An Extension of the Notion of Zero-Epi Maps to the Context of Topological Spaces

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Abstract. We introduce the class of hyper-solvable equations whose concept may be regarded as an extension to the context of topological spaces of the known notion of 0-epi maps. After collecting some notation, definitions and preliminary results we give a homotopy principle for hyper-solvable equations. We provide examples showing how these equations arise in the framework of Leray-Schauder degree, Lefschetz number theory and essential compact vector fields in the sense of A. Granas.

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1. Introduction

The main purpose of this work is to extend the definition and properties of 0-epi (zero-epi) maps to the context of topological spaces. For readers's convenience we recall here the definition as expounded in [4]:

Given two Banach spaces E and F, and a bounded open subset U of E, a continuous map $f : \overline{U} \to F$ defined on the closure $\overline{U}$ of U is called 0-*admissible* (zero-admissible) provided that $f(x) \neq 0$ for $x \in \partial U$ – the boundary of U. Now, a 0-admissible map f is said to be 0-*epi* (zero-epi) if the equation $f(x) = h(x)$ admits a solution in U *for any* (continuous) compact map $h : \overline{U} \to F$ such that $h(x) = 0$ for $x \in \partial U$.

The class of 0-epi maps enjoys properties akin to those satisfied both by Brouwer and Leray-Schauder (topological) degree. In particular, 0-epi maps satisfy the following *continuation* (or *homotopy*) *principle*, which shows that if

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f is 0-epi, then the solvability of the equation $f(x) = 0$ is stable under a class of perturbations which is considerably larger than that used in the definition of 0-epi map.

Theorem 1.1. *Let $f : \overline{U} \to F$ be 0-epi and let $H : \overline{U} \times [0,1] \to F$ be a (continuous) compact homotopy such that $H(x, 0) = 0$ for any $x \in \overline{U}$. Assume that $f(x) + H(x, t) \neq 0$ for all $(x, t) \in \partial U \times [0, 1]$. Then, the map $f(\cdot) + H(\cdot, 1)$ is 0-epi.*

We refer the reader to [4] for the proof of this result and some of its applications.

The equations we consider in this paper are of the type $f(x) = h(x)$, where f and h are continuous maps between two topological spaces, the second one being locally compact. Given such an equation, our purpose is to provide conditions ensuring that its solvability is not destroyed under a sufficiently large class of perturbations of h. In our opinion, a reasonable class, call it $\mathcal{C}$, is represented by those maps that can be joined to h through a locally compact homotopy which, loosely speaking, maintains a priori bounds on the set of solutions during the deformation (Theorem 3.1 will clarify the exact nature of $\mathcal{C}$).

Now, a comparison with the theory of 0-epi maps leads us to look for a conveniently small subclass $\mathcal{C}_0$ of $\mathcal{C}$ such that when the solvability of $f(x) = h(x)$ is preserved under perturbations in $\mathcal{C}_0$, it is still preserved under perturbations in the larger class $\mathcal{C}$. Clearly, the smaller is $\mathcal{C}_0$, the easier is to check whether or not the solvability of the equation $f(x) = h(x)$ is stable under perturbations in $\mathcal{C}_0$.

It seems to us that, given $\mathcal{C}$ as above, a convenient subclass $\mathcal{C}_0$ of $\mathcal{C}$ with the above requirements (and probably the smallest one) is given by those maps obtained (at $\lambda = 1$) by means of a homotopy satisfying conditions (a) - (c) of Definition 2.6 below. An equation whose solvability is stable under perturbations in $\mathcal{C}_0$ will be called *hyper-solvable*.

In Section 3 we prove a continuation result, Theorem 3.1, which implies that hyper-solvable equations are actually invariant under perturbations belonging to the class $\mathcal{C}$. Examples showing how these equations arise naturally in the framework of topological fixed point theory are given in Section 4.

2. Preliminaries

In what follows X and Y will stand for arbitrary topological spaces, and any map considered in this paper is assumed to be continuous. In some statements the space X is assumed to be T_4, i.e. Hausdorff and normal.

Some of the definitions that follow are standard. We include them for completeness sake.

