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A hyperbolic Out(F_n)-complex

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Abstract. For any finite collection f_i of fully irreducible automorphisms of the free group F_n we construct a connected δ-hyperbolic Out(F_n)-complex in which each f_i has positive translation length.

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1. Introduction

The study of the outer automorphism group Out(F_n) of a free group F_n of rank n has very successfully been driven by analogies with mapping class groups. At the foundation of the theory is Culler–Vogtmann's Outer space [CV86], which plays the role of Teichmüller space. The topology of Outer space is very well understood, but its geometry is still very much a mystery. This is to be contrasted with the rich theory of the geometry of Teichmüller space. An instance of this contrast is the celebrated result of Masur and Minsky [MM99] that the curve complex is hyperbolic. There is no analogous result in the Out(F_n) category, although candidates for such a complex abound, see [KL09].

In this paper we prove the following, where $\overline{\mathcal{PT}}$ denotes the compactified Outer space.

Main Theorem. *For any finite collection $f_1, \dots, f_k$ of fully irreducible elements of* Out(F_n) *there is a connected δ-hyperbolic graph $\mathcal{X}$ equipped with an* (*isometric*) *action of* Out(F_n) *such that*

- *the stabilizer in* Out(F_n) *of a simplicial tree in $\overline{\mathcal{PT}}$ has bounded orbits,*
- *the stabilizer in* Out(F_n) *of a proper free factor $F \subset F_n$ has bounded orbits, and*
- *$f_1, \dots, f_k$ have nonzero translation lengths.*

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The situation is much less than ideal, not only because of the dependence of $\mathcal{X}$ on choices, but also because there is no "intrinsic" description of the complexes in the style of the curve complex.

However, the complexes are useful in that they allow construction of many quasi-homomorphisms on $\mathrm{Out}(F_n)$, a result recently announced by Ursula Hamenstädt.

The construction follows an idea of Brian Bowditch, who used it to show that convergence groups are hyperbolic [Bow98]. In Section 2 we review Bowditch's construction, in Section 3 we sketch the analogous construction of a hyperbolic complex for mapping class groups and in Section 4 we carry out this program for $\mathrm{Out}(F_n)$. The construction for $\mathrm{Out}(F_n)$ relies on the dynamics of the action of $\mathrm{Out}(F_n)$ on spaces of trees and currents, and we review the necessary material. Some of the results we need are slight variations of the ones found in the literature, and we sketch the proofs of these.

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2. Bowditch's construction

The goal of this section is to show that if a group Γ acts on a space M satisfying some simple axioms, then Γ also acts on a δ-hyperbolic space. The model situation is that of a convergence group action on a compact space, discussed by Bowditch [Bow98]. He proved that a group Γ that acts on a compact metrizable space as a convergence group (i.e., properly and cocompactly on the space of triples of distinct points) is hyperbolic and the compact space is equivariantly homeomorphic to the boundary $\partial\Gamma$. In fact, with very little modification, Bowditch's construction applies to noncompact spaces. For example, by looking at the action of the mapping class group on the (suitable subset of the) Thurston boundary, this gives its action on a hyperbolic graph (it is not clear how this graph is related to the curve complex).

We will outline Bowditch's construction. First, we recall some definitions. Fix an action (by homeomorphisms) of a group Γ on a (metrizable) space M. We will assume that M has no isolated points.

2.1. Annulus systems. An *annulus* in M is a pair $A = (A^-, A^+)$ of disjoint closed subsets of M whose union is not all of M. For a subset $K \subset M$ write $K < A$ if $K \subset \mathrm{int}\, A^-$ and write $A < K$ if $K < -A := (A^+, A^-)$. For annuli A and $B = (B^-, B^+)$ write $A < B$ if $\mathrm{int}\, A^+ \cup \mathrm{int}\, B^- = M$.

An *annulus system* $\mathcal{A}$ is a Γ-invariant set of annuli such that $A \in \mathcal{A}$ implies $-A \in \mathcal{A}$.

