Stability Phenomenon for Generalizations of Algebraic Differential Equations

G. Barsegian, H. Begehr and I. Laine

Abstract. Certain stability properties for meromorphic solutions $w(z) = u(x,y) + i\,v(x,y)$ of partial differential equations of the form $\sum_{t=0}^{m} f_t\,(w')^{m-t} = 0$ are considered. Here the coefficients f_t are functions of x, y, of u, v and the partial derivatives of u, v. Assuming that certain growth conditions for the coefficients f_t are valid in the preimage under w of five distinct complex values, we find growth estimates, in the whole complex plane, for the order $\rho(w)$ and the unintegrated Ahlfors-Shimizu characteristic $A(r, w)$.

Keywords: *Algebraic differential equations, growth of meromorphic functions, stability of growth*

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1. Introduction

Recently, a stability phenomenon has been described for meromorphic solutions of algebraic differential equations of first order (see [4]). Obviously, a similar stability phenomenon may appear for other types of differential or functional equations as well as for related practical situations. In general, let $f(z)$ be a solution of a given equation $P(z, f) = 0$ in a domain D, f satisfying a property $\mathcal{P}$ in D. We now ask whether this property $\mathcal{P}$ remains true in D if we only know that $f(z)$ satisfies the equation $P(z, f) = 0$ on a "small" subset $\gamma \subset D$? In other words, we ask whether we may infer (in some sense) complete information of solution(s) for all $z \in D$ from a partial information only. The importance of such a problem setting is clear in applications: In

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practice, it is usual to make conclusions for complete data in a domain from observation data found along some curves, on a lattice or on a discrete point set in the domain. Obviously, the stability problem described above reveals a similarity with uniqueness problems too.

It is natural to expect that the stability phenomenon takes place for solutions which admit some kind of analytic nature. In the first line, we have in mind meromorphic solutions of algebraic differential equations in a domain D of the complex plane $\mathbb{C}$ (see [11] for a background). Similar stability for solutions of complex elliptic partial differential equations [6] apparently seems to be more difficult to study. However, such investigations appear most promising, due to the wide applicability of this topic.

2. Stability phenomenon for first order generalizations

The classical result due to A.A. Gol'dberg [10] tells that meromorphic solutions of a first order algebraic differential equation

$$P(z, w, w') = 0$$

in $\mathbb{C}$ are of finite Nevanlinna order. More precisely, suppose the coefficients $P_0(z, w), \ldots, P_m(z, w)$ in

$$P(z, w, w') := P_0(z, w)(w')^m + P_1(z, w)(w')^{m-1} + \ldots + P_m(z, w) = 0 \quad (2.1)$$

are polynomials in both variables with degree $c(w, t) = \deg_z P_t(z, w)$ $(t = 0, \ldots, m)$ with respect to z. Then for any meromorphic solution w of (2.1), the Nevanlinna order $\rho(w)$ satisfies

$$\rho(w) \leq 2 \max_{1 \leq t \leq m} \frac{c(w, t) - c(w, 0)}{t} + 2$$

(see [1]).

We first recall that the Gol'dberg result follows if we only require that w satisfies equation (1.1) on a certain type of small counting set of values z. In what follows, let w be meromorphic in $\mathbb{C}$, and let γ stand for the set of all points $z \in \mathbb{C}$ such that $w(z) = a_j$ $(j = 1, \ldots, 5)$, where $a_1, \ldots, a_5$ are distinct complex values. In [4] the following theorem was proved:

Theorem A. *If w satisfies differential equation (2.1) on γ, then $\rho(w) < \infty$.*

The proof of Theorem A in [4] was based on the method of derivatives (see [1] where it was first applied to first order differential equations). Later on,

the method was developed in [2, 3, 5] for applying it to higher order algebraic differential equations.