Definition 2.1. A map $f : X \to Y$ is said to be

(i) *compact* if $f(X)$ is a relatively compact subset of Y (i.e., the closure $\overline{f(X)}$ of $f(X)$ is a compact set);

(ii) *locally compact* if for any $x \in X$ there exists a neighborhood U_x of x with $f(U_x)$ relatively compact in Y.

Obviously, if X is a locally compact space, then any map $f : X \to Y$ is locally compact.

We recall that a subset A of a (real) topological vector space E is said to be *bounded* if for any neighborhood U of the origin there exists $\delta > 0$ such that $\lambda A \subset U$ for $|\lambda| < \delta$.

Definition 2.2. Assume that X is a subset of a topological vector space E and let Y be as above. A map $f : X \to Y$ is said to be *completely continuous* if it maps bounded subsets of X into relatively compact subsets of Y.

Clearly, if E is a normed space and f is completely continuous, then f is also locally compact. Notice that this is not true in general when E is a topological vector space, since E need not be locally bounded. For example, an infinite dimensional Banach space with its weak topology is never locally bounded, because any weak neighborhood of the origin contains a straight line.

The following definition is not standard. As a matter of fact it is at the heart of this paper, together with the other two definitions that follow.

Definition 2.3. Let $H : X \times [0,1] \to Y$ be a homotopy. By $\mathrm{Dis}\, H$ we denote the set of those points of X which are *displaced* by H. Namely,

$$\mathrm{Dis}\, H = \Big\{ x \in X : H(x,0) \neq H(x,\lambda) \text{ for some } \lambda \in [0,1] \Big\}.$$

Definition 2.4. A locally compact homotopy $H : X \times [0,1] \to Y$ is said to be *conditionally compact* if $H(\mathrm{Dis}\, H \times [0,1])$ is a relatively compact subset of Y. In other words, H is conditionally compact if it is locally compact on $X \times [0,1]$ and actually compact on $\mathrm{Dis}\, H \times [0,1]$.

The following example shows that a conditionally compact homotopy need not be compact.

Example 2.5. Let E and F be Banach spaces. Define a homotopy $H : E \times [0,1] \to F$ by $H(x,\lambda) = h(x) + \lambda k(x)$, where h is completely continuous, but not compact, and k is compact with bounded support. Now, $\mathrm{Dis}\, H$ is obviously contained in the support of k. Thus H, being completely continuous, sends $\mathrm{Dis}\, H \times [0,1]$ into a relatively compact set.

Definition 2.6. Let $f, h : X \to Y$ be two maps, with h locally compact. Given a (possibly empty) subset A of X, we say that the coincidence equation $f(x) = h(x)$ is

(i) *admissible rel* A (or simply *admissible*, when $A = \emptyset$) if the (coincidence) set

$$\Sigma_0 = \{x \in X : f(x) = h(x)\}$$

is compact (possibly empty) and does not intersect A;

(ii) *hyper-solvable* (*from* X *into* Y) *rel* A (or simply *hyper-solvable*, when $A = \emptyset$), if it is admissible rel A and the equation $f(x) = H(x, 1)$ has a solution whenever $H : X \times [0, 1] \to Y$ is a homotopy satisfying the following conditions:

(a) $H(x, 0) \equiv h(x)$ (i.e., H starts from h)

(b) H is conditionally compact

(c) the set $\Sigma = \{(x, \lambda) \in X \times [0, 1] : f(x) = H(x, \lambda)\}$ is compact and does not intersect $A \times [0, 1]$.

Remark 2.7. If the equation $f(x) = h(x)$ is hyper-solvable rel A, then it has a solution in $X \setminus A$. To see this define $H(x, \lambda) = h(x)$ for any $(x, \lambda) \in X \times [0, 1]$ and observe that properties (a) - (c) of Definition 2.6 are fulfilled.

Clearly, any compact homotopy is conditionally compact. The following result provides a sufficient (and, obviously, necessary) condition for the converse to hold true.

