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# Oscillations for Certain Difference Equations with Continuous Variable

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**Abstract.** In this paper, we investigate some nonlinear difference equations with continuous variable. A linearized oscillation result is established and oscillation criteria for some forced difference equations are obtained.

**Keywords:** *Difference equations, positive solutions, oscillation*

**AMS subject classification:** 39A

## 0. Introduction

Recently, there has been an increasing interest in the study of the oscillatory behavior of the solutions of delay difference equations [1]. In [3], authors consider the oscillation of the delay difference equation

$$y(t) - y(t-\tau) + p(t)H(y(t-\sigma)) = f(t) \qquad (t \geq 0)$$

where $\tau, \sigma > 0$, and $p \in C(\mathbb{R}_+, \mathbb{R}_+), f \in C(\mathbb{R}_+, \mathbb{R})$ and $H \in C(\mathbb{R}, \mathbb{R})$. In the present paper we use some ideas from [3] to consider the oscillation of the equation

$$y(t-\tau) - y(t) + \sum_{i=1}^{m} p_i f_i(y(t+\sigma_i)) = 0 \tag{1}$$

where $\tau > 0, \sigma_m \geq ... \geq \sigma_1 > 0, p_i > 0, f_i \in C(\mathbb{R}, \mathbb{R}), uf_i(u) > 0$ for $u \neq 0$ and $\lim_{u\to\infty} \frac{f_i(u)}{u} = 1$, and of the equation

$$y(t) - a(t)y(t-\tau) + G(t, y(t-\sigma)) = f(t) \tag{2}$$

where $\tau, \sigma > 0$ and $a \in C(\mathbb{R}_+, \mathbb{R}_+), G \in C(R_+ \times \mathbb{R}, \mathbb{R})$ and $f \in C(\mathbb{R}_+, \mathbb{R})$. As usuall, a solution of equation (1) or (2) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise the solution is called non-oscillatory. In Section 1, we obtain a linearized oscillation result for equation (1) and in Section 2 we obtain some oscillation criteria for the forced equation (2).

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## 1. Linearized oscillation for equation (1)

Consider equation (1) together with the associated linear difference equation

$$y(t-\tau) - y(t) + \sum_{i=1}^{m} p_i y(t+\sigma_i) = 0. \tag{3}$$

The first lemma is borrowed from [2].

**Lemma 1.** *The following statements are equivalent:*

**(a)** *Every solution of equation* (3) *oscillates.*

**(b)** *The characteristic equation*

$$e^{-\lambda\tau} - 1 + \sum_{i=1}^{m} p_i e^{\lambda\sigma_i} = 0 \tag{4}$$

*has no real roots.*

**Lemma 2.** *Every solution of equation* (3) *oscillates if and only if the inequality*

$$y(t-\tau) - y(t) + \sum_{i=1}^{m} p_i y(t+\sigma_i) \le 0 \tag{5}$$

*has no eventually positive solutions.*

**Proof.** *Necessity.* Suppose $y > 0$ is an eventually positive solution of equation (5). Then

$$y(t) \ge y(t-\tau) + \sum_{i=1}^{m} p_i y(t+\sigma_i) \qquad (t \ge T-\tau > 0). \tag{6}$$

The further proof is simple and based on the following Knaster Fixed Point Theorem [4]:

*Let* $(X, \le)$ *be an ordered set, let for every subset* $M$ *of* $X$ *there exist* $\inf M$ *and* $\sup M$*, and let* $T : M \to M$ *be an increasing mapping, that is,* $x \le y$ *implies* $Tx \le Ty$*. Then there exists at least one element* $x \in X$ *such that* $Tx = x$*.*