We now proceed to show that similar conclusions hold in a much larger collection of equations which may be described as certain kind of generalizations of algebraic differential equations. To this end, and throughout of this paper, we consider partial differential equations of the form

$$P\left(x, y, u, v, u'_x, u'_y, v'_x, v'_y\right) = 0. \tag{2.2}$$

As the example following Theorem 1 shows, a suitable equation of type (2.2) may admit a meromorphic solution. In such a case, the Cauchy-Riemann equations simultaneously hold. On the other hand, it is easy to see that constants are the only meromorphic solutions of $(u_x - 3v_y) + i(v_x + 3u_y) = 0$. In this paper, we are interested on meromorphic solutions only.

Suppose first that a meromorphic function $w = w(z) = u(x, y) + iv(x, y)$ is a solution of

$$P\left(x, y, u, v, u'_x, u'_y, v'_x, v'_y\right) = f_0(w')^m + f_1(w')^{m-1} + \ldots + f_m = 0 \tag{2.3}$$

where

$$f_t = f_t\left(x, y, u, v, u'_x, u'_y, v'_x, v'_y\right) \qquad (t = 0, \ldots, m). \tag{2.4}$$

Moreover, assume (2.4) satisfies the following

Condition 1. For $z = (x, y) \in \gamma$,

$$\left|f_t(x, y, u, v, u'_x, u'_y, v'_x, v'_y)\right| \le c_t\left(\sqrt{x^2 + y^2}\right)^{c(t)} \qquad (t = 1, \ldots, m) \tag{2.5}$$

and

$$\left|f_0(x, y, u, v, u'_x, u'_y, v'_x, v'_y)\right| \ge \frac{c_0}{\left(\sqrt{x^2 + y^2}\right)^{c(0)}} \qquad \left(\sqrt{x^2 + y^2} > 1\right) \tag{2.6}$$

where $c_t > 0$ and $c(t) > 0$ $(t = 0, \ldots, m)$ are constants.

Theorem 1. *Suppose w is a meromorphic solution of equation* (2.3) *satisfying Condition* 1. *Then*

$$\rho(w) \le 2 \max_{1 \le t \le m} \frac{c(t) + c(0)}{t} + 2. \tag{2.7}$$

Remark. Obviously, Theorem 1 contains the Gol'dberg result.

Example. Let $P(z)$ be a polynomial of degree n. Take an arbitrary complex-valued function $\eta_0(x, y)$ such that

$$\left(\sqrt{x^2 + y^2}\right)^N \ge |\eta_0(x, y)| \ge \frac{c_0}{\left(\sqrt{x^2 + y^2}\right)^{c(0)}}$$

for real constants N, c_0 and $c(0)$. Then, for a meromorphic function w, assume $w(z) \in \{a_1, \ldots, a_5\}$ with $w(z) \neq 0$ and $w(z) \neq \infty$. For $\eta_1(x, y) = P'(z)\eta_0(x, y)$ we notice that

$$|\eta_1(x, y)| \, |w(z)| \leq C \Big(\max_{1 \leq j \leq 5} |a_j| \Big) \big(\sqrt{x^2 + y^2} \big)^{N+n-1}$$

for a constant $C > 0$. Consequently, the equation $f_0 w' = f_1$ is of type (2.3), and $f_0 = \eta_0(x, y)$ and $f_1 = \eta_1(x, y)w$ are of type (2.4) satisfying (2.5) and (2.6). Moreover, $w(z) = \exp(P(z))$ is a solution of $f_0 w' = f_1$.

We next replace Condition 1 with

Condition 2. Given $\omega_t(x, y) \geq 0$ $(t = 0, \ldots, m)$, assume for $z = (x, y) \in \gamma$ that

$$\big| f_t(x, y, u, v, u'_x, u'_y, v'_x, v'_y) \big| \leq \omega_t(x, y) \qquad (t = 1, \ldots, m) \qquad (2.8)$$

and

$$\big| f_0(x, y, u, v, u'_x, u'_y, v'_x, v'_y) \big| \geq \omega_0(x, y) > 0 \qquad \big(\sqrt{x^2 + y^2} > 1 \big). \qquad (2.9)$$

Condition 2 enables us to get estimates for the unintegrated Ahlfors-Shimizu characteristic $A(r, w)$:

Theorem 2. *Suppose w is a meromorphic solution of equation (2.3) satisfying Condition 2, and let $\varphi(r)$ be a monotone increasing function, $\varphi(r) \to \infty$ as $r \to \infty$. Then*

$$A(r, w) \leq \varphi^2(r) r^2 \max_{(x,y) \in \gamma \cap \{|z| < r\}} \left(\frac{m \omega_t(x, y)}{\omega_0(x, y)} \right)^{\frac{2}{t}} \qquad (2.10)$$

for all r outside of an exceptional set E of finite logarithmic measure.

