tau \Big(\int_0^t (t-s)^{-1-\beta p} \Big\| \int_s^t \big(h(\sigma), k(t,t-\sigma)\big)d\sigma \Big\|_Z^p ds\Big) dt\Big)^{\frac{1}{p}} \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Using the Hölder inequality, I_1 can be majorized as (the following constant C being independent of h, k, τ)

$$\begin{aligned}
I_1 &\le C\|h\|_{L^1(0,\tau;Y)}^{\frac{1}{p'}} \Big(\int_0^\tau \Big(\int_0^t (t-s)^{-1-\beta p} \\
&\qquad \times \Big(\int_0^s \|h(\sigma)\|_Y \big\|k(t,t-\sigma) - k(s,s-\sigma)\big\|_W^p d\sigma\Big) ds\Big) dt\Big)^{\frac{1}{p}} \\
&\le C\|h\|_{L^1(0,\tau;Y)} \sup_{0<\sigma<\tau} [k(\cdot+\sigma,\cdot)]_{W^{\beta,p}(0,\tau-\sigma;W)}.
\end{aligned}$$

In the same way,

$$\begin{aligned}
I_2 &\le C_1\|h\|_{L^1(0,\tau;Y)}^{\frac{1}{p'}} \Big(\int_0^\tau\Big(\int_0^t (t-s)^{-1-\beta p} \int_s^t \|h(\sigma)\|_Y \|k(t,t-\sigma))\|_W^p d\sigma\Big) ds\Big) dt\Big)^{\frac{1}{p}} \\
&\le C_2\|h\|_{L^1(0,\tau;Y)}^{\frac{1}{p'}} \Big(\int_0^\tau \Big(\int_0^{\tau-\sigma} t^{-\beta p} \|k(t+\sigma,t)\|_W^p dt\Big) \|h(\sigma)\|_Y d\sigma\Big)^{\frac{1}{p}}.
\end{aligned}$$

So we have

$$[z]_{W^{\beta,p}(0,\tau,Z)} \le C\|h\|_{L^1(0,\tau;Y)} \sup_{0<\sigma<\tau} \|k(\cdot+\sigma,\cdot)\|_{W^{\beta,p}(0,\tau-\sigma;W)}$$

where $C > 0$ is a constant independent of τ. To conclude the proof, we assume $\beta < \frac{1}{p}$ and estimate

$$\begin{aligned}
&\left(\int_0^\tau \|t^{-\beta} z(t)\|^p dt \right)^{\frac{1}{p}} \\
&\quad \le \|h\|_{L^1(0,\tau;Y)}^{\frac{1}{p'}} \left(\int_0^\tau t^{-\beta p} \left(\int_0^t \|h(s)\|_Y \|k(t,t-s)\|_W^p ds \right) dt \right)^{\frac{1}{p}} \\
&\quad = \|h\|_{L^1(0,\tau;Y)}^{\frac{1}{p'}} \left(\int_0^\tau \left(\int_s^\tau t^{-\beta p} \|k(t,t-s)\|_W^p dt \right) \|h(s)\|_Y ds \right)^{\frac{1}{p}} \\
&\quad \le \|h\|_{L^1(0,\tau;Y)} \sup_{0<\sigma<\tau} \|k(\cdot+\sigma,\cdot)\|_{W^{\beta,p}(0,\tau-\sigma;W)}.
\end{aligned}$$

The proof is complete ∎

Lemma 4.7. *Let D and X be Banach spaces, with D continuously embedded into X, and let $A \in \mathcal{L}(D,X)$ satisfying condition* (h1). *Further, let $h \in W^{\beta,p}(a,b)$ $\ (p > 1, \beta \in (0,1) \setminus \{\frac{1}{p}\})$ with $h(a) = 0$ if $\beta > \frac{1}{p}$ and $y_0 \in \overline{D}$, and set*

$$z(t) = \int_a^t S(t-s)h(s)y_0 ds.$$

Then

$$\|z\|_{W^{\beta,p}(a,b;D)} \le \eta(b-a)\|h\|_{W^{\beta,p}(a,b)}\|y_0\|$$

with $\lim_{\tau\to 0}\eta(\tau) = 0$.

Proof. By Theorem 4.1,

$$\|z\|_{W^{\beta,p}(0,\tau;D)} \le C(1)\|h\|_{W^{\beta,p}(0,\tau)}\|y_0\|$$

if, for example, $\tau \le 1$. To get the conclusion, it suffices to consider the case $y_0 \in D$; the general case will follow with a density argument. Therefore, we assume $y_0 \in D$. It is well known (and easy to show) that the part of A in D satisfies again condition (h1). So, applying again Theorem 4.1 with D replacing X, we obtain, for $\tau \le 1$,

$$\|z'\|_{W^{\beta,p}(0,\tau;D)} \le C(1)\|h\|_{W^{\beta,p}(0,\tau)}\|y_0\|_D.$$

Applying now Lemma 4.5 we get the conclusion ∎

Given $f \in W^{\beta,p}(0,T;X)$ $\ (1 < p \le +\infty$ and $0 < \beta < 1$ with $\beta \ne \frac{1}{p}; T > 0)$ and $\tau \in (0,T)$, we shall write simply $\|f\|_{W^{\beta,p}(\tau,T;X)}$ instead of $\|f_{|(\tau,T)}\|_{W^{\beta,p}(\tau,T;X)}$.

Lemma 4.8. *Let $f \in W^{\beta,p}(0,T;X)$ $(p \in (1,+\infty]$ and $\beta \in (0,1)$ with $\beta \neq \frac{1}{p}; 0 < \tau < T \leq T_0 < +\infty)$ be such that $f(t) = 0$ for $t \in (0,\tau)$. Then there exist constants $C_1, C_2 > 0$ independent of f, τ, T such that*

$$\|f\|_{W^{\beta,p}(\tau,T;X)} \leq C_1 \|f\|_{W^{\beta,p}(0,T;X)} \tag{4.14}$$

$$\|f\|_{W^{\beta,p}(0,T;X)} \leq C_2 \|f\|_{W^{\beta,p}(\tau,T;X)}. \tag{4.15}$$

Proof. It follows from the identity

$$\begin{aligned} &[f]^p_{W^{\beta,p}(0,T;X)} + (\beta p)^{-1} \int_0^T t^{-\beta p} \|f(t)\|^p dt \\ &= [f]^p_{W^{\beta,p}(\tau,T;X)} + (\beta p)^{-1} \int_\tau^T (t-\tau)^{-\beta p} \|f(t)\|^p dt \end{aligned} \tag{4.16}$$

and Proposition 4.1 ∎

5. Proofs of the main abstract results

We want to study system (2.7) under conditions (k1) - (k7) in Section 2. Since the proofs are complicated we precede by steps, proving intermediate lemmata. We are interested in the existence and the uniqueness of a solution (u,h) of problem (2.7) belonging to

$$(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)) \times W^{\beta,p}(0,\tau)$$

with $p \in (1,+\infty]$ and $\beta \in (0,1) \setminus \{\frac{1}{p}\}$, for some $\tau > 0$.