Proposition 2.8. *Let* $H : X \times [0, 1] \to Y$ *be a conditionally compact homotopy. Assume that the starting map* $h(x) = H(x, 0)$ *is compact. Then,* H *is compact.*

Proof. We have

$$H(X \times [0, 1]) \subset H(\mathrm{Dis}\, H \times [0, 1]) \cup H\big((X \backslash \mathrm{Dis}\, H) \times [0, 1]\big)$$
$$= H(\mathrm{Dis}\, H \times [0, 1]) \cup h(X \backslash \mathrm{Dis}\, H)$$
$$\subset \overline{H(\mathrm{Dis}\, H \times [0, 1])} \cup \overline{h(X \backslash \mathrm{Dis}\, H)}$$

and the assertion is proved ∎

The following result has been suggested to us by Jorge Ize.

Proposition 2.9. *Let* Y *be a topological vector space and* $H : X \times [0, 1] \to Y$ *be a homotopy. Define* $h(x) = H(x, 0)$ *and* $G(x, \lambda) = H(x, \lambda) - h(x)$. *Then, the set* $H(\mathrm{Dis}\, H \times [0, 1])$ *is relatively compact if and only if the starting map* h *is compact on* $\mathrm{Dis}\, H$ *and the perturbing homotopy* G *is compact.*

Proof. Assume that h is compact on $\mathrm{Dis}\, H$ and G is compact. We have to show that $H(\mathrm{Dis}\, H \times [0, 1])$ is a relatively compact set. This follows at once from the inclusion

$$H(\mathrm{Dis}\, H \times [0, 1]) \subset h(\mathrm{Dis}\, H) + G(\mathrm{Dis}\, H \times [0, 1]).$$

Assume now that $H = h + G$ is conditionally compact. We have to show that $h(\operatorname{Dis} H)$ and $G(X \times [0, 1])$ are relatively compact sets. The first set is relatively compact since

$$h(\operatorname{Dis} H) = H(\operatorname{Dis} H \times \{0\}) \subset H(\operatorname{Dis} H \times [0, 1]).$$

As far as the second set regards notice that

$$G(X \times [0, 1]) = G(\operatorname{Dis} H \times [0, 1]) \cup \{0\}$$

and

$$G(\operatorname{Dis} H \times [0, 1]) \subset H(\operatorname{Dis} H \times [0, 1]) + (-h(\operatorname{Dis} H)).$$

Thus the assertion is proved ∎

3. Continuation principle

The following result can be regarded as a sort of homotopy invariance for hyper-solvable equations. As indicated in Introduction, such a result shows that the solvability of hyper-solvable equations is stable under a large class of perturbations. In fact, the perturbing homotopy need not be conditionally compact. The only requirement is its local compactness together with a sort of *a priori* bounds on the set of solutions of the coincidence equation.

Tneorem 3.1 (Continuation principle for hyper-solvable equations). *Let $f : X \to Y$ be a map between two topological spaces, where X is T_4, and let A be a closed subset of X. Assume that $G : X \times [0, 1] \to Y$ is a locally compact homotopy such that the equation $f(x) = G(x, 0)$ is hyper-solvable* rel *A. If the set*

$$\Gamma = \big\{(x, \lambda) \in X \times [0, 1] : f(x) = G(x, \lambda)\big\}$$

is compact and does not intersect $A \times [0, 1]$, then the equation

$$f(x) = G(x, 1)$$

is hyper-solvable rel *A as well.*

Proof. Let us show first that the equation

$$f(x) = G(x, 1)$$

has a solution. Let $\pi_1 : X \times [0, 1] \to X$ be the projection onto the first factor and consider the compact set $S = \pi_1(\Gamma)$, which by assumption does not intersect A. Since G is locally compact and A is closed, there exists an open

neighborhood W of S such that $W \cap A = \emptyset$ and $G(W \times [0,1])$ is relatively compact in Y. Since X is Hausdorff, S is closed in X. Thus, being X normal, by Urysohn's lemma there exists a continuous function $\sigma : X \to [0,1]$ such that $\sigma(x) = 0$ if $x \notin W$ and $\sigma(x) = 1$ if $x \in S$. Consider the homotopy $H(x,\lambda) = G(x, \lambda\sigma(x))$, which is conditionally compact, since $\mathrm{Dis}\, H \subset W$ and consequently

$$H(\mathrm{Dis}\, H \times [0,1]) \subset H(W \times [0,1]) \subset G(W \times [0,1]),$$

the last set being relatively compact.