Remark 1. The set $\gamma \cap \{|z| < r\}$ is finite, hence we have max in (2.10) instead of sup.

Remark 2. Theorem 2 generalizes Theorem 1. Indeed, if $\omega_t(x, y)$ has growth limits as in Theorem 1, then (2.10) implies $A(r, w) \leq \varphi(r) r^k$ for some $k \in \mathbb{N}$ as $r \to \infty$, $r \notin E$. Since E is of finite logarithmic measure, the exceptional set can be eliminated in a standard way to conclude that $\rho(w) < \infty$. Here we take into account that $\varphi(r)$ may tend to infinity as slowly as we please.

Remark 3. If $w(z)$ is a solution of equation (2.3) satisfying Condition 2, and the set γ is arbitrarily defined, then the right-hand side of (2.10) grows to infinity faster than any polynomial.

3. Stability phenomenon for higher order generalizations▮

The method of estimating derivatives has been applied (see [2, 3, 5]) to study-
ing higher order algebraic differential equations of type

$$\left.\begin{array}{r} F_0(z,w)(w')^m + F_1(z,w,w'',\ldots,w^{(k)})(w')^{m-1} \\ \vdots \\ + F_m(z,w,w'',\ldots,w^{(k)}) = 0 \end{array}\right\} \tag{3.1}$$

where $F_t \ (t = 0,\ldots,m)$ are polynomials in each of their variables. We make
use of the notations

$$F_t = \sum_{j(t)} a_{j(t)} z^{c(z,j,t)} w^{c(w,j,t)} (w'')^{c(w'',j,t)} \cdots (w^{(k)})^{c(w^{(k)},j,t)} \tag{3.2}$$

and

$$p_t = \max_{j(t)} \left\{ 2c(w'',j,t) + 3c(w^{(3)},j,t) + \ldots + kc(w^{(k)},j,t) \right\} \quad (t = 1,\ldots,m).$$

Observe that for $k = 2$ the value of $\frac{p_t}{2}$ is just the maximal degree of w'' in
the polynomial F_t. Recall that a quantity similar to p_t occurs in [9], playing
a crucial role there.

Theorem B (see [2, 3]). *Any meromorphic solution w of equation (3.1)
is of finite order of growth provided $p_\nu < \nu$ for $\nu = 1,\ldots,m$.*

Remark 1. Theorem B is sharp. In fact, it is easy to see that $w(z) = \exp(\exp z)$ satisfies an equation of the form

$$(w')^k + a_1(w')^{k-1}w + \ldots + a_{k-1}w'w^{k-1} - w^{(k)}w^{k-1} = 0.$$

Here $p_k = k$ and $\rho(w) = \infty$.

Remark 2. Later on, a new proof based on the Zalcman lemma [13] has
been offered by W. Bergweiler [7]. Simultaneously, G. Frank and Y. Wang
also applied the Zalcman lemma, considering also some cases with $p_\nu = \nu$ [8].
In [5], G. Barsegian, I. Laine and C. Yang applied the method of estimating
derivatives to most general cases of equations of the form (3.1).