Assume that, for some $\tau > 0$,

$$(u,h) \in (W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;X)) \times W^{\beta,p}(0,\tau)$$

is a solution of system (2.7), for some $p \in (1,+\infty]$ and $\beta \in (0,1)$. We set, for $t \in [0,\tau]$,

$$A(t) = G'(u(t)) + \int_0^t h(t-s) d_1 F\big(u(t), u(s)\big)\, ds. \tag{5.1}$$

Clearly, $A(t) \in \mathcal{L}(D,X)$ and $A(0) = G'(u_0) = A$. If $v := \partial_t u$, we set also

$$\mathcal{U}(v,h)(t) = \int_0^t h(t-s) d_2 F\big(u_0 + 1 * v(t), u_0 + 1 * v(s)\big) v(s)\, ds \tag{5.2}$$

$$\chi(t) = \Phi\big(F(u(t), u_0)\big). \tag{5.3}$$

Observe that $v \in W^{1+\beta,p}(0,\tau;X) \cap W^{\beta,p}(0,\tau;D)$ and

$$\left.\begin{aligned} &v'(t) = A(t)v(t) + h(t)F(u(t),u_0) + \mathcal{U}(v,h)(t) + f'(t) \quad \big(t \in (0,\tau)\big) \\ &v(0) = G(u_0) + f(0) \end{aligned}\right\}. \tag{5.4}$$

We start with certain simple necessary conditions concerning the data:

Lemma 5.1. *Under assumptions* (k1) - (k7)*, the following conditions are necessary in order that problem* (2.7) *has a solution*

$$(u,h) \in \big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$$

for some $\tau > 0$ and some $p > 1$ and $\beta \in (0,1) \setminus \{\frac{1}{p}\}$:

(i) $G(u_0) + f(0) \in \begin{cases} D_A(1+\beta-\frac{1}{p},p) & \text{if } \beta < \frac{1}{p} \\ D & \text{if } \beta > \frac{1}{p}. \end{cases}$

(ii) $g \in W^{2+\beta,p}(0,\tau)$.

(iii) $\Phi(u_0) = g(0)$ *and* $\Phi(G(u_0) + f(0)) = g'(0)$.

Assume further that $\chi(0) \neq 0$ and $\beta > \frac{1}{p}$. Then the following further condition holds:

(iv) $A[G(u_0)+f(0)] + f'(0) + \mathcal{H}F(u_0,u_0) \in D_A(\beta - \frac{1}{p}, p)$ *with $\mathcal{H}$ defined as in* (2.8)*.*

Proof. Assertion (ii) is obvious. Concerning assertion (i), we have already observed that, if $v := \partial_t u$, then v lies in $W^{1+\beta,p}(0,\tau;X) \cap W^{\beta,p}(0,\tau;D)$ and solves (5.4). Therefore, assertion (i) follows from Theorem 4.1. Assertion (iii) follows immediately from the previous considerations. Finally, if $\beta > \frac{1}{p}$, we have

$$v'(0) = A[G(u_0) + f(0)] + h(0)F(u_0,u_0) + f'(0). \tag{5.5}$$

It remains to determine $h(0)$. Applying Φ to (5.5), we obtain

$$g''(0) = \Phi\big[A(G(u_0) + f(0)) + f'(0)\big] + \chi(0)h(0)$$

which implies

$$h(0) = \mathcal{H}. \tag{5.6}$$

Then the conclusion follows from Theorem 4.1 ∎

Remark 5.1. We observe explicitly the fact that, under the assumption $\chi(0) \neq 0$ depending only on u_0, in the case $\beta > \frac{1}{p}$, $h(0)$ is uniquely determined by (2.8).

Let now $v \in W^{\beta,p}(0,\tau;D)$ $(\tau > 0, p \in (1,+\infty]$ and $\beta \in (0,1)$ with $\beta \neq \frac{1}{p})$ and $h \in W^{\beta,p}(0,\tau)$, and set for a fixed $u_0 \in D$

$$A_{v,h}(t) = G'\big(u_0 + 1*v(t)\big) + \int_0^t h(t-s)d_1F\big(u_0+1*v(t), u_0+1*v(s)\big)ds. \tag{5.7}$$

The following three lemmata have in common the following assumptions, which we write once and for all and indicate globally with (H):

(H) There exists $\tau > 0$ and $R > 0$ such that, if $\delta \in [0,\tau)$, $v, v_1, v_2 \in W^{\beta,p}(0,\tau;D)$ and $h, h_1, h_2 \in W^{\beta,p}(0,\tau)$ $(1 < p \le +\infty, \beta \in (0,1) \setminus \{\frac{1}{p}\})$, then

$$\max\left\{\begin{array}{l} \|v\|_{W^{\beta,p}(0,\tau;D)}, \|v_1\|_{W^{\beta,p}(0,\tau;D)}, \|v_2\|_{W^{\beta,p}(0,\tau;D)}, \\ \|v_1\|_{W^{\beta,p}(\delta,\tau;D)}, \|v_2\|_{W^{\beta,p}(\delta,\tau;D)}, \|h\|_{W^{\beta,p}(0,\tau)}, \\ \|h_1\|_{W^{\beta,p}(0,\tau)}, \|h_2\|_{W^{\beta,p}(0,\tau)}, \|h_1\|_{W^{\beta,p}(\delta,\tau)}, \|h_2\|_{W^{\beta,p}(\delta,\tau)} \end{array}\right\} \le R.$$

Further, $v_1(t) = v_2(t)$ and $h_1(t) = h_2(t)$ if $0 < t < \delta$ (this condition is dropped if $\delta = 0$).

Lemma 5.2. *Under condition* (H) *we have:*

(i) $[A_{v,h} - G'(u_0)]v \in W^{\beta,p}(0,\tau;X)$.

(ii) *There exist $M(R) > 0$ and $\alpha > 0$ independent of $\tau \in (0,\tau_0]$ such that*

$$\big\|[A_{v,h} - G'(u_0)]v\big\|_{W^{\beta,p}(0,\tau;X)} \le M(R)\tau^\alpha.$$

(iii)
$$\left\{\begin{array}{l} \Big\|\big[A_{v_1,h_1} - A_{v_1,h_1}(\delta)\big]v_1 - \big[A_{v_2,h_2} - A_{v_2,h_2}(\delta)\big]v_2\Big\|_{W^{\beta,p}(\delta,\tau;X)} \\ \le M(R)(\tau-\delta)^\alpha\Big[\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)} + \|h_1 - h_2\|_{W^{\beta,p}(\delta,\tau)}\Big]. \end{array}\right.$$

Proof. The mapping $u_0 + 1 * v$ belongs to $W^{1+\beta,p}(0,\tau;D)$. So, owing to Lemma 4.4, $t \to G'(u_0 + 1 * v(t)) - G'(u_0)$ belongs to $W^{\gamma,p}\big(0,\tau;\mathcal{L}(D,X)\big)$ for every $\gamma \in (0,1)$. Therefore from Lemma 4.3 we have

$$\big[G'(u_0 + 1 * v(t)) - G'(u_0)\big]v \in W^{\beta,p}(0,\tau;X).$$

Fix now $\gamma \in (\beta \vee \frac{1}{p}, 1)$. From Lemma 4.3 we get

$$\begin{aligned} &\big\|\big[G'(u_0 + (1 * v)) - G'(u_0)\big]v\big\|_{W^{\beta,p}(0,\tau;X)} \\ &\qquad \le CR\tau^{\gamma - \beta \vee \frac{1}{p}}\big[G'(u_0 + (1 * v)) - G'(u_0)\big]_{W^{\gamma,p}(0,\tau;\mathcal{L}(D,X))} \end{aligned}$$

with $C > 0$ independent of $\tau \le \tau_0$ and R. Further, from Lemmata 4.1, 4.4 and 4.5 we have

$$\begin{aligned} \big[G'(u_0 + (1 * v)) - G'(u_0)\big]_{W^{\gamma,p}(0,\tau;\mathcal{L}(D,X))} &\le C(R)[1 * v]_{W^{\gamma,p}(0,\tau;D)} \\ &\le C(R)\tau^{1-\gamma+\beta\wedge\frac{1}{p}}\|v\|_{W^{\beta,p}(0,\tau;D)}. \end{aligned}$$