If $K, L \subset M$ write $(K|L) = n \in \{0, 1, \dots, \infty\}$ if n is the maximal number of annuli A_i in $\mathcal{A}$ such that

$$K < A_1 < A_2 < \cdots < A_n < L.$$

For finite sets we drop braces, e.g., $(ab|cd)$ means $(\{a,b\}|\{c,d\})$.

Consider the following axioms.

(A1) If $x \neq y$ and $z \neq w$ then $(xy|zw) < \infty$.

(A2) There is $k \geq 0$ such that for any $x, y, z, w \in M$ either $(xz|yw) \leq k$ or $(xw|yz) \leq k$.

2.2. Hyperbolic crossratio. A *crossratio* on M is a function $M^4 \to [0,\infty]$, $(x,y,z,w) \mapsto (xy|zw)$ such that $(xy|zw) = (yx|zw) = (zw|xy)$. A crossratio is *$k$-hyperbolic* if:

(C1) If $F \subset M$ is a 4-element subset, we can write $F = \{x,y,z,w\}$ such that $(xz|yw) \simeq_k 0$ and $(xw|yz) \simeq_k 0$.

(C2) If $F \subset M$ is a 5-element subset, we can write $F = \{x,y,z,w,u\}$ such that $(xy|zu) \simeq_k (xy|wu)$, $(xu|zw) \simeq_k (yu|zw)$, $(xy|zw) \simeq_k (xy|zu) + (xu|zw)$.

Here $a \simeq_k b$ means $|a-b| \leq k$. The intuition is that if M is a tree, then letting $(xy|zw)$ be the distance between $[x,y]$ and $[z,w]$ defines a 0-hyperbolic crossratio.

Note that (C1) implies that for any 4 distinct points at most one of the crossratios $(xy|zw)$, $(xz|yw)$, $(xw|yz)$ is $> k$. We write $(xy:zw)$ to mean that $(xz|yw)$ and $(xw|yz)$ are $\leq k$. We also write $(xy:u:zw)$ to mean $(xy:zw)$, $(yu:zw)$, $(xu:zw)$, $(xy:uw)$, $(xy:uz)$.

A hyperbolic crossratio is a *path crossratio* if for any distinct $x,y,z,w \in M$ and any $p \leq (xy|zw)$ there is $u \in M$ with $(xy:u:zw)$ and $(xy|zu) \simeq p$, where we write $\simeq$ for $\simeq_k$ when k is understood.

Proposition 2.1. *Suppose that the annulus system $\mathcal{A}$ satisfies* (A1) *and* (A2). *Then the crossratio $(xy|zw)$ defined by counting annuli is a hyperbolic path crossratio.*

Proof. This is [Bow98], Proposition 6.5. Note that Bowditch assumes that M is compact, but in fact he does not use this assumption in the proof. □

2.3. Hyperbolic path quasi-metric. Let Q be the set of ordered triples of distinct points in M. We assume that we are given hyperbolic path crossratio on M.

If $A = (a_1,a_2,a_3) \in Q$ and $B = (b_1,b_2,b_3) \in Q$ define

$$\rho(A,B) = \max(a_i a_j | b_k b_l)$$

over $i \neq j$ and $k \neq l$.

The intuition is that one can embed the 6 points a_i, b_j into a metric tree so that the crossratios get distorted a bounded amount. Then $\rho(A,B)$ (up to a bounded number) is the distance between the centers of the tripods spanned by a_i and by b_j, respectively.