To use this theorem, define the set

$$X = \{x \in C : 0 \le x(t) \le y(t) \ \ (t \ge T-\tau)\}$$

endowed with usual pointwise ordering, i.e. $x_1 \le x_2$ if $x_1(t) \le x_2(t)$ for every $t \ge T - \tau$. It is easy to see that every $A \subseteq X$ has a supremum which belongs to $X$. Define an operator $S$ on $X$ by

$$(Sx)(t) = \begin{cases} x(t-\tau) + \sum_{i=1}^{m} p_i x(t+\sigma_i) & \text{if } t \ge T \\ (1 - \frac{t}{T})y(t) + \frac{ty(t)Sx(T)}{Ty(T)} & \text{if } T - \tau \le t < T. \end{cases} \tag{7}$$

For any $x \in X$, from

$$0 \le (Sx)(t) = x(t-\tau) + \sum_{i=1}^{m} p_i x(t+\sigma_i) \le y(t-\tau) + \sum_{i=1}^{m} p_i y(t+\sigma_i) \le y(t)$$

for $t \ge T$ and

$$0 \le (Sx)(t) = \Big(1 - \frac{t}{T}\Big)y(t) + \frac{ty(t)Sx(T)}{Ty(T)} \le \Big(1 - \frac{t}{T}\Big)y(t) + \frac{t}{T}y(t) = y(t)$$

for $T - \tau \le t < T$ we know that $SX \subseteq X$. Moreover, $S$ is obviously non-decreasing. By the Knaster Fixed Point Theorem, there exists an $x^* \in X$ such that $Sx^* = x^*$. As $T \le t \le T + \tau$,

$$\begin{aligned} x^*(t) &= x^*(t-\tau) + \sum_{i=1}^{m} p_i x^*(t+\sigma_i) \\ &\ge x^*(t-\tau) \\ &= Sx^*(t-\tau) \\ &= \Big(1 - \frac{t-\tau}{T}\Big)y(t-\tau) + \frac{(t-\tau)y(t-\tau)Sx^*(T)}{Ty(T)} \\ &\ge \Big(1 - \frac{t-\tau}{T}\Big)y(t-\tau) \\ &> 0. \end{aligned}$$

Repeating this procedure, we get $x^*(t) > 0$ as $t \ge T$. So $x^*$ is an eventually positive solution of equation (3), which is a contradiction.

*Sufficiency.* Suppose equation (3) has an eventually positive solution $y > 0$ or eventually negative solution $x < 0$. Because the latter means $-x > 0$ is an eventually positive solution of equation (3), we only discuss the former case. It is easy to show that equation (5) has an eventually positive solution $y > 0$. This is a contradiction ∎

**Lemma 3.** *Assume that* $uf_i(u) > 0$ *for* $u \neq 0, \lim_{u\to\infty} \frac{f_i(u)}{u} = 1$ *and* $f_i$ *is convex for* $u > 0$ *and concave for* $u < 0$ $(i = 1, 2, ..., m)$. *Further, assume that every solution of the equation*

$$y(t-\tau) - y(t) + (1-\varepsilon)\sum_{i=1}^{m} p_i y(t+\sigma_i) = 0 \qquad (\varepsilon \in (0,1)) \tag{8}$$

*oscillates. Then every solution of equation* (1) *oscillates.*

**Proof.** Suppose equation (1) has an eventually positive solution $y$ and set $z(t) = \frac{1}{\tau}\int_{t-\tau}^{t} y(s)\,ds > 0$. Then

$$z'(t) = \frac{1}{\tau}\big(y(t) - y(t-\tau)\big) = \frac{1}{\tau}\sum_{i=1}^{m} p_i f_i(y(t+\sigma_i)) > 0$$

eventually. Hence $\lim_{t\to\infty} z(t) = \beta > 0$ exists. We claim that $\beta = \infty$. Otherwise, $0 < \beta < \infty$. We integrate equation (1) from $t-\tau$ to $t$ and get

$$\int_{t-\tau}^{t} y(s-\tau)\,ds - \int_{t-\tau}^{t} y(s)\,ds + \sum_{i=1}^{m} p_i \int_{t-\tau}^{t} f_i(y(s+\sigma_i))\,ds = 0. \tag{9}$$