Next, we have

$$\begin{aligned} &\int_0^t h(t-s)d_1F\big(u_0 + 1 * v(t), u_0 + 1 * v(s)\big)v(t)\,ds \\ &\quad = (1 * h)(t)d_1F\big(u_0 + 1 * v(t), u_0\big)v(t) \\ &\qquad + \int_0^t h(t-s)\Big[d_1F\big(u_0 + 1 * v(t), u_0 + 1 * v(s)\big) \\ &\qquad - d_1F\big(u_0 + 1 * v(t), u_0\big)\Big]ds\; v(t). \end{aligned}$$

Considering the first summand, we have for a fixed $\gamma > \beta \vee \frac{1}{p}$

$$
\begin{aligned}
&\big\|(1*h)d_1F(u_0+1*v,u_0)v\big\|_{W^{\beta,p}(0,\tau;X)}\\
&\quad\le C\tau^{\gamma-\beta\vee\frac{1}{p}}[1*h]_{W^{\gamma,p}(0,\tau)}\big\|d_1F(u_0+1*v,u_0)v\big\|_{W^{\beta,p}(0,\tau;X)}\\
&\qquad\text{(with C} > 0 \text{ independent of } \tau,\ \text{by Lemma 4.3)}\\
&\quad\le C(R)\tau^{1-|\beta-\frac{1}{p}|}\big\|d_1F(u_0+1*v,u_0)\big\|_{W^{\gamma,p}(0,\tau;\mathcal{L}(D,X))}\\
&\qquad\text{(by Lemmata 4.3 and 4.5)}\\
&\quad\le C(R)\tau^{1-|\beta-\frac{1}{p}|}\\
&\qquad\text{(by Lemma 4.4).}
\end{aligned}
$$

Now we pass to estimate

$$
I := \int_0^{\cdot} h(\cdot-s)\Big[d_1F\big(u_0+1*v(\cdot),u_0+1*v(s)\big)-d_1F(u_0+1*v(\cdot),u_0)\Big]ds\ v(\cdot).
$$

Using Lemmata 4.2 - 4.6 we get

$$
\begin{aligned}
\|I\|_{W^{\beta,p}(0,\tau;X)} &\le C(R)\|h\|_{L^1(0,\tau)}\sup_{0<\sigma<\tau}\Big[d_1F(u_0+1*v(\cdot+\sigma),u_0+1*v)\\
&\qquad - d_1F(u_0+1*v(\cdot+\sigma),u_0)\Big]_{W^{\gamma,p}(0,\tau-\sigma;\mathcal{L}(D,X))}\\
&\le C(R)\tau^{1\wedge(\beta+\frac{1}{p'})}\sup_{0<\sigma<\tau}[1*v(\cdot+\sigma)]_{W^{\gamma,p}(0,\tau-\sigma;D)}\\
&\le C(R)\tau^{1\wedge(\beta+\frac{1}{p'})}.
\end{aligned}
$$

So assertions (i) and (ii) are proved.

To prove assertion (iii), for simplicity we set

$$
\begin{aligned}
A_j(t) &= A_{v_j,h_j}(t)\\
u_j(t) &= u_0 + 1*v_j(t)\\
B_j(t) &= \textstyle\int_0^t h_j(t-s)d_1F\big(u_j(t),u_j(s)\big)ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\Big\|\big[A_1(\cdot)-A_1(\delta)\big]v_1(\cdot)-\big[A_2(\cdot)-A_2(\delta)\big]v_2(\cdot)\Big\|_{W^{\beta,p}(\delta,\tau;X)}\\
&\quad\le \Big\|\big[G'(u_1(\cdot))-G'(u(\delta))\big]v_1(\cdot)-\big[G'(u_2(\cdot))-G'(u(\delta))\big]v_2(\cdot)\Big\|_{W^{\beta,p}(\delta,\tau;X)}\\
&\qquad+\Big\|\big[B_1(\cdot)-B_1(\delta)\big]v_1(\cdot)-\big[B_2(\cdot)-B_2(\delta)\big]v_2(\cdot)\Big\|_{W^{\beta,p}(\delta,\tau;X)}.
\end{aligned}
$$

With the previous arguments we get

$$\Big\|\big[G'(u_1(\cdot)) - G'(u(\delta))]v_1(\cdot) - \big[G'(u_2(\cdot)) - G'(u(\delta))]v_2(\cdot)\Big\|_{W^{\beta,p}(\delta,\tau;X)}$$
$$\le C(R)(\tau-\delta)^\alpha \|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)}$$

for some $\alpha > 0$. Next,

$$\begin{aligned}
&\Big\|\big[B_1(\cdot) - B_1(\delta)\big]v_1(\cdot) - \big[B_2(\cdot) - B_2(\delta)\big]v_2(\cdot)\Big\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\quad \le \big\|\big[B_1(\cdot) - B_2(\cdot)\big]v_1(\cdot)\big\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\qquad + \big\|\big[B_2(\cdot) - B_2(\delta)\big]\ \big[v_1(\cdot) - v_2(\cdot)\big]\big\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\quad \le C(\tau-\delta)^{\gamma-\beta\vee\frac{1}{p}}\Big\{R\big[B_1(\cdot) - B_2(\cdot)\big]_{W^{\gamma,p}(\delta,\tau;\mathcal{L}(D,X))} \\
&\qquad + \big[B_2(\cdot) - B_2(\delta)\big]_{W^{\gamma,p}(\delta,\tau;\mathcal{L}(D,X))}\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)}\Big\}
\end{aligned}$$

for a fixed $\gamma \in (\frac{1}{p}, 1)$ where we have used Lemma 4.3. Then, using Lemma 4.8 and the arguments of the proof of assertion (ii),

$$\begin{aligned}
&\big[B_1(\cdot) - B_2(\cdot)\big]_{W^{\gamma,p}(\delta,\tau;\mathcal{L}(D,X))} \\
&\qquad \le C_1\big[B_1(\cdot) - B_2(\cdot)\big]_{W^{\gamma,p}(0,\tau;\mathcal{L}(D,X))} \\
&\qquad \le C_2\big[\|v_1 - v_2\|_{W^{\beta,p}(0,\tau;D)} + \|h_1 - h_2\|_{W^{\beta,p}(0,\tau)}\big] \\
&\qquad \le C_3\big[\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)} + \|h_1 - h_2\|_{W^{\beta,p}(\delta,\tau)}\big]
\end{aligned}$$

applying again Lemma 4.8. Finally, from the first part of the proof,

$$\big[B_2(\cdot) - B_2(\tau)\big]_{W^{\gamma,p}(\delta,\tau;\mathcal{L}(D,X))} \le C(R)$$

and assertion (iii) is also proved ∎

Lemma 5.3. *Under condition* (H) *we have:*

(i) $h(\cdot)\big[F(u_0 + 1 * v(\cdot), u_0) - F(u_0, u_0)\big] \in W^{\beta,p}(0,\tau;X)$.