Proposition 2.2. *(Q, ρ) is a hyperbolic path quasi-metric space. This means that for some $k \geq 0$,*

- *(quasi-metric) $\rho(A, C) \leq \rho(A, B) + \rho(B, C) + k$,*
- *(hyperbolic) the 4-point definition of k-hyperbolicity holds (via Gromov products),*
- *(path) Any two points A, B can be connected by a finite sequence $A = z_0, z_1, \dots, z_N = B$ so that $\rho(z_i, z_j) \simeq_k |i - j|$.*

Proof. Proposition 4.2 and Lemma 4.3 of [Bow98]. □

Proposition 2.3. *If (Q, ρ) is a hyperbolic path quasi-metric, for large $r > 0$ define the graph $G_r(Q)$ whose vertices are the points of Q and two vertices A, B are connected by an edge if $\rho(A, B) \leq r$. Then $G_r(Q)$ is a connected δ-hyperbolic graph quasi-isometric to (Q, ρ).*

Proof. See the discussion before Lemma 3.1 in [Bow98]. □

We shall refer to the graph $\mathcal{X} = G_r(Q)$ as the *Bowditch complex*. Note that the set of vertices of $\mathcal{X}$ is equipped with the edge-path metric d as well as with the quasi-metric ρ.

3. A hyperbolic complex for *MCG*

Let *MCG* denote the mapping class group of a fixed compact connected surface. The standard reference for the material in this section is [FLP91]. To motivate some of the arguments in the Main Theorem, we start by discussing the (somewhat simpler) version for *MCG*.

Theorem 3.1. *For any finite collection $f_1, \dots, f_k$ of pseudo-Anosov mapping classes there is connected δ-hyperbolic graph $\mathcal{X}$ and an action of MCG such that*

- *the stabilizer of a simple closed curve has bounded orbits, and*
- *$f_1, \dots, f_k$ have nonzero translation lengths.*

We will call $\mathcal{X}$ the *Bowditch complex for MCG*.

3.1. Verifying (A1) and (A2). For now the space M is Thurston's boundary $\mathcal{PML}$, i.e., the space of projective measured laminations; it will be made smaller later. Let $\Lambda_i^{\pm}$ be the stable and unstable laminations for f_i and choose small neighborhoods $D_i^{\pm}$ of $\Lambda_i^{\pm}$ forming an annulus A_i. The annulus system consists of the translates of

$\pm A_i$ for $i = 1, 2, \ldots, k$. By $I(-,-)$ denote the intersection number and by $L(\cdot)$ the length with respect to a fixed hyperbolic structure.

In this section we verify (A1) and (A2) (after passing to a smaller M). To simplify notation, we assume $k = 1$ and drop subscripts.

Lemma 3.2. *If a and b are simple closed curves then $(a|b) < \infty$.*

Proof. Suppose $g \in MCG$ such that $a < g(A) < b$. Then $g^{-1}(a) \in D^-$ and $g^{-1}(b) \in D^+$. The expression

$$\frac{I(g^{-1}(a), g^{-1}(b))}{L(g^{-1}(a))L(g^{-1}(b))}$$

does not change if we scale $g^{-1}(a)$ or $g^{-1}(b)$. It follows, by the continuity of I and L, that this expression is close to

$$\mu = \frac{I(\Lambda^+, \Lambda^-)}{L(\Lambda^+)L(\Lambda^-)} > 0,$$

and in particular it is bounded away from 0. But $I(g^{-1}(a), g^{-1}(b)) = I(a, b)$ is fixed, so it follows that both $L(g^{-1}(a))$ and $L(g^{-1}(b))$ are uniformly bounded. Since a and b fill, there are only finitely many such g. □

Remark 3.3. Note that when a, b are disjoint simple closed curves then $(a|b) = 0$. This is because $u \in D^-$, $v \in D^+$ implies $I(u, v) > 0$.