The homotopy H satisfies also property (c) of Definition 2.6, since the set

$$\Sigma = \big\{ (x,\lambda) \in X \times [0,1] : f(x) = H(x,\lambda) \big\}$$

coincides with Γ. Indeed, if (x,λ) belongs to either Σ or Γ, then $x \in S$, and therefore $\sigma(x) = 1$. Since the equation $f(x) = H(x,0)$ is hyper-solvable rel A, there exists $\bar{x} \in X \setminus A$ such that $f(\bar{x}) = H(\bar{x}, 1)$. Consequently, the equality $\Sigma = \Gamma$ implies $f(\bar{x}) = G(\bar{x}, 1)$.

It remains to show that the equation $f(x) = G(x, 1)$ is actually hyper-solvable rel A. To this end consider a locally compact homotopy

$$\widehat{H} : X \times [0,1] \to Y$$

with the following properties:

(a) $\widehat{H}(x,0) \equiv G(x,1)$

(b) $\widehat{H}$ is conditionally compact

(c) the set $\big\{ (x,\lambda) \in X \times [0,1] : f(x) = \widehat{H}(x,\lambda) \big\}$ is compact and does not intersect $A \times [0,1]$.

We have to show that the equation $f(x) = \widehat{H}(x,1)$ is solvable. To see this define the homotopy

$$\widehat{G}(x,\lambda) = \begin{cases} G(x, 2\lambda) & \text{if } \lambda \in [0, \frac{1}{2}] \\ \widehat{H}(x, 2\lambda - 1) & \text{if } \lambda \in [\frac{1}{2}, 1] \end{cases}$$

which is clearly locally compact, and conclude as above that the equation $f(x) = \widehat{G}(x,1)$ has a solution (in $X \setminus A$) ∎

The following result is a useful consequence of the above Continuation Principle.

Corollary 3.2. *Let X be a T_4 space, A a closed subset of X and Y a topological vector space. Then, $f(x) = h(x)$ is hyper-solvable rel A if and only if so is $f(x) - h(x) = 0$.*

Proof. Assume that $f(x) - h(x) = 0$ is hyper-solvable rel A. Let us show that $f(x) = h(x)$ is hyper-solvable rel A as well. We need to prove that if $H : X \times [0,1] \to Y$ is a locally compact homotopy satisfying properties (a) - (c) of Definition 2.6, then the equation $f(x) = H(x,1)$ has a solution. To this end observe first that the homotopy

$$G(x, \lambda) = H(x, \lambda) - h(x)$$

is locally compact as well. Moreover, the set

$$\Gamma = \big\{(x, \lambda) \in X \times [0,1] : f(x) - h(x) = G(x, \lambda)\big\}$$

coincides with the set

$$\Sigma = \big\{(x, \lambda) \in X \times [0,1] : f(x) = H(x, \lambda)\big\}$$

which is compact because of assumption c). Since $f(x) - h(x) = 0$ is hyper-solvable rel A, by Theorem 3.1 so is the equation $f(x) - h(x) = G(x, 1)$. In particular, $f(x) = H(x, 1)$ has a solution. Thus, $f(x) = h(x)$ is hyper-solvable rel A. The converse implication follows analogously ∎

4. Examples of hyper-solvable equations

The following result shows that the concept of hyper-solvable equation may be regarded as an extension of the notion of 0-epi map introduced in [4].

Theorem 4.1. *Let $f : \overline{U} \to F$ be a map from the closure of a bounded open set U of a Banach space E into a Banach space F. If f is 0-epi, then the equation $f(x) = 0$ is hyper-solvable rel ∂U. If, moreover, f is proper, then the converse implication holds.*

Proof. Assume that f is 0-epi and consider a conditionally compact homotopy $H : \overline{U} \to F$ with $H(x, 0) \equiv 0$ and

$$\Sigma = \big\{(x, \lambda) \in \overline{U} \times [0,1] : f(x) = H(x, \lambda)\big\}$$

compact and disjoint from $\partial U \times [0,1]$. By Proposition 2.8, H is a compact homotopy. Thus, because of the Homotopy Property of 0-epi maps (see [4]), the equation $f(x) = H(x, 1)$ has a solution (in U). Hence, $f(x) = 0$ is hyper-solvable rel ∂U.