(ii) *There exist* $M(R) > 0$ *and* $\alpha > 0$ *independent of* $\tau \in (0, \tau_0]$ *such that*

$$\big\|h(\cdot)\big[F\big(u_0 + 1 * v(\cdot), u_0\big) - F(u_0, u_0)\big]\big\|_{W^{\beta,p}(0,\tau;X)} \le M(R)\tau^\alpha.$$

$$\textbf{(iii)}\ \begin{cases}
\Big\|h_1(\cdot)\big[F(u_0 + 1 * v_1(\cdot), u_0) - F(u_0 + 1 * v_1(\delta), u_0)\big] \\
- h_2(\cdot)\big[F(u_0 + 1 * v_2(\cdot), u_0) - F(u_0 + 1 * v_2(\delta), u_0)\big]\Big\|_{W^{\beta,p}(\delta,\tau;X)} \\
\le M(R)(\tau-\delta)^\alpha\big[\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)} + \|h_1 - h_2\|_{W^{\beta,p}(\delta,\tau)}\big].
\end{cases}$$

Proof. The proof follows the same arguments as in the proof of Lemma 5.2 and will not be worked out ∎

Lemma 5.4. *Under condition* (H) *we have:*

(i) $\mathcal{U}(v,h) \in W^{\beta,p}(0,\tau;X)$.

(ii) *There exist $M(R) > 0$ and $\alpha > 0$ independent of $\tau \in (0,\tau_0]$ such that*

$$\|\mathcal{U}(v,h)\|_{W^{\beta,p}(0,\tau;X)} \leq M(R)\tau^{\alpha}.$$

(iii) $$\begin{cases} \big\|\mathcal{U}(v_1,h_1) - \mathcal{U}(v_2,h_2)\big\|_{W^{\beta,p}(\delta,\tau;X)} \\ \leq M(R)(\tau-\delta)^{\alpha}\big[\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)} + \|h_1 - h_2\|_{W^{\beta,p}(\delta,\tau)}\big]. \end{cases}$$

Proof. We have

$$\begin{aligned} \mathcal{U}(v,h)(t) = {} & d_2F\big(u_0 + 1*v(t), u_0\big)(h*v)(t) \\ & + \int_0^t h(t-s)\Big[d_2F\big(u_0 + 1*v(t), u_0 + 1*v(s)\big) \\ & - d_1F\big(u_0 + 1*v(t), u_0\big)\Big]v(s)\,ds. \end{aligned}$$

By Lemmata 4.3 and 4.4,

$$\big\|d_2F(u_0 + 1*v, u_0)(h*v)\big\|_{W^{\beta,p}(0,\tau;X)} \leq C(R)\|h*v\|_{W^{\beta,p}(0,\tau;D)}.$$

If $\beta < \frac{1}{p}$, then owing to Lemma 4.6

$$\|h*v\|_{W^{\beta,p}(0,\tau;D)} \leq C\|h\|_{L^1(0,\tau)}\|v\|_{W^{\beta,p}(0,\tau;D)} \leq C(R)\tau^{\beta+\frac{1}{p'}}$$

by Lemma 4.2. If contrary $\beta > \frac{1}{p}$, then

$$\begin{aligned} \|h*v\|_{W^{\beta,p}(0,\tau;D)} & \leq \|h*(v - v(0))\|_{W^{\beta,p}(0,\tau;D)} + \|1*h\|_{W^{\beta,p}(0,\tau)}\|v(0)\|_D \\ & \leq C(\|h\|_{L^1(0,\tau)}[v]_{W^{\beta,p}(0,\tau;D)} + \|v(0)\|_D\|1*h\|_{W^{\beta,p}(0,\tau)}) \\ & \leq C(R)\tau^{1+\frac{1}{p}-\beta} \end{aligned}$$

owing to Lemmata 4.2, 4.5 and 4.6.

Next, we consider

$$I_2 = \int_0^{\cdot} h(\cdot - s)\Big[d_2F\big(u_0 + 1*v(\cdot), u_0 + 1*v(s)\big) - d_2F\big(u_0 + 1*v(\cdot), u_0\big)\Big]v(s)\,ds.$$

We have, by Lemmata 4.2 - 4.4 and 4.6, for a fixed $\gamma \in (\beta, 1)$ with $\gamma > \frac{1}{p}$,

$$\begin{aligned} \|I_2\|_{W^{\beta,p}(0,\tau;X)} & \leq C\|h\|_{L^1(0,\tau)} \sup_{0<\sigma<\tau}\Big\|\Big[d_2F\big(u_0 + 1*v(\cdot+\sigma), u_0 + 1*v\big) \\ & \qquad - d_2F\big(u_0 + 1*v(\cdot+\sigma), u_0\big)\Big]v\Big\|_{W^{\beta,p}(0,\tau-\sigma;X)} \\ & \leq C(R)\tau^{1\wedge(\beta+\frac{1}{p'})} \sup_{0<\sigma<\tau}\Big[d_2F\big(u_0 + 1*v(\cdot+\sigma), u_0 + 1*v\big) \\ & \qquad - d_2F\big(u_0 + 1*v(\cdot+\sigma), u_0\big)\Big]_{W^{\gamma,p}(0,\tau-\sigma;\mathcal{L}(D,X))} \\ & \leq C(R)\tau^{1\wedge(\beta+\frac{1}{p'})}. \end{aligned}$$

Finally we show assertion (iii). For simplicity we set $u_j(t) = u_0 + 1 * v_j(t)$. We have

$$\mathcal{U}(v_1, h_1) - \mathcal{U}(v_2, h_2) = \mathcal{U}(v_1, h_1 - h_2) + \big[\mathcal{U}(v_1, h_2) - \mathcal{U}(v_2, h_2)\big].$$

With the same method used in the proof of assertion (i) we get, employing preliminarly Lemma 4.8,

$$\begin{aligned}
&\|\mathcal{U}(v_1, h_1 - h_2)\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\quad \le C\|\mathcal{U}(v_1, h_1 - h_2)\|_{W^{\beta,p}(0,\tau;X)} \\
&\quad \le C(R)(\|h_1 - h_2\|_{L^1(\delta,\tau)} + \|1 * (h_1 - h_2)\|_{W^{\beta,p}(\delta,\tau)}) \\
&\quad \le C(R)(\tau - \delta)^{1-|\beta-\frac{1}{p}|}\|h_1 - h_2\|_{W^{\beta,p}(\delta,\tau)}.
\end{aligned}$$

Next we have

$$\begin{aligned}
&\big\|\mathcal{U}(v_1, h_2) - \mathcal{U}(v_2, h_2)\big\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\quad \le \bigg\| \int_0^{\cdot} h_2(\cdot - s)\big[d_2F(u_1(\cdot), u_1(s)) - d_2F(u_2(\cdot), u_2(s))\big]v_1(s)\,ds\bigg\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\qquad + \bigg\| \int_0^{\cdot} h_2(s)d_2F\big(u_2(\cdot), u_2(\cdot - s)\big)\big[v_1(\cdot - s) - v_2(\cdot - s)\big]ds\bigg\|_{W^{\beta,p}(\delta,\tau;X)}
\end{aligned}$$