Lemma 3.4. *Let a_1, a_2, b_1, b_2 be laminations in $\mathcal{PML}$ with $I(a_1, a_2) > 0$ and $I(b_1, b_2) > 0$. Then $(a_1 a_2 | b_1 b_2) < \infty$.*

Proof. Consider some $g \in MCG$ with $a_i < g(A) < b_j$, $i, j = 1, 2$. Then $g^{-1}(a_i) \in D^-$ and $g^{-1}(b_j) \in D^+$. As above we have

$$I(g^{-1}(a_i), g^{-1}(b_j)) \geq K\ L(g^{-1}(a_i))L(g^{-1}(b_j)).$$

If there are infinitely many such g then one of the following cases occurs:

Case 1. $L(g^{-1}(a_i))$ and $L(g^{-1}(b_j))$ are bounded above for some choice $i, j \in \{1, 2\}$ (and a subsequence of the g's). Choose a curve c and note that both intersection numbers $I(c, g^{-1}(a_i))$ and $I(c, g^{-1}(b_j))$ are bounded, i.e., $I(g(c), a_i)$ and $I(g(c), b_j)$ are both bounded (note that $I(\Lambda, \Lambda') \leq C\ L(\Lambda)L(\Lambda')$ for any two laminations, where C is a constant that depends only on the underlying hyperbolic surface). Since a_i and b_j fill, this implies that $L(g(c))$ is bounded. Since this is true for any c it follows that there are only finitely many g's, contradiction.

Case 2. Either $L(g^{-1}(a_i)) \to 0$ for both $i = 1, 2$ or $L(g^{-1}(b_j)) \to 0$ for both $j = 1, 2$ (over a subsequence of g's). Say the former. Then $I(g^{-1}(a_1), g^{-1}(a_2)) \to 0$, i.e., $I(a_1, a_2) = 0$, contradiction. □

There is also a hybrid situation:

Lemma 3.5. *If a is a curve and b_1, b_2 are laminations with $I(b_1, b_2) > 0$ then $(a|b_1 b_2) < \infty$.*

Proof. Similar to the other two lemmas. We have

$$I(g^{-1}(a), g^{-1}(b_i)) \geq K\, L(g^{-1}(a))L(g^{-1}(b_i))$$

for $j = 1, 2$. There are now two cases.

Case 1. $L(g^{-1}(a))$ stays bounded. Then both $L(g^{-1}(b_i))$, $i = 1, 2$, are bounded as well by the above inequality Since a and b_1 fill, this restricts g to a finite set, as in Case 1 of Lemma 3.4.

Case 2. $L(g^{-1}(a)) \to \infty$. Then $L(g^{-1}(b_i)) \to 0$, $i = 1, 2$, and hence $I(b_1, b_2) = 0$ as in Case 2 of Lemma 3.4. □

Lemma 3.6. *If $D^{\pm}$ are chosen to be small enough then for any $a, b, c, d \in \mathcal{PML}$ we have $(ac|bd) = 0$ or $(ad|bc) = 0$.*

Proof. Scale each lamination in $\mathcal{PML}$ so that its length is 1 (with respect to a fixed hyperbolic metric). Choose $D^{\pm}$ so that when $x, y \in D^+$ (or D^-) then $I(x, y) < \varepsilon$ and if $x \in D^+$ and $y \in D^-$ then $|I(x, y) - I(\Lambda^+, \Lambda^-)| < \varepsilon$. This is possible by the continuity of the intersection number. Note that we could also write e.g.

$$\frac{I(x, y)}{L(x)L(y)} < \varepsilon$$

for the first inequality and this is invariant under scaling x and y.

Now assume $(ac|bd) > 0$ and $(ad|bc) > 0$. Then for some $g_1, g_2 \in MCG$ we have $a_1, c_1, a_2, d_2 \in D^-$, $b_1, d_1, b_2, c_2 \in D^+$, where $a_i = g_i(a)$ etc. Thus we have

$$\frac{I(a_1, c_1)}{L(a_1)L(c_1)} < \varepsilon$$

and

$$\frac{I(a_2, c_2)}{L(a_2)L(c_2)} \sim \mu$$

where $\mu = \frac{I(\Lambda^+, \Lambda^-)}{L(\Lambda^+)L(\Lambda^-)}$. Dividing the two inequalities and noting that $I(a_1, c_1) = I(a_2, c_2)$ gives

$$\frac{L(a_2)L(c_2)}{L(a_1)L(c_1)} \lesssim \varepsilon/\mu.$$

Similarly we have

$$\frac{L(b_2)L(d_2)}{L(b_1)L(d_1)} \lesssim \varepsilon/\mu, \quad \frac{L(a_1)L(d_1)}{L(a_2)L(d_2)} \lesssim \varepsilon/\mu, \quad \frac{L(b_1)L(c_1)}{L(b_2)L(c_2)} \lesssim \varepsilon/\mu.$$