Since $f_i$ is convex for $u > 0$, by Jensen's inequality we have

$$z(t-\tau) - z(t) + \sum_{i=1}^{m} p_i f_i(z(t+\sigma_i)) \leq 0. \tag{10}$$

Letting $t \to \infty$, from (10) we obtain the inequality $\sum_{i=1}^{m} p_i f_i(\beta) \leq 0$ which is a contradiction. By $\lim_{u\to\infty} \frac{f_i(u)}{u} = 1$ for any $\varepsilon \in (0,1)$ there exists $\alpha > 0$ such that

$$(1-\varepsilon)u < f_i(u) < (1+\varepsilon)u \qquad (u \geq \alpha). \tag{11}$$

Thus, from (10) we have

$$z(t-\tau) - z(t) + (1-\varepsilon)\sum_{i=1}^{m} p_i z(t+\sigma_i) \leq 0.$$

By Lemma 2, equation (8) has a positive solution. This is a contradiction. Similarly, we can prove that equation (1) have no eventually negative solutions ∎

**Lemma 4.** *If*

$$y(t-\tau)-y(t)+(1+\varepsilon)\sum_{i=1}^{m}p_i y(t+\sigma_i)=0 \qquad (\varepsilon\in(0,1)) \tag{12}$$

*has positive solutions and $f_i$ is non-decreasing in $u$, so does equation* (1).

**Proof.** By Lemma 1 and the fact that equation (12) has an eventually positive solution, the characteristic equation

$$e^{-\lambda\tau}-1+(1+\varepsilon)\sum_{i=1}^{m}p_i e^{\lambda\sigma_i}=0 \tag{13}$$

has a real root $\eta$. Clearly, $\eta>0$. Thus $e^{\eta t}$ is a solution of equation (12) which tends to infinity as $t\to\infty$. Suppose $y(t)\to\infty$ as $t\to\infty$ is a positive solution of equation (12). From (11) we have

$$y(t-\tau)-y(t)+\sum_{i=1}^{m}p_i f_i(y(t+\sigma_i))\le y(t-\tau)-y(t)+(1+\varepsilon)\sum_{i=1}^{m}p_i y(t+\sigma_i)=0.$$

Then

$$y(t)\ge y(t-\tau)+\sum_{i=1}^{m}p_i f_i(y(t+\sigma_i)). \tag{14}$$

Define

$$Y=\big\{a\in C:\, 0\le a(t)\le y(t) \text{ for } t\ge T-\tau\big\}$$

and an operator $E$ on $Y$ by

$$Ea(t)=\begin{cases} a(t-\tau)+\sum_{i=1}^{m}p_i f_i(a(t+\sigma_i)) & \text{if } t\ge T\\ (1-\frac{t}{T})y(t)+\frac{ty(t)Ex(T)}{Ty(T)} & \text{if } T-\tau\le t<T.\end{cases}$$

Similar to the proof of Lemma 2, we can prove that there exists a fixed point $a\in Y$ and $a(t)>0$ for $t\ge T$. Since $a=Ea$, $a$ is a positive solution of equation (1) ∎

**Lemma 5.** *The equation*

$$F(\lambda)=e^{-\lambda\tau}-1+\sum_{i=1}^{m}p_i e^{\lambda\sigma_i}=0 \tag{15}$$

*has real roots if and only if there exists $\varepsilon_0\in(0,1)$ such that*

$$e^{-\lambda\tau}-1+(1+\varepsilon)\sum_{i=1}^{m}p_i e^{\lambda\sigma_i}=0 \qquad (|\varepsilon|<\varepsilon_0) \tag{16}$$

*has real roots.*

**Proof.** *Sufficiency.* If there exists $\varepsilon_0 \in (0,1)$ such that equation (16) has real roots, then let $\varepsilon = 0$ and we obtain that the equation

$$F(\lambda) = e^{-\lambda\tau} - 1 + \sum_{i=1}^{m} p_i e^{\lambda\sigma_i} = 0$$

has real roots.