In the case $\beta < \frac{1}{p}$, the first summand can be majorized, for a fixed $\gamma \in (\frac{1}{p}, 1)$, with

$$\begin{aligned}
&C_1\bigg\| \int_0^{\cdot} h_2(\cdot - s)\Big[d_2F\big(u_1(\cdot), u_1(s)\big) - d_2F\big(u_2(\cdot), u_2(s)\big)\Big]v_1(s)\,ds\bigg\|_{W^{\beta,p}(0,\tau;X)} \\
&\quad \le C(R) \sup_{0<\sigma<\tau} \Big\|d_2F\big(u_1(\cdot + \sigma), u_1(\cdot)\big) - d_2F\big(u_2(\cdot + \sigma), u_2(\cdot)\big)\Big\|_{W^{\gamma,p}(0,\tau-\sigma;\mathcal{L}(D,X))} \\
&\quad \le C(R) \sup_{0<\sigma<\tau} \big\|u_1(\cdot + \sigma) - u_2(\cdot + \sigma)\big\|_{W^{\gamma,p}(0,\tau-\sigma;D)} \\
&\quad \le C(R)\bigg[\sup_{\delta<\sigma<\tau} \|u_1(\sigma) - u_2(\sigma)\|_D + [u_1 - u_2]_{W^{\gamma,p}(0,\tau;D)}\bigg].
\end{aligned}$$

By Lemmata 4.1 and 4.5,

$$\begin{aligned}
\sup_{\delta<\sigma<\tau} \|u_1(\sigma) - u_2(\sigma)\|_D &\le C(\tau - \delta)^{\gamma-\frac{1}{p}}[u_1 - u_2]_{W^{\gamma,p}(\delta,\tau;D)} \\
&\le C(\tau - \delta)^{1+\beta-\frac{1}{p}}\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)}
\end{aligned}$$

and, from Lemmata 4.8 and 4.5,

$$\begin{aligned}
[u_1 - u_2]_{W^{\gamma,p}(0,\tau;D)} &\le C_2[u_1 - u_2]_{W^{\gamma,p}(\delta,\tau;D)} \\
&\le C(\tau - \delta)^{1+\beta-\gamma}\|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)}.
\end{aligned}$$

The case $\beta > \frac{1}{p}$ can be treated similarly, observing that

$$\begin{aligned}
&\Big\| \int_0^{\cdot} h_2(\cdot - s)\big[d_2F(u_1(\cdot), u_1(s)) - d_2F(u_2(\cdot), u_2(s))\big] v_1(s)\, ds \Big\|_{W^{\beta,p}(0,\tau;X)} \\
&\quad \le \Big\| \int_0^{\cdot} h_2(\cdot - s)\Big[d_2F(u_1(\cdot), u_1(s)) - d_2F(u_2(\cdot), u_2(s)) \\
&\qquad - d_2F(u_1(\cdot), u_0) + d_2F(u_2(\cdot), u_0)\Big] v_1(s)\, ds \Big\|_{W^{\beta,p}(0,\tau;X)} \\
&\qquad + \Big\| \big[d_2F(u_1(\cdot), u_0) - d_2F(u_1(\cdot), u_0)\big]\, \big[h_2 * (v_1 - v_1(0))\big] \Big\|_{W^{\beta,p}(0,\tau;X)} \\
&\qquad + \Big\| \big[d_2F(u_1(\cdot), u_0) - d_2F(u_1(\cdot), u_0)\big] (1 * h_2) v_1(0) \Big\|_{W^{\beta,p}(0,\tau;X)}
\end{aligned}$$

and choosing $\gamma = \beta$.

Finally, from Lemmata 4.8 and 4.6, if $\beta < \frac{1}{p}$,

$$\begin{aligned}
&\Big\| \int_0^{\cdot} h_2(s) d_2F\big(u_2(\cdot), u_2(\cdot - s)\big)\big[v_1(\cdot - s) - v_2(\cdot - s)\big] ds \Big\|_{W^{\beta,p}(\delta,\tau;X)} \\
&\quad \le C_1 \Big\| \int_0^{\cdot} h_2(s) d_2F\big(u_2(\cdot), u_2(\cdot - s)\big)\big[v_1(\cdot - s) - v_2(\cdot - s)\big] ds \Big\|_{W^{\beta,p}(0,\tau;X)} \\
&\quad \le C \|v_1 - v_2\|_{L^1(\delta,\tau;D)} \sup_{0<\sigma<\tau} \big\| h_2(\cdot) d_2F\big(u_2(\cdot + \sigma), u_2(\sigma)\big)\big\|_{W^{\beta,p}(0,\tau-\sigma;\mathcal{L}(D,X))} \\
&\quad \le C(R)(\tau - \delta)^{\beta + \frac{1}{p'}} \|v_1 - v_2\|_{W^{\beta,p}(\delta,\tau;D)}.
\end{aligned}$$

The case $\beta > \frac{1}{p}$ can be treated with the same techniques, observing that

$$\begin{aligned}
&\int_0^t h_2(s) d_2F\big(u_2(t), u_2(t - s)\big)\big[v_1(t - s) - v_2(t - s)\big] ds \\
&\quad = \int_0^t h_2(s)\big[d_2F\big(u_2(t), u_2(t - s)\big) - d_2F\big(u_2(t), u_2(t)\big)\big]\, \big[v_1(t - s) - v_2(t - s)\big] ds \\
&\qquad + d_2F\big(u_2(t), u_2(t)\big)\big[h_2 * (v_1 - v_2)\big](t).
\end{aligned}$$

Thus Lemma 5.4 is proved ∎

We set now

$$\begin{aligned}
\mathcal{R}(v,h)(t) = {} & \big[A_{v,h}(t) - A_{v,h}(0)\big] v(t) \\
& + h(t)\big[F(u_0 + 1 \circ v(t), u_0) - F(u_0, u_0)\big] \qquad (5.8) \\
& + \mathcal{U}(v,h)(t).
\end{aligned}$$

Lemma 5.5. *Let $p \in (1, +\infty]$ and $\beta \in (0,1) \setminus \{\frac{1}{p}\}$. Assume that conditions* (k1) - (k7) *and, moreover, conditions* (i) - (iii) *of Lemma* 5.1 *together with condition* (iv) *in the case $\beta > \frac{1}{p}$ hold. Indicate with $\{S(t)\}_{t \geq 0}$ the semigroup generated by $A = G'(u_0)$. Let $\tau > 0$ and (u, h) a solution of problem* (2.7) *belonging to*

$$\big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$$

for some $\tau > 0$ and set $v = \partial_t u$. Then $(v, h) \in W^{\beta,p}(0,\tau;D) \times W^{\beta,p}(0,\tau)$ and (v, h) solves the system

$$\left.\begin{aligned} v(t) &= S(t)[G(u_0) + f(0)] + \int_0^t S(t-s) f'(s)\, ds \\ &\quad + \int_0^t S(t-s) h(s) F(u_0, u_0)\, ds + \int_0^t S(t-s) \mathcal{R}(v,h)(s)\, ds \\ h(t) &= \chi(0)^{-1} \big[g''(t) - \Phi\big(Av(t) + \mathcal{R}(v,h)(t) + f'(t)\big)\big] \end{aligned}\right\} \tag{5.9}$$

with $\mathcal{R}$ defined in (5.8).

On the other hand, let $(v, h) \in W^{\beta,p}(0,\tau;D) \times W^{\beta,p}(0,\tau)$ be a solution of system (5.9) *such that, if $\beta > \frac{1}{p}$, $h(0) = \mathcal{H}$, and set $u = u_0 + 1 * v$. Then $u \in W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)$ and (u, h) solves problem* (2.7).