Multiplying gives the contradiction $1 \lesssim \varepsilon^4/\mu^4$ (for small ε). □

Proof of Theorem 3.1. Let $M \subset \mathcal{PML}$ be the subset consisting of stable laminations Λ_g^+ as g varies over all pseudo-Anosov homeomorphisms in MCG. The annulus system will be the restriction to M of the annulus system considered above. Since distinct elements of M have nonzero intersection number, Lemma 3.4 satisfies (A1), and Lemma 3.6 satisfies (A2). The resulting Bowditch complex $\mathcal{X}$ is hyperbolic. The statements about orbits and translation lengths are verified in the next section. □

3.2. Orbits in $\mathcal{X}$

Proposition 3.7. *The stabilizer in MCG of a simple closed curve has bounded orbits. The original pseudo-Anosov homeomorphisms f_i have nonzero translation lengths.*

Proof. By construction, $(\Lambda_f^+|\Lambda_f^-) > 0$. Then by pumping (i.e., using north-south dynamics) $(\Lambda_f^+|\Lambda_f^-) = \infty$ and in fact

$$d((a,b,c),(f^m(a), f^m(b), f^m(c)))$$

grows linearly. This proves that f has nonzero translation length.

Now consider the stabilizer S_a of a curve a. Fix a triple $(p_1, p_2, p_3) \in \mathcal{X}$. By Lemma 3.5 we know that $N = \max_{i \neq j}(a|p_i p_j) < \infty$. If $g \in S_a$, consider a collection of D disjoint annuli separating p_i, p_j from $g(p_u)$, $g(p_v)$ for some $i \neq j$ and $u \neq v$. At most one of these contains a ($B = (B^-, B^+)$ *contains* a if $a \in M - (B^- \cup B^+)$); remove it. Thus at least $\frac{D-1}{2}$ separate a from p_i, p_j or from $g(p_u), g(p_v)$ and we deduce $N \geq (a|p_i p_j) \geq \frac{D-1}{2}$ or $(a|g(p_u)g(p_v)) \geq \frac{D-1}{2}$. But $(a|g(p_u)g(p_v)) = (a|p_u p_v) \leq N$ since $g(a) = a$, so in any case $D \leq 2N + 1$. □

Remark 3.8. This argument shows that the orbit of $A = (p_1, p_2, p_3)$ under S_a has (ρ-)diameter at most $2\max_{i \neq j}(a|p_i p_j) + 1$. Note also that $\max_{i \neq j}(a|p_i p_j)$ is a lower bound for the diameter of the orbit, by considering iterations by the Dehn twist in a and using the fact that the iterates of p_1, p_2, p_3 converge to a.

Remark 3.9. In the above proof we used the triangle type inequality

$$(A|B) \leq (A|x) + (x|B) + 1$$

where $x \in M$ and $A, B \subset M$. This is [Bow98], Lemma 6.1.

3.3. Comparing the Bowditch complex $\mathcal{X}$ and the curve complex $\mathcal{C}$. Recall that if a group acts isometrically on a δ-hyperbolic geodesic space with bounded orbits, then there is an orbit of diameter $\leq 8\delta$. Thus there are some $K_\rho > 0$ and $K_d > 0$ such that the stabilizer $S_a \subset MCG$ of a has an orbit of vertices in $\mathcal{X}$ of ρ-diameter $\leq K_\rho$ and d-diameter $\leq K_d$, for any curve a (e.g., K_d can be taken to be $8\delta + 1$ if $\mathcal{X}$ is δ-hyperbolic with respect to d).