*Necessity.* Suppose $F(\lambda) = 0$ has a real root $\eta, i.e. F(\eta) = 0$. Define a function $H$ as

$$H(\varepsilon, \lambda) = e^{-\lambda\tau} - 1 + (1+\varepsilon)\sum_{i=1}^{m} p_i e^{\lambda\sigma_i} \qquad (|\varepsilon| < 1).$$

It is easy to see that $H \in C\big((-1,1) \times \mathbb{R}, \mathbb{R}\big)$ and

$$H(0,\eta) = e^{-\eta\tau} - 1 + \sum_{i=1}^{m} p_i e^{\eta\sigma_i} = F(\eta) = 0.$$

In a small neighbourhood of $(0,\eta)$ the equation $H(\varepsilon, \lambda(\varepsilon)) = 0$ defines a continuous function $\lambda = \lambda(\varepsilon)$ which satisfies $H(\varepsilon, \lambda(\varepsilon)) = 0, \lambda(0) = \eta$ and $\lim_{\varepsilon\to 0} \lambda(\varepsilon) = \eta$. So there exists $\varepsilon_0 \in (0,1)$ such that equation (16) has real roots ∎

From the above lemmas, we can describe the first main result in this paper.

**Theorem 1.** *Assume that $p_i, \tau, \sigma_i > 0$ and $f_i \in C(\mathbb{R}, \mathbb{R}), uf_i(u) > 0$ for $u \neq 0, \lim_{u\to\infty} \frac{f_i(u)}{u} = 1$, $f_i$ is non-decreasing in $u$, convex for $u > 0$ and concave for $u < 0$ $(i = 1, 2, ..., m)$. Then every solution of equation* (1) *oscillates if and only if every solution of equation* (3) *oscillates.*

**Proof.** *Sufficiency.* If the solution of equation (3) oscillates, by Lemma 1 equation (4) and hence by Lemma 5 equation (16) has no real roots. By Lemma 1, every solution of equation (8) oscillates. From Lemma 3 we show that every solution of equation (1) oscillates.

*Necessity.* Suppose equation (3) has an eventually positive solution. By Lemma 1 equation (4) and by Lemma 5 equation (16) has real roots. By Lemma 1

$$y(t-\tau) - y(t) + (1+\varepsilon)\sum_{i=1}^{m} p_i y(t+\sigma_i) = 0 \qquad (|\varepsilon| < |\varepsilon_0|)$$

and by Lemma 4 equation (1) has eventually positive solutions, which is a contradiction ∎

## 2. Oscillations for equation (2)

The following lemma will be used to state the main results in Section 2.

**Lemma 6.** *Assume that $f \in C(\mathbb{R}_+, \mathbb{R})$ and $a(t) \neq 0$ as $t \geq T$ where $T \geq \tau$. Then there exists a continuous function $F = F(t)$ as $t \geq T - \tau$ such that $F(t) - F(t-\tau)\,a(t) = f(t)$ for $t \geq T$.*

**Proof.** Define

$$a_1(t) = \begin{cases} a(t) & \text{if } t \geq T \\ \frac{t-T+\tau}{\tau}\, a_1(T) & \text{if } T-\tau \leq t < T \\ 0 & \text{if } t < T-\tau. \end{cases}$$

Then $a_1 \in C(\mathbb{R}, \mathbb{R})$ and $a_1(t) = a(t)$ for $t \geq T$. Define

$$r(t) = \begin{cases} f(t)\, a^{-1}(t) & \text{if } t \geq T \\ \frac{t-T+\tau}{\tau}\, r(T) & \text{if } T-\tau \leq t < T \\ 0 & \text{if } t < T-\tau. \end{cases}$$

Then $r \in C(\mathbb{R}, \mathbb{R})$. Let

$$F(t) = \sum_{i=0}^{\infty} r(t-i\tau) \prod_{j=0}^{i} a_1(t-j\tau) \qquad (t \geq T).$$