Assume now that $f(x) = 0$ is hyper-solvable rel ∂U and let $h : \overline{U} \to F$ be any compact map such that $h(x) = 0$ for all $x \in \partial U$. Define $H(x, \lambda) = \lambda h(x)$ for $(x, \lambda) \in \overline{U} \times [0, 1]$. The homotopy H is clearly conditionally compact, being h a compact map. Moreover, the set

$$\Sigma = \left\{ (x, \lambda) \in \overline{U} \times [0, 1] : f(x) = H(x, \lambda) \right\}$$

does not intersect $\partial U \times [0, 1]$, because for $x \in \partial U$ one has $f(x) \neq 0$ (being $f(x) = 0$ hyper-solvable rel ∂U) and $h(x) = 0$ (by assumption). We show now that Σ is compact. In fact, let K denote the closure of the relatively compact set $H(\overline{U} \times [0, 1])$. Since f is proper, the set $f^{-1}(K)$ is compact. Now, observe that $\Sigma \subset f^{-1}(K) \times [0, 1]$. Thus, Σ is compact, as claimed.

Being $f(x) = 0$ hyper-solvable rel ∂U, there exists $\bar{x} \in U$ such that $f(\bar{x}) = H(\bar{x}, 1) = h(\bar{x})$. Hence, f is 0-epi ∎

Example 4.2. The equation $x = 0$ is hyper-solvable in $\mathbb{R}^n$ (i.e., from $\mathbb{R}^n$ into itself). To see this, take a homotopy $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$ according to Definition 2.6. By Proposition 2.8, H is a compact map. Take a closed ball D containing the closure of $H(\mathbb{R}^n \times [0, 1])$ and apply the Brouwer fixed point theorem to the map $g : D \to D$ given by $g(x) = H(x, 1)$.

Remark 4.3. The assertion that the equation $x = 0$ is hyper-solvable in $\mathbb{R}^n$ may be regarded as a reformulation of the Brouwer fixed point theorem.

With the same argument as in Example 4.2, replacing the Brouwer fixed point theorem by that of Schauder, we get the following

Example 4.4. The equation $x = 0$ is hyper-solvable in any normed space.

The following result extends that in Example 4.4 and will be deduced from the Schauder fixed point theorem, without any fixed-point index theory.

Theorem 4.5. *Let X be a metrizable Absolute Neighborhood Retract (ANR) and let $p \in X$. Then, the equation $x = p$ is hyper-solvable in X. In particular, a continuous map $h : X \to X$ has a fixed point whenever it is homotopic to a constant map through a compact homotopy.*

Proof. Let $H : X \times [0, 1] \to X$ be a conditionally compact homotopy such that $H(x, 0) \equiv p$. It is enough to show that the equation $x = H(x, 1)$ has a solution. By the Arens-Eells embedding theorem (see [1]), X can be regarded as a closed subset of a normed space E. Since X is an ANR, there exists an open neighborhood U of X in E and a retraction $r : \overline{U} \to X$. Define $\sigma : E \to [0, 1]$ by

$$\sigma(x) = \frac{d(x, E \backslash U)}{d(x, X) + d(x, E \backslash U)}$$

where d stands for the distance in the normed space E. Because of Proposition 2.8, H is a compact homotopy. Thus, the continuous map $h : E \to E$ given by

$$h(x) = \begin{cases} p & \text{if } x \in E\backslash U \\ H(r(x), \sigma(x)) & \text{if } x \in \overline{U} \end{cases}$$

is compact as well. Therefore, as a consequence of Schauder's fixed point theorem, there exists $\bar{x} \in E$ such that $\bar{x} = h(\bar{x})$. From the definition of h it follows $\bar{x} \in X$. Hence, $r(\bar{x}) = \bar{x}$, $\sigma(\bar{x}) = 1$ and, consequently, $\bar{x} = H(\bar{x}, 1)$ ∎

The following is a consequence and an extension of the previous result.