Proof. Let (u, h) be a solution of problem (2.7) belonging to

$$\big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau)$$

for some $\tau > 0$. Then $v := \partial_t u \in W^{1+\beta,p}(0,\tau;X) \cap W^{\beta,p}(0,\tau;D)$ and we have already observed that v is a solution of problem (5.4). It follows immediately from Theorem 4.1 that the first equation in system (5.9) is satisfied. The second equation can be obtained applying Φ to both terms of (5.4) and using the fact that $\Phi(v'(t)) = g''(t)$.

On the other hand, let $(v, h) \in W^{\beta,p}(0,\tau;D) \times W^{\beta,p}(0,\tau)$ be a solution of system (5.9) and define

$$g(t) = f'(t) + h(t) F(u_0, u_0) + \mathcal{R}(v,h)(t). \tag{5.10}$$

Then $g \in W^{\beta,p}(0,\tau;X)$, owing to Lemmata 5.2 - 5.4. Owing to condition (i) of Lemma 5.1 and condition (iv) in the case $\beta > \frac{1}{p}$, applying Theorem 4.1 we obtain that v belongs to $W^{1+\beta,p}(0,\tau;X) \cap W^{\beta,p}(0,\tau;D)$ and solves problem (5.4). Then $u \in W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)$. The equation in (5.4) can be written in the form

$$u''(t) = \frac{d}{dt}\Big(G(u(t)) + \int_0^t h(t-s) F(u(t), u(s))\, ds + f(t)\Big). \tag{5.11}$$

The condition $u'(0) = v(0) = G(u_0)+f(0)$ implies that the two first conditions in problem (2.7) are satisfied. It remains to show that $\Phi \circ u = g$. From the second equation in system (5.9) we have

$$\begin{aligned} g''(t) &= \Phi\big[Av(t) + h(t)F(u_0, u_0) + \mathcal{R}(v,h)(t) + f'(t)\big] \\ &= \Phi(v'(t)) = \Phi(u''(t)) = (\Phi \circ u)''(t). \end{aligned}$$

Then we get the conclusion from condition (iii) of Lemma 5.1 ∎

We are now in position to prove the main result of the paper. We define, preliminarily,

$$\begin{aligned} v_0(t) = S(t)[G(u_0) + f(0)] \\ + \int_0^t S(t-s) f'(s)\, ds \\ + \mathcal{H}_{p,\beta} \int_0^t S(t-s) F(u_0, u_0)\, ds \end{aligned} \tag{5.12}$$

with

$$\mathcal{H}_{p,\beta} = \begin{cases} 0 & \text{if } 0 < \beta < \frac{1}{p} \\ \mathcal{H} & \text{if } \frac{1}{p} < \beta < 1 \end{cases} \tag{5.13}$$

and

$$h_0(t) = \chi(0)^{-1}\big[g''(t) - \Phi\big(Av_0(t) + f'(t)\big)\big]. \tag{5.14}$$

Proof of Theorem 2.1. By Lemma 5.5 we can get solutions of prescribed regularity of problem (2.7) looking for solutions in $B(\tau, R)$ of system (5.9). We put

$$\mathcal{S}(v,h)(t) = \int_0^t S(t-s)\big[\mathcal{R}(v,h)(s) + (h(s) - \mathcal{H}_{p,\beta})F(u_0, u_0)\big] ds, \tag{5.15}$$

write system (5.9) in the form

$$\left.\begin{aligned} v(t) &= v_0(t) + \mathcal{S}(v,h)(t) \\ h(t) &= h_0(t) - \chi(0)^{-1}\Phi\big(A\mathcal{S}(v,h)(t) + \mathcal{R}(v,h)(t)\big) \end{aligned}\right\} \tag{5.16}$$

and define

$$\Gamma(v,h) = \Big(v_0 + \mathcal{S}(v,h), h_0 - \chi(0)^{-1}\Phi\big(A\mathcal{S}(v,h) + \mathcal{R}(v,h)\big)\Big). \tag{5.17}$$

Then, if $(v,h) \in B(\tau, R)$, owing to Lemmata 5.2 - 5.4 and Theorem 4.1, $\Gamma(v,h) \in W^{\beta,p}(0,\tau; D) \times W^{\beta,p}(0,\tau)$. Moreover, if $\beta > \frac{1}{p}$, then

$$h_0(0) - \chi(0)^{-1}\Phi\big(A\mathcal{S}(v,h)(0) + \mathcal{R}(v,h)(0)\big) = \mathcal{H}.$$

Next, using Theorem 4.1, Lemmata 5.2 - 5.4 and 4.7, we have

$$\|\mathcal{S}(v,h)\|_{W^{\beta,p}(0,\tau;D)} \le C(T)\big[M(R')\tau^{\alpha} + \eta(\tau)R'\big]$$

with

$$R' = R + \max\big\{\|v_0\|_{W^{\beta,p}(0,T;D)},\ \|h_0\|_{W^{\beta,p}(0,T)}\big\}. \tag{5.18}$$

In the same way, for some $\alpha > 0$,

$$\begin{aligned}&\big\|\chi(0)^{-1}\Phi\big(A\mathcal{S}(v,h) + \mathcal{R}(v,h)\big)\big\|_{W^{\beta,p}(0,\tau)}\\ &\qquad\le |\chi(0)|^{-1}\|\Phi\|_{X'}\big[C(T)M(R')\tau^{\alpha} + \eta(\tau)R'\big].\end{aligned}$$

We deduce that, if τ is sufficiently small, then

$$\Gamma(B(\tau,R)) \subseteq B(\tau,R). \tag{5.19}$$

With the same arguments one can show that there exists $C(R) > 0$ such that, if (v_1,h_1) and (v_2,h_2) belong to $B(\tau,R)$, one has

$$\begin{aligned}&\max\left\{\begin{array}{l}\big\|\mathcal{S}(v_1,h_1) - \mathcal{S}(v_2,h_2)\big\|_{W^{\beta,p}(0,\tau;D)}\\ \big\|\chi(0)^{-1}\Phi\big(A\mathcal{S}(v_1,h_1) + \mathcal{R}(v_1,h_1) - A\mathcal{S}(v_2,h_2) - \mathcal{R}(v_2,h_2)\big)\big\|_{W^{\beta,p}(0,\tau)}\end{array}\right\}\\ &\quad\le C(R)(\tau^{\alpha} + \eta(\tau))\max\big\{\|v_1 - v_2\|_{W^{\beta,p}(0,\tau;D)}, \|h_1 - h_2\|_{W^{\beta,p}(0,\tau)}\big\}. \quad (5.20)\end{aligned}$$

The contraction mapping theorem gives the conclusion ∎

Next, we give the

Proof of Theorem 2.2. Let (u_2,h_2) be another solution of problem (2.7) belonging to

$$\big(W^{2+\beta,p}(0,T;X) \cap W^{1+\beta,p}(0,T;X)\big) \times W^{\beta,p}(0,T).$$

To get the conclusion, we can show that, if $\delta \in [0,T)$ is such that $u_1(t) = u_2(t)$ for all $t \in [0,\delta]$ and $h_1(t) = h_2(t)$ almost everywhere in $[0,\delta]$, then there exists $\tau \in (\delta,T]$ such that $u_1(t) = u_2(t)$ for all $t \in [0,\tau]$ and $h_1(t) = h_2(t)$ almost everywhere in $[0,\tau]$.