Obviously, $F \in C(\mathbb{R}_+, \mathbb{R})$. When $t \geq T$, we know

$$\begin{aligned} &F(t) - a(t)F(t-\tau) \\ &= \sum_{i=0}^{\infty} r(t-i\tau) \prod_{j=0}^{i} a_1(t-j\tau) - a(t) \sum_{i=0}^{\infty} r(t-\tau-i\tau) \prod_{j=0}^{i} a_1(t-\tau-j\tau) \\ &= \sum_{i=0}^{\infty} r(t-i\tau) \prod_{j=0}^{i} a_1(t-j\tau) - \sum_{i=1}^{\infty} r(t-i\tau) \prod_{j=0}^{i} a_1(t-j\tau) \\ &= r(t)a(t) \\ &= f(t) \end{aligned}$$

and the proof is complete ∎

Set

$$\overline{y}(t) = \int_{T'}^{t} y(s)\,ds + \int_{t-\tau}^{T'} a(s+\tau)y(s)\,ds$$
$$\overline{F}(t) = \int_{T'}^{t} F(s)\,ds + \int_{t-\tau}^{T'} a(s+\tau)F(s)\,ds$$

where

$$T' = \begin{cases} t - \frac{\tau}{2} & \text{if } a(s+\tau) \in (0,1] \\ t - \frac{3\tau}{2} & \text{if } a(s+\tau) \in (1,+\infty) \end{cases} \qquad (s \in [t-\tau, t])$$

and set $\overline{F}_{\pm}(t) = \max\{\pm\overline{F}(t), 0\}$.

**Theorem 2.** *Assume the following:*

**(a)** $g(t,u) = \min_{t-\tau \le s \le t} G(s,u)$ *for* $u > 0$.

**(b)** $G(t,u)$ *is an odd function in* $u$, $uG(t,u) > 0$ *for* $u \neq 0$, $g(t,0) = 0$, *and* $g(t,u)$ *is non-decreasing and is convex in* $u > 0$.

**(c)** *For any number* $N > 0$, *there exist two sequences* $\{t_i\}$ *and* $\{t'_i\}$ *such that* $t_{i+1} - t_i \ge \tau$ *and* $t'_{i+1} - t'_i \ge \tau$, *and* $a(t) \in (0,1]$ *or* $a(t) \in (1,+\infty)$ *as* $t \in [t_i - \sigma - 2\tau, t_i - \sigma - \tau]$ $(i = 1,2,\ldots)$, *and*

$$\sum_{i=1}^{\infty} \tau g\left(t_i, \frac{1}{\tau}\overline{F}_+(t_i - \sigma)\right) > N \tag{17}$$

$$\sum_{i=1}^{\infty} \tau g\left(t'_i, \frac{1}{\tau}\overline{F}_-(t'_i - \sigma)\right) > N. \tag{18}$$

*Then every solution of equation* (2) *oscillates.*

**Proof.** From Lemma 6, equation (2) can be rewritten in the form

$$(y(t) - F(t)) - a(t)\big(y(t-\tau) - F(t-\tau)\big) + G(t, y(t-\sigma)) = 0. \tag{19}$$

Suppose the contrary, let $y > 0$ be an eventually positive solution of equation (19) and let $z = \overline{y} - \overline{F}$. Then equation (19) becomes

$$z'(t) + G(t, y(t-\sigma)) = 0. \tag{20}$$

So $z'(t) < 0$ for $t \ge T$. If $z(t) < 0$ eventually, then $0 < \overline{y}(t) < \overline{F}(t)$ eventually and hence $\overline{F}_-(t) = 0$ and $g(t, \frac{1}{\tau}\overline{F}_-(t-\sigma)) = 0$ which contradicts (18). Therefore $z(t) > 0$ and $\lim_{t\to\infty} z(t) = \alpha \ge 0$ exists. Integrating equation (20) from $T$ to $\infty$ we obtain

$$\int_T^{\infty} G(t, y(t-\sigma))\,dt = z(T) - \alpha < \infty. \tag{21}$$

Since $z(t) > 0$, we have $\overline{y}(t) \ge \overline{F}(t)$ and hence $\overline{y}(t) \ge \overline{F}_+(t)$ for $t \ge T$. There exists $k > 0$ such that $t_k - \tau \ge T + \sigma$ and so by Jensen's inequality