Arguing as in the first order case we now assume that a meromorphic
function $w = w(z) = u(x,y) + iv(x,y)$ is a solution of

$$P\left(x,y,u,u'_x,\ldots,u_x^{(k)},u'_y,\ldots,u_y^{(k)},v'_x,\ldots,v_x^{(k)},v'_y,\ldots,v_y^{(k)}\right)$$
$$= f_0(w')^m + f_1(w')^{m-1} + \ldots + f_m = 0 \tag{3.3}$$

where

$$f_t = f_t\left(x,y,u,\ldots,v_y^{(k)}\right) \quad (t = 0,\ldots,m). \tag{3.4}$$

Moreover, we assume that (3.4) satisfies

Condition 3. For $(x, y) \in \gamma$,

$$|f_t| \le c_t \left(\sqrt{x^2 + y^2} \right)^{c(t)} \sum_{j(t)} |w''|^{c(w'', j, t)} \cdots |w^{(k)}|^{c(w^{(k)}, j, t)} \tag{3.5}$$

for $t = 1, \ldots, m$, and

$$|f_0| \ge \frac{c_0}{\left(\sqrt{x^2 + y^2} \right)^{c(0)}} \qquad \left(\sqrt{x^2 + y^2} > 1 \right) \tag{3.6}$$

where $c_t > 0$ and $c(t) > 0$ $(t = 0, \ldots, m)$ are constants.

Theorem 3. *Suppose w is a meromorphic solution of equation (3.3) satisfying Condition 3 and the inequality*

$$k := \max_{1 \le t \le m} \frac{p_t}{t} < 1. \tag{3.7}$$

Then w is of finite order

$$\rho(w) \le 2 \max_{1 \le t \le m} \frac{c(t) + c(0)}{t(1 - k)} + 2. \tag{3.8}$$

Remark. Theorem 3 offers a stability phenomenon, simultaneously generalizing previous results: the coefficients are not necessarily polynomials. Examples similar to the first order case can be easily constructed.

Similarly as to Condition 2, we also consider

Condition 4. Given $\omega_t(x, y) \ge 0$ $(t = 0, \ldots, m)$, assume for $z = (x, y) \in \gamma$ that

$$|f_t| \le \omega_t(x, y) \sum_{j(t)} |w''|^{c(w'', j, t)} \cdots |w^{(k)}|^{c(w^{(k)}, j, t)} \qquad (t = 1, \ldots, m) \tag{3.9}$$

and

$$|f_0| \ge \omega_0(x, y) \qquad \left(\sqrt{x^2 + y^2} > 1 \right). \tag{3.10}$$

Theorem 4. *Suppose w is a meromorphic solution of equation (3.3) satisfying Condition 4 and inequality (3.7). Then, for a constant $c \in (1, \infty)$,*

$$A(r, w) \le c \, \varphi^2(r) \, r^2 \left(1 + \max_{(x, y) \in \gamma \cap \{|z| < r\}} \left(\frac{m \, \omega_t(x, y)}{\omega_0(x, y)} \right)^{\frac{2}{t(1 - k)}} \right) \tag{3.11}$$

for all r outside of an exceptional set of finite logarithmic measure.

4. Proofs using the method of estimating derivatives

The essential contribution of [1] to the main conclusions of the value distribution theory has been, qualitatively speaking, that at "good" a-points of a meromorphic function w the modulus $|w'|$ is "big"; estimates from below for $|w'|$ are given in terms of the characteristic functions. The estimates have been applied to algebraic differential equations of first order in the complex plane, having let to what has been called as the method of estimating derivatives. This permits to make conclusions about meromorphic solutions w by considering the equation at "good" a-points of w and using the corresponding estimates for $|w'|$. Later on, the method was extended by working out estimates of higher derivatives $|w^{(k)}|$ from above; these have been applied to prove Theorem B (see, e.g., [5]). For our purposes below, we recall

Theorem C (see [1, 3]). *Suppose w is meromorphic, $a_1, \ldots, a_q$ $(q > 4)$ are distinct complex constants, and $\varphi(r)$ is a monotone increasing function with $\varphi(r) \to \infty$ as $r \to \infty$. Then there exists a set E of finite logarithmic measure such tat, for every $r \notin E$, there exists a subset*

$$\left\{ z_j(a_\nu) \big| \nu = 1, \ldots, q \text{ and } j = 1, \ldots, n_0(r, a_\nu) \right\}$$

of the a_ν-points of w in $|z| \leq r$ such that

$$\sum_{n=1}^{q} n_0(r, a_\nu) \geq (q-4)A(r, w) - o(A(r, w)) \qquad (r \to \infty, \, r \notin E)$$

and

$$\left| w'(z_j(a_\nu)) \right| \geq \frac{A(r, w)^{\frac{1}{2}}}{\varphi(r)\, r} \qquad \left(\nu = 1, \ldots, q; \, j = 0, \ldots, n_0(r, a_\nu); \, r \notin E \right).$$