Set $v_2 = \partial_t u_2$ and $A = A_{v_1,h_1}(\delta) = A_{v_2,h_2}(\delta)$, indicate with $\{S(t)\}_{t\ge 0}$ the semigroup generated by A in X and finally set

$$R = \max\left\{\begin{array}{l}\|v_1\|_{W^{\beta,p}(0,T;D)},\ \|v_2\|_{W^{\beta,p}(0,T;D)},\ \|v_1\|_{W^{\beta,p}(\delta,T;D)},\ \|v_2\|_{W^{\beta,p}(\delta,T;D)}\\ \|h_1\|_{W^{\beta,p}(0,T)},\ \|h_2\|_{W^{\beta,p}(0,T)},\ \|h_1\|_{W^{\beta,p}(\delta,T)},\ \|h_2\|_{W^{\beta,p}(\delta,T)}\end{array}\right\}.$$

We have

$$\left.\begin{aligned}(v_1-v_2)'(t) = {}& A(v_1-v_2)(t)\\ &+\Big\{\big[A_{v_1,h_1}(t)-A_{v_1,h_1}(\delta)\big]v_1(t)\\ &-\big[A_{v_2,h_2}(t)-A_{v_2,h_2}(\delta)\big]v_2(t)\Big\}\\ &+[h_1(t)-h_2(t)]F(u_1(\delta),u_0)\\ &+\Big\{h_1(t)\big[F(u_1(t),u_0)-F(u_1(\delta),u_0)\big]\\ &-h_2(t)\big[F(u_2(t),u_0)-F(u_2(\delta),u_0)\big]\Big\}\\ &+\mathcal{U}(v_1,h_1)(t)-\mathcal{U}(v_2,h_2)(t)\quad\big(t\in(\delta,T)\big)\\ (v_1-v_2)(\delta) = {}& 0\end{aligned}\right\} \tag{5.21}$$

Using Theorem 4.1 and Lemmata 4.7, 5.2/(ii), 5.3/(iii) and 5.4/(iii), we deduce

$$\begin{aligned}&\|v_1-v_2\|_{W^{\beta,p}(\delta,\tau;D)}\\ &\quad\le C(R)\eta(\tau-\delta)\big[\|v_1-v_2\|_{W^{\beta,p}(\delta,\tau;D)}+\|h_1-h_2\|_{W^{\beta,p}(\delta,\tau)}\big]\end{aligned} \tag{5.22}$$

with $\lim_{r\to 0}\eta(r)=0$. This estimate implies that, if $\tau-\delta$ is sufficiently small,

$$\|v_1-v_2\|_{W^{\beta,p}(\delta,\tau;D)}\le C(R)\eta(\tau-\delta)\|h_1-h_2\|_{W^{\beta,p}(\delta,\tau)}. \tag{5.23}$$

Observe now that

$$\begin{aligned}h_1(t)-h_2(t) = {}&\Phi\big(F(u_1(\delta),u_0)\big)^{-1}\Big\{\Phi\Big[A\big(v_2(t)-v_1(t)\big)\\ &+\big(A_{v_2,h_2}(t)-A_{v_2,h_2}(\delta)\big)v_2(t)\\ &-\big(A_{v_1,h_1}(t)-A_{v_1,h_1}(\delta)\big)v_1(t)\\ &+h_2(t)\big(F(u_2(t),u_0)-F(u_2(\delta),u_0)\big)\\ &-h_1(t)\big(F(u_1(t),u_0)-F(u_1(\delta),u_0)\big)\\ &+\mathcal{U}(v_2,h_2)(t)-\mathcal{U}(v_1,h_1)(t)\Big]\Big\}\end{aligned} \tag{5.24}$$

implying

$$\begin{aligned}&\|h_1-h_2\|_{W^{\beta,p}(\delta,\tau)}\\ &\quad\le C\Big(\|v_1-v_2\|_{W^{\beta,p}(\delta,\tau;D)}+\eta(\tau-\delta)\|h_1-h_2\|_{W^{\beta,p}(\delta,\tau)}\Big).\end{aligned} \tag{5.25}$$

Clearly, (5.25) and (5.23) imply together that, if $\tau-\delta$ is sufficiently small, $h_1(t)=h_2(t)$ almost everywhere in $[0,\tau]$. This fact and (5.23) together allow to get the desired conclusion ∎

Proof of Corollary 2.1. By Theorem 2.1 there exists $\tau > 0$ such that problem (2.7) has a solution

$$(u,h) \in \big(W^{2+\beta,p}(0,\tau;X) \cap W^{1+\beta,p}(0,\tau;D)\big) \times W^{\beta,p}(0,\tau).$$

If τ is sufficiently small, by continuity, all assumptions a) - c) of Theorem 2.2 are satisfied (setting $(u_1,h_1) = (u,h)$). So Theorem 2.2 implies the result ∎

6. Appendix: Proof of Theorem 4.1

In this section will shall give a detailed proof of Theorem 4.1. Another proof of this maximal regularity result (without estimate (4.6)) was obtained also applying the extrapolation techniques in [6]. We shall always assume that A is a linear operator in X satisfying assumption $(h1)$ and we shall indicate with $\{S(t)\}_{t\geq 0}$ the semigroup (possibly not strongly continuous in 0) generated by A.

We begin with some lemmata.

Lemma 6.1. *Let $1 < p \leq +\infty$ and for $x \in X$ set $v_0(t) = S(t)x$. Then:*

a) *If $0 < \beta < \frac{1}{p}$, then $v \in W^{\beta,p}(0,T;X)$ for every $x \in X$ and every $T > 0$.*

b) *If $\frac{1}{p} < \beta < 1$ and $T > 0$, then $v_0 \in W^{\beta,p}(0,T;X)$ if and only if $x \in D_A(\beta - \frac{1}{p}, p)$.*

Proof. Statement a) is proved in [5: Section 4/Theorem 7]. In [5: Section 4/Theorem 8] it is shown that, if $x \in D_A(\beta - \frac{1}{p}, p)$ with $\beta > \frac{1}{p}$ and $p < +\infty$, then $v_0 \in W^{\beta,p}(0,T;X)$. To prove the inverse statement, observe that by Proposition 4.1, if $v_0 \in W^{\beta,p}(0,T;X)$ for some $\beta \in (\frac{1}{p}, 1)$, then

$$\int_0^T t^{-\beta p}\|S(t)x - x\|^p dt < +\infty.$$

This is another characterization of $D_A(\beta - \frac{1}{p}, p)$ (see [17: Proposition 2.2.4]). This argument is valid also in the case $p = +\infty$. Finally, in the case $p = +\infty$, if $x \in D_A(\beta, \infty)$, then

$$\begin{aligned}
\|v_0(t) - v_0(s)\| &= \left\| \int_s^t A v_0(\sigma)\, d\sigma \right\| \\
&\leq C_1 \int_s^t \sigma^{\beta-1} d\sigma \; \|x\|_{D_A(\beta,\infty)} \\
&\leq C_2 (t-s)^{\beta} \|x\|_{D_A(\beta,\infty)}
\end{aligned}$$

for $0 \leq s < t \leq T$ ∎

Lemma 6.2. *Let* $1 < p \leq +\infty$ *and set*

$$v_0(t) = S(t)x \tag{6.1}$$

for $x \in X$. *Then:*

a) *If* $0 < \beta < \frac{1}{p}$ *and* $T > 0$, *then* $v_0 \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$ *if and only if* $x \in D_A(1+\beta-\frac{1}{p},p)$.

b) *If* $\frac{1}{p} < \beta < 1$ *and* $T > 0$, *then* $v_0 \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$ *if and only if* $x \in D$ *and* $Ax \in D_A(\beta-\frac{1}{p},p)$.