$$\begin{aligned}\int_{T+\sigma}^{\infty} G(t, y(t-\sigma))\,dt &\ge \sum_{i=k}^{\infty} \int_{t_i-\tau}^{t_i} G(t, y(t-\sigma))\,dt \\ &\ge \sum_{i=k}^{\infty} \int_{t_i-\tau}^{t_i} g(t, y(t-\sigma))\,dt \\ &\ge \sum_{i=k}^{\infty} \tau g\bigg(t_i, \frac{1}{\tau}\int_{t_i-\tau}^{t_i} y(t-\sigma)\,dt\bigg).\end{aligned} \tag{22}$$

Setting $A_i = [t_i - 2\tau - \sigma, t_i - \tau - \sigma]$ we obtain

$$
\begin{aligned}
&\int_{t_i-\tau}^{t_i} y(s-\sigma)\,ds \\
&\quad = \begin{cases} \int_{t_i-\frac{\tau}{2}}^{t_i} y(s-\sigma)\,ds + \int_{t_i-\tau}^{t_i-\frac{\tau}{2}} y(s-\sigma)\,ds & \text{if } a \in C(A_i,(0,1]) \\ \int_{t_i-\frac{3\tau}{2}}^{t_i} y(s-\sigma)\,ds + \int_{t_i-\tau}^{t_i-\frac{3\tau}{2}} y(s-\sigma)\,ds & \text{if } a \in C(A_i,(1,+\infty)) \end{cases} \\
&\quad \geq \begin{cases} \int_{t_i-\frac{\tau}{2}-\sigma}^{t_i-\sigma} y(s)\,ds + \int_{t_i-\tau-\sigma}^{t_i-\frac{\tau}{2}-\sigma} a(s+\tau)y(s)\,ds & \text{if } a \in C(A_i,(0,1]) \\ \int_{t_i-\frac{3\tau}{2}-\sigma}^{t_i-\sigma} y(s)\,ds - \int_{t_i-\frac{3\tau}{2}-\sigma}^{t_i-\tau-\sigma} a(s+\tau)y(s)\,ds & \text{if } a \in C(A_i,(1,+\infty)) \end{cases} \\
&\quad = \overline{y}(t_i-\sigma) \\
&\quad \geq \overline{F}_+(t_i-\sigma).
\end{aligned} \tag{23}
$$

In view of (21) - (23) and that $g(t,u)$ is non-decreasing in $u > 0$ we have

$$z(T) > \int_{T+\sigma}^{\infty} G(t, y(t-\sigma))\,dt \geq \sum_{i=k}^{\infty} \tau g\left(t_i, \frac{1}{\tau}\overline{F}_+(t_i-\sigma)\right)$$

which contradicts (17). Suppose $y < 0$ is an eventually negative solution of equation (19). Then similarly we can prove that $z' > 0, z < 0$, hence $\lim_{t\to\infty} z(t) = \beta \leq 0$ and

$$
\begin{aligned}
\infty &> \beta - z(T) \\
&= -\int_{T}^{\infty} G(t, y(t-\sigma))\,dt \\
&> -\int_{T+\sigma}^{\infty} G(t, y(t-\sigma))\,dt \\
&\geq \sum_{i=k}^{\infty} \left( -\int_{t_i'-\tau}^{t_i'} G(t, y(t-\sigma))\,dt \right) \\
&= \sum_{i=k}^{\infty} \int_{t_i'-\tau}^{t_i'} G(t, -y(t-\sigma))\,dt \\
&\geq \sum_{i=k}^{\infty} \int_{t_i'-\tau}^{t_i'} g(t_i', -y(t-\sigma))\,dt \\
&\geq \sum_{i=k}^{\infty} \tau g\Big(t_i', -\frac{1}{\tau}\overline{y}(t_i'-\sigma)\Big).
\end{aligned} \tag{24}
$$