Moreover, for any integer $k \geq 2$, there exists a constant $C \in [1, \infty)$, depending on k and $a_1, \ldots, a_q$ such that

$$\left| w^{(k)}(z_j(a_\nu)) \right| \leq C \left| w'(z_j(a_\nu)) \right|^k \qquad \left(\nu = 1, \ldots, q; \, j = 0, \ldots, n_0(r, a_\nu), \, r \notin E \right).$$

To make application of Theorem C easier below, we give the following easy consequence of it:

Lemma 4.1. *Suppose $q = 5$ in Theorem C. Then at least for one of $a_1, \ldots, a_5$, say a_1, there exists a subset $\left\{ z_j^*(a_1) \big| j = 1, \ldots, n^*(r, a_1) \right\}$ of the a_1-points of w in $|z| \leq r$ such that*

$$n^*(r, a_1) \geq \tfrac{1}{5} A(r, w) - o(A(r, w)) \qquad (r \to \infty, \, r \notin E)$$

and

$$\left|w'(z_j^*(a_1))\right| \geq \frac{A(r,w)^{\frac{1}{2}}}{\varphi(r)r} \qquad \left(j = 1,\ldots,n^*(r,a_1); \ r \notin E\right). \qquad (4.1)$$

Moreover, for any integer $k \geq 2$ there exists a constant $C \in [1,\infty)$ depending on k and $a_1,\ldots,a_5$ such that

$$\left|w^{(k)}(z_j^*(a_1))\right| \leq C\left|w'(z_j^*(a_1))\right|^k \qquad \left(j = 1,\ldots,n^*(r,a_1); \ r \notin E\right). \qquad (4.2)$$

We now proceed to prove Theorems 1 - 4. To this end, recall that all roots of an algebraic equation

$$z^m + b_1 z^{m-1} + \ldots + b_m = 0$$

lie in the disk $|z| \leq M = \max_{1 \leq t \leq m}(m|b_t|)^{1/t}$ (see [12]). Consider first equation (2.3) with (2.4) satisfying Condition 2 on the set of a_1-points determined by Lemma 4.1. We obtain for any $z_j^*(a_1) = x_j^* + iy_j^*$ the inequality

$$\left|w'(z_j^*(a_1))\right| \leq \max_{1 \leq t \leq m}\left(\frac{m\,\omega_t(x_j^*,y_j^*)}{\omega_0(x_j^*,y_j^*)}\right)^{\frac{1}{t}} \leq \max_{(x,y)\in\gamma\cap\{|z|<r\}}\left(\frac{m\,\omega_t(x,y)}{\omega_0(x,y)}\right)^{\frac{1}{t}} \ (r \notin E).\ \blacksquare$$

Therefore, (4.1) implies

$$A(r,w) \leq \varphi^2(r)\,r^2 \max_{(x,y)\in\gamma\cap\{|z|<r\}}\left(\frac{m\,\omega_t(x,y)}{\omega_0(x,y)}\right)^{\frac{2}{t}} \qquad (r \notin E)$$

and Theorem 2 has been proved.

In the case of Condition 1, we similarly obtain

$$\left|w'(z_j^*(a_1))\right| \leq \max_{1 \leq t \leq m}\left(\left(\frac{m\,c_t}{c_0}\right)^{\frac{1}{t}}|z_j^*(a_1)|^{\frac{c(t)+c(0)}{t}}\right) \qquad (r \notin E).$$

Taking into account that $|z_j^*(a_1)| < r$, (4.1) now implies

$$A(r,w) \leq \max_{1 \leq t \leq m}\left(\frac{m\,c_t}{c_0}\right)^{\frac{2}{t}}\varphi^2(r)\,r^{2K+2} \qquad (r \notin E)$$

where $K = \max_{1 \leq t \leq m}\frac{c(t)+c(0)}{t}$. Now, the reasoning described in Remark 2 after Theorem A results in $\rho(w) \leq 2K+2$, and we are done with the proof of Theorem 1.