Theorem 4.6. *Let X be a metrizable* ANR *and let $U \subset X$ be open. Then, given $p \in X\backslash\partial U$, the equation $x = p$ is hyper-solvable from $\overline{U}$ into X rel ∂U if and only if $p \in U$ (or, more generally, from U into X if and only if $p \in U$).*

Proof. We may assume $p \in U$, since by Remark 2.7 this condition is necessary for the equation $x = p$ to be hyper-solvable from $\overline{U}$ into X rel ∂U. Let $H : \overline{U} \times [0, 1] \to X$ be a conditionally compact homotopy starting from the constant map $x \mapsto p$ and such that the set

$$\Sigma = \left\{ (x, \lambda) \in \overline{U} \times [0, 1] : x = H(x, \lambda) \right\}$$

is compact and does not intersect $\partial U \times [0, 1]$. We need to show that the equation $x = H(x, 1)$ has a solution. Denote by S the projection of Σ onto the first factor of $\overline{U} \times [0, 1]$. Since Σ does not intersect $\partial U \times [0, 1]$, S is a compact subset of U, which clearly contains p since $(p, 0) \in \Sigma$. Let V be any open neighborhood of S with closure in U. Define $\sigma : X \to [0, 1]$ by

$$\sigma(x) = \frac{d(x, X\backslash V)}{d(x, S) + d(x, X\backslash V)}$$

where d is any metric on X compatible with the topology. Define the homotopy $G : X \times [0, 1] \to X$ by

$$G(x, \lambda) = \begin{cases} p & \text{if } x \in X\backslash V \\ H(x, \lambda\sigma(x)) & \text{if } x \in \overline{V}. \end{cases}$$

Because of Proposition 2.8, H is a compact homotopy. Thus, so is G. Consequently, from Theorem 4.5 we may deduce that the equation $x = G(x, 1)$ has a solution $\bar{x} \in X$, since $x \mapsto G(x, 1)$ is homotopic to the constant map $x \mapsto p$ through G. Observe that $\bar{x} \in V$, since the contrary would lead to the contradiction $\bar{x} = p \in S \subset V$. Hence, $\bar{x} = G(\bar{x}, 1) = H(\bar{x}, \sigma(\bar{x}))$, and this implies that $\bar{x} \in S$. The assertion that the equation $x = H(x, 1)$ has a solution now follows from the fact that $\sigma(x) = 1$ for all $x \in S$.

With the same argument, and just taking the homotopy H merely defined on $U \times [0, 1]$, one can show that when $p \in U$ the equation $x = p$ is actually hyper-solvable from U into X ∎

As an immediate consequence of the homotopy invariance of the Lefschetz number (see, e.g., [2]) we get the following

Example 4.7. Let $h : K \to K$ be a map from a compact polyhedron into itself. If the Lefschetz number $\Lambda(h)$ of h is different from zero, then $x = h(x)$ is hyper-solvable.

Example 4.8. Let $h : \overline{U} \to E$ be a compact map from the closure of a bounded open subset U of a Banach space E into E. Assume that $x \neq h(x)$ for all $x \in \partial U$. If the Leray-Schauder degree $\deg(I - h, U, 0)$ is different from zero, then $x = h(x)$ is hyper-solvable rel ∂U. To see this, let $H : \overline{U} \times [0, 1] \to E$ be a homotopy as in Definition 2.6. By Proposition 2.8, H is a compact map, and by property (c) of Definition 2.6, $x \neq H(x, \lambda)$ for all $x \in \partial U$. The assertion now follows from the homotopy invariance of the Leray-Schauder degree.

The following example follows at once from the homotopy property of essential compact vector fields introduced by Granas in [3].

Example 4.9. Let $h : \overline{U} \to E$ be a compact map from the closure of a bonded open subset U of a Banach space E into E. Assume that $x \neq h(x)$ for all $x \in \partial U$. If the compact vector field $I - h$ is essential, then the equation $x = h(x)$ is hyper-solvable rel ∂U.

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