Since $z < 0, \overline{y} < \overline{F}$ and hence $-\overline{y} > -\overline{F}$, therefore $-\overline{y}(t) > (-\overline{F}(t))_+ = \overline{F}_-(t)$. From (24) we have

$$\sum_{i=k}^{\infty} \tau g\Big(t_i', \frac{1}{\tau}\overline{F}_-(t_i'-\sigma)\Big) < -z(T)$$

which contradicts (18) ∎

**Example 1.** Consider the difference equation

$$y(t) - ty(t-\pi) + (1+t)y^3(t - \tfrac{\pi}{2}) = (1+t)(\cos t + \sin^3 t). \tag{25}$$

In this case $G(t,u) = (1+t)u^3$, $f(t) = (1+t)(\cos t + \sin^3 t)$, $\sigma = \frac{\pi}{2}$, $\tau = \pi$, $a(t) = t$ and $F(t) = \cos t + \sin^3 t$. Let $T = \frac{5\pi}{2} + 1$. So $a(t) > 1$ as $t \geq T$. Thus

$$\begin{aligned}
\overline{F}(t) &= \int_{t-\frac{3\pi}{2}}^{t} (\cos s + \sin^3 s)\,ds + \int_{t-\pi}^{t-\frac{3\pi}{2}} (s+\pi)(\cos s + \sin^3 s)\,ds \\
&= \left(\tfrac{4}{9} + \tfrac{\pi}{6} - \tfrac{t}{3}\right)\sin^3 t + \left(\tfrac{4}{9} + \tfrac{t}{3}\right)\cos^3 t + \left(2t - \tfrac{\pi}{2} - \tfrac{1}{3}\right)\sin t - \left(\tfrac{\pi}{2} + \tfrac{1}{3}\right)\cos t \\
g(t,u) &= \min_{t-\pi \leq s \leq t} \{(1+s)u^3\} = (1+t-\pi)u^3 \quad (u > 0).
\end{aligned}$$

It is easy to see that the former two coditions of Theorem 2 hold. We only need to show that (17) and (18) could be fulfilled. In fact, let $t_1 = 3\pi, t_n = t_{n-1} + 2\pi$ and $t'_1 = \frac{9\pi}{2}, t'_n = t'_{n-1} + 2\pi$, i.e. two sequences $\{t_i\}, \{t'_i\}$ $(i \geq 1)$ exist and

$$\begin{aligned}
&\overline{F}_+(t_i - \tfrac{\pi}{2}) = \tfrac{5}{3}t_i + \tfrac{1}{9} - \tfrac{7}{6}\pi \geq 5\pi + \tfrac{1}{9} - \tfrac{7}{6}\pi > 3\pi \\
&\sum_{i=1}^{\infty} \pi g(t_i, \tfrac{1}{\pi}\overline{F}_+(t_i - \sigma)) = \tfrac{1}{\pi^2} \sum_{i=1}^{\infty} (1 + t_i - \pi)(\overline{F}_+(t_i - \sigma))^3.
\end{aligned}$$

In view of $(1 + t_i - \pi)(\overline{F}_+(t_i - \sigma))^3 > 54\pi^4$, $\sum_{i=1}^{\infty} \pi g\big(t_i, \frac{1}{\pi}\overline{F}_+(t_i - \sigma)\big) = \infty$. Analogously, $\overline{F}_-(t'_i - \frac{\pi}{2}) = \frac{t'_i}{3} - \frac{2}{3}\pi + \frac{1}{9} > \frac{5}{6}\pi$ and so $\sum_{i=1}^{\infty} \pi g\big(t'_i, \frac{1}{\pi}\overline{F}_-(t'_i - \frac{\pi}{2})\big) = \infty$. Therefore (17) and (18) are satisfied. By Theorem 2, the solutions of equation (25) oscillate. Actually, $y = \cos t$ is a such solution of equation (25).