To prove Theorem 4, we consider equation (3.3) with (3.4) satisfying Condition 4 on the set of a_1-points determined by Lemma 4.1. Making use of (4.2) for any $z_j^*(a_1) = x_j^* + iy_j^*$ such that $|z_j^*(a_1)| < r \notin E$, we obtain the inequality

$$\left|w'(z_j^*(a_1))\right| \leq Cb \max_{1 \leq t \leq m}\left(\frac{m\,\omega_t(x_j^*,y_j^*)}{\omega_0(x_j^*,y_j^*)}\right)^{\frac{1}{t}} \max\left\{1, \left|w'(z_j^*(a_1))\right|\right\}^k \qquad (4.3)$$

where b is the maximal number of monomials occurring in (3.9). Now, the case when the inequality $\left|w'(z_j^*(a_1))\right| \leq 1$ holds for at least of $z_j^*(a_1)$ under consideration is trivial. In fact, from (4.1) we immediately conclude

$$A(r, w) \leq \varphi^2(r)\, r^2. \tag{4.4}$$

Therefore, we assume that $\left|w'(z_j^*(a_1))\right| > 1$ for all $z_j^*(a_1)$ under consideration. Then (4.3) takes the form

$$\left|w'(z_j^*(a_1))\right| \leq Cb \max_{1 \leq t \leq m} \left(\frac{m\,\omega_t(x_j^*, y_j^*)}{\omega_0(x_j^*, y_j^*)}\right)^{\frac{1}{t}} \left|w'(z_j^*(a_1))\right|^k$$

and so

$$\left|w'(z_j^*(a_1))\right| \leq \left(Cb \max_{1 \leq t \leq m} \left(\frac{m\,\omega_t(x_j^*, y_j^*)}{\omega_0(x_j^*, y_j^*)}\right)^{\frac{1}{t}}\right)^{\frac{1}{1-k}}$$

$$\leq \left(Cb \max_{(x,y) \in \gamma \cap \{|z| < r\}} \left(\frac{m\,\omega_t(x, y)}{\omega_0(x, y)}\right)^{\frac{1}{t}}\right)^{\frac{1}{1-k}}.$$

Hence, (4.1) results in

$$A(r, w) \leq (Cb)^{\frac{1}{1-k}} \varphi^2(r)\, r^2 \left(\max_{(x,y) \in \gamma \cap \{|z| < r\}} \left(\frac{m\,\omega_t(x, y)}{\omega_0(x, y)}\right)^{\frac{1}{t}}\right)^{\frac{2}{1-k}} \quad (r \notin E). \tag{4.5}$$

Theorem 4 now follows from (4.4) and (4.5).

When we take Condition 3 instead of Condition 4, we similarly get

$$\left|w'(z_j^*(a_1))\right| \leq Cb \max_{1 \leq t \leq m} \left(\left(\frac{m\,c_t}{c_0}\right)^{\frac{1}{t}} |z_j^*(a_1)|^{\frac{c(t)+c(0)}{t}}\right) \left|w'(z_j^*(a_1))\right|^k \qquad (r \notin E).$$

Similarly as to above we obtain

$$A(r, w) \leq (Cb)^{\frac{2}{1-k}} \varphi^2(r)\, r^2 \left(\max_{1 \leq t \leq m} \left(\left(\frac{m\,c_t}{c_0}\right)^{\frac{1}{t}} |z_j^*(a_1)|^{\frac{c(t)+c(0)}{t}}\right)\right)^{\frac{2}{1-k}}$$

$$\leq (Cb)^{\frac{2}{1-k}} \varphi^2(r)\, r^{2K_1+2} \qquad (r \notin E)$$

where $K_1 = \max_{1 \leq t \leq m} \frac{c(t)+c(0)}{t(1-k)}$. Parallel to previous reasoning for Theorem 1, the exceptional set may be easily eliminated, and Theorem 3 follows.

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