Proof. Observe that, as $v_0'(t) = Av_0(t)$, $v_0 \in W^{1+\beta,p}(0,T;X)$ if and only if $v_0 \in W^{\beta,p}(0,T;D)$. Consider first the case $\beta > \frac{1}{p}$. Then, if $v_0 \in W^{\beta,p}(0,T;D)$, v_0 is continuous with values in D. This implies $x \in D$. Moreover, $t \to S(t)Ax = AS(t)x$ belongs to $W^{\beta,p}(0,T;X)$. Owing to Lemma 6.1, this can happen if and only if $Ax \in D_A(\beta-\frac{1}{p},p)$.

Consider next the case $\beta < \frac{1}{p}$. If $x \in D_A(1+\beta-\frac{1}{p},p)$, then $v_0 \in W^{1+\beta,p}(0,T;X)$ owing to [5: Section 4/ Theorem 10]. On the other hand, if $v_0 \in W^{\beta,p}(0,T;D)$, by Proposition 4.1 one has

$$\int_0^T t^{-\beta p}\|AS(t)x\|^p dt < +\infty$$

which implies $x \in D_A(1+\beta-\frac{1}{p},p)$ $\blacksquare$

Proof of Theorem 4.1. It is clearly not restrictive to consider the case $a = 0$ and $b = T$ with $T > 0$. We start by showing that conditions (i) - (iii) are necessary and sufficient in order to get a solution $u \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$.

It is clear that (i) is a necessary condition. Set

$$v_1(t) = \int_0^t S(t-s)f(s)\,ds. \tag{6.2}$$

Then, owing to [5: Theorem 24] and [17: Theorem 4.3.1(III)], if $f \in W^{\beta,p}(0,T;X)$ and either $\beta < \frac{1}{p}$ or $\frac{1}{p} < \beta$ and $f(0) = 0$, then $v_1 \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$. So, if $\beta < \frac{1}{p}$ and $f \in W^{\beta,p}(0,T;X)$, then $u \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$ if and only if $v_0 \in W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$. Therefore in this case we get the conclusion from Lemma 6.2.

Consider now the case $\beta > \frac{1}{p}$. As $W^{\beta,p}(0,T;D) \subseteq C([0,T];D)$, it is necessary that $u_0 \in D$. We have

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(0)\,ds + \int_0^t S(t-s)[f(s)-f(0)]\,ds. \tag{6.3}$$

Owing to the quoted results in [5, 17], if $f \in W^{\beta,p}(0,T;X)$, then the last summand in (6.3) belongs to $W^{1+\beta,p}(0,T;X) \cap W^{\beta,p}(0,T;D)$. For $t \in (0,T)$ we have

$$A\Big(S(t)u_0 + \int_0^t S(t-s)f(0)\,ds\Big) = S(t)\big(Au_0 + f(0)\big) - f(0)$$

so that, owing to Lemma 6.1, the sum of the two first summands in (6.3) belongs to $W^{\beta,p}(0,T;D)$ if and only if $Au_0 + f(0) \in D_A(\beta - \frac{1}{p}, p)$.

It remains only to consider estimates (4.25) - (4.26). Consider the operator TA with $T \in (0,T_0]$. Then

$$\Big\{\lambda \in \mathbb{C} :\ |\lambda| \geq T_0R_0 \text{ and } |\mathrm{Arg}\lambda| \leq \frac{\pi}{2}\Big\} \subseteq \rho(TA).$$

Moreover, if $|\lambda| \geq T_0R_0$ and $|\mathrm{Arg}\lambda| \leq \frac{\pi}{2}$, then

$$\|(\lambda - TA)^{-1}\|_{\mathcal{L}(X)} \leq M|\lambda|^{-1}.$$

This means that there exists $C > 0$ independent of $T \in (0,T_0]$ such that, if v solves the problem

$$\left.\begin{aligned} &v'(t) = TAv(t) + g(t) \quad (t \in [0,1]) \\ &v(0) = 0 \end{aligned}\right\} \tag{6.4}$$

with $g \in W^{\beta,p}(0,1;X)$ and $g(0) = 0$ in the case $\beta > \frac{1}{p}$, then

$$\|v\|_{W^{\beta,p}(0,1;X)} + T\|Av\|_{W^{\beta,p}(0,1;X)} \leq C\|g\|_{W^{\beta,p}(0,1;X)}. \tag{6.5}$$

Consider first the case $u_0 = 0$ and, if $\beta > \frac{1}{p}$, $f(0) = 0$. Then $v(s) = u(Ts)$ $(s \in [0,1])$ is the solution of the problem

$$\left.\begin{aligned} &v'(s) = TAv(s) + Tf(Ts) \quad (s \in [0,1]) \\ &v(0) = 0 \end{aligned}\right\}. \tag{6.6}$$

It follows

$$\begin{aligned} \|u\|_{W^{\beta,p}(0,T;D)} &= T^{\frac{1}{p}-\beta}\|v\|_{W^{\beta,p}(0,1;D)} \\ &\leq C(T_0)T^{\frac{1}{p}-\beta-1}\big[\|v\|_{W^{\beta,p}(0,1;X)} + T\|Av\|_{W^{\beta,p}(0,1;X)}\big] \\ &\leq C_1(T_0)T^{\frac{1}{p}-\beta}\|f(T\cdot)\|_{W^{\beta,p}(0,1;X)} \\ &= C_1(T_0)\|f\|_{W^{\beta,p}(0,T;X)}. \end{aligned}$$

Now we consider the general case. We start from the case $\beta < \frac{1}{p}$. Recalling notations (6.1) - (6.2) we have

$$\begin{aligned}\|u\|_{W^{\beta,p}(0,T;D)} &\le \|v_0\|_{W^{\beta,p}(0,T;D)} + \|v_1\|_{W^{\beta,p}(0,T;D)}\\ &\le \|v_0\|_{W^{\beta,p}(0,T_0;D)} + C_1(T_0)\|f\|_{W^{\beta,p}(0,T;X)}\\ &\le C(T_0)\big[\|u_0\|_{D_A(\beta+1-\frac{1}{p},p)} + \|f\|_{W^{\beta,p}(0,T;X)}\big]\end{aligned}$$

applying Lemma 6.2.

Now we pass to the case $\frac{1}{p} < \beta < 1$ and set

$$v_2(t) = v_0(t) + \int_0^t S(t-s)f(0)\,ds.$$

Then

$$\begin{aligned}\|u\|_{W^{\beta,p}(0,T;D)} &\le \|v_2\|_{W^{\beta,p}(0,T_0;D)} + C_1(T_0)\|f - f(0)\|_{W^{\beta,p}(0,T;X)}\\ &\le C_2(T_0)\Big[\|u_0\|_D + \|f(0)\| + \|Au_0 + f(0)\|_{D_A(\beta-\frac{1}{p})} + [f]_{W^{\beta,p}(0,T;X)}\Big]\\ &= C_2(T_0)\Big[\|u_0\|_D + \|Au_0 + f(0)\|_{D_A(\beta-\frac{1}{p})} + \|f\|_{W^{\beta,p}(0,T;X)}\Big].\end{aligned}$$

With this the proof is complete ∎

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Received 06.08.2001