If $a(t) = 1$ and $G(t, y(t-\sigma)) = p(t)y(t-\sigma)$, then equation (2) becomes

$$y(t) - y(t-\tau) + p(t)y(t-\sigma) = f(t). \tag{26}$$

**Corollary 1.** *Suppose* $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ *and for any number* $N > 0$ *there exist two sequences* $\{t_i\}$ *and* $\{t'_i\}$ *such that* $t_{i+1} - t_i, t'_{i+1} - t'_i \geq \tau$ $(i \geq 1)$ *and*

$$\sum_{i=1}^{\infty} q(t_i)\overline{F}_+(t_i - \sigma) > N \qquad \text{and} \qquad \sum_{i=1}^{\infty} q(t'_i)\overline{F}_-(t'_i - \sigma) > N$$

*where* $q(t) = \min_{t-\tau \leq s \leq t} p(s)$. *Then every solution of equation* (26) *oscillates.*

Corollary 1 is [3: Theorem 2.5]. Similar to [3], from Theorem 2 we can obtain the following conclusion.

**Corollary 2.** *Assume conditions* (a) - (b) *in Theorem* 2 *and either* $a(t) \in (0,1]$ *or* $a(t) \in (1,+\infty)$ *as* $t \geq T$. *Furthermore, let* $\int_T^\infty g\big(t, \frac{1}{\tau}\overline{F}_\pm(t-\sigma)\big)dt = \infty$. *Then every solution of equation* (2) *oscillates.*

By the bivariate Jensen inequality we can get the next oscillation criterion.

**Theorem 3.** *Assume the following:*

**(i)** *$G(t,u)$ is non-decreasing and is an odd function in $u$, $G(t,0) = 0$, $uG(t,u) > 0$ for $u \neq 0$ and $G(t,u)$ is convex in $(t,u)$ as $t,u > 0$.*

**(ii)** *Condition* (c) *of Theorem* 2 *holds where* (17) *and* (18) *are replaced by*

$$\sum_{i=1}^{\infty} \tau G\Big(t_i - \frac{\tau}{2}, \frac{1}{\tau}\overline{F}_+(t_i-\sigma)\Big) > N \quad \textit{and} \quad \sum_{i=1}^{\infty} \tau G\Big(t_i' - \frac{\tau}{2}, \frac{1}{\tau}\overline{F}_-(t_i'-\sigma)\Big) > N$$

*respectively. Then every solution of equation* (2) *oscillates.*

The proof of Theorem 3 is similar to that of Theorem 2. We only need to pay attention to Jensen's inequality for functions in two variables, i.e.

$$\frac{1}{\tau}\int_{t-\tau}^{t} G(s,y(s))\,ds \geq G\left(t - \frac{\tau}{2}, \frac{1}{\tau}\int_{t-\tau}^{t} y(s)\,ds\right)$$

since $G(t,u)$ is convex in $(t,u)$.

Similar to Corollary 1, for the linear equation (26) we have

**Corollary 3.** *Assume $p \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $up(t)$ is convex in $(t,u)$ as $t,u > 0$. Furthermore, for any number $N > 0$ let there exist two sequences $\{t_i\}$ and $\{t_i'\}$ such that $t_{i+1} - t_i, t_{i+1}' - t_i' \geq \tau \quad (i \geq 1)$,*

$$\sum_{i=1}^{\infty} p(t_i - \tfrac{\tau}{2})\overline{F}_+(t_i - \sigma) > N \quad \textit{and} \quad \sum_{i=1}^{\infty} p(t_i' - \tfrac{\tau}{2})\overline{F}_-(t_i' - \sigma) > N.$$

*Then every solution of equation* (26) *oscillates.*

**Corollary 4.** *Assume condition* (i) *of Theorem* 3 *holds, either* $a(t) \in (0,1]$ *or* $a(t) \in (1,+\infty)$ *as* $t \geq T$, *and* $\int_T^\infty G\big(t - \frac{\tau}{2}, \frac{1}{\tau}\overline{F}_\pm(t-\sigma)\big)dt = \infty$. *Then every solution of equation* (2) *oscillates.*

We can also extend the above methods to investigate the oscillation of the solution of the difference equation

$$a(t)y(t-\tau) - y(t) + G(t, y(t+\sigma)) = f(t).$$

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