Define $\Phi\colon \mathcal{C} \to \mathcal{X}$ by the rule that $\Phi(a)$ is a triple (p, q, r) that belongs to such an orbit.

Lemma 3.10. *Φ is coarsely well defined and it is Lipschitz.*

Proof. We need to check that different choices for $\Phi(a)$ are close, but this is a special case of the Lipschitz condition. Suppose that a, b are disjoint curves and let $\Phi(a) = (p_1, p_2, p_3)$, $\Phi(b) = (q_1, q_2, q_3)$. Then $\max_{i \neq j}(a|p_i p_j) \leq K_\rho$ and $\max_{u \neq v}(b|q_u q_v) \leq K_\rho$ and since $(a|b) = 0$ (see Remark 3.3) we have

$$\max_{i \neq j}(p_i p_j|b) \leq K_\rho + 1$$

by the triangle inequality. Thus

$$\begin{aligned}\rho(p_1 p_2 p_3, q_1 q_2 q_3) &= \max_{i \neq j, u \neq v}(p_i p_j|q_u q_v) \\ &\leq \max_{i \neq j}(p_i p_j|b) + \max_{u \neq v}(b|q_u q_v) + 1 \leq 2K_\rho + 2.\end{aligned}$$

□

Lemma 3.11. *If $D^\pm$ are chosen small enough then Φ is coarsely onto.*

Proof. Let (p_1, p_2, p_3) be a triple in M. Scale them so that all 3 intersection numbers are equal, say to 1. Then by Bowditch's lemma (see Lemmas 5.1 and 5.3 of [Bow06], or for another exposition see [BF07], Lemma 5.7) there is a curve a such that $I(a, p_i) \leq R$, $i = 1, 2$ for a constant R that depends only on the surface, and moreover, $I(a, z) \leq R(I(p_1, z) + I(p_2, z))$ for all laminations z (this result is stated in the above references only for multicurves, but it extends easily to laminations). By putting $z = p_3$ we see that $I(a, p_3) \leq 2R$. (By Bowditch's proof of hyperbolicity of $\mathcal{C}$, a is near the center of the ideal triangle in the curve complex with vertices at infinity corresponding to p_1, p_2, p_3.)

We now claim that $(a|p_i p_j) = 0$ for $i \neq j$ if $D^\pm$ are chosen small. Thus $\Phi(a)$ is close to (p_1, p_2, p_3).

Suppose that $(a|p_i p_j) > 0$. Then there is $g \in MCG$ so that $\tilde{a} = g^{-1}(a) \in D^-$ and $\tilde{p}_i = g^{-1}(p_i) \in D^+$, $\tilde{p}_j = g^{-1}(p_j) \in D^+$. As in Lemma 3.6 we have

$$\frac{I(\tilde{a}, \tilde{p}_i)}{L(\tilde{a})L(\tilde{p}_i)} \sim \mu \quad \frac{I(\tilde{a}, \tilde{p}_j)}{L(\tilde{a})L(\tilde{p}_j)} \sim \mu \tag{1}$$

and

$$\frac{I(\tilde{p}_i, \tilde{p}_j)}{L(\tilde{p}_i)L(\tilde{p}_j)} < \varepsilon. \tag{2}$$

Dividing (2) by each of the equations in (1) and taking into account that $I(\tilde{p}_i, \tilde{p}_j) = I(p_i, p_j) = 1$ and $I(\tilde{a}, \tilde{p}_i) = I(a, p_i) \leq 2R$, $I(\tilde{a}, \tilde{p}_j) \leq 2R$ gives

$$L(\tilde{a}) \stackrel{\sim}{<} 2R\sqrt{\varepsilon}/\mu,$$

which is a contradiction for small ε (we are on a fixed hyperbolic surface). □

Remark 3.12. Suppose that f and g are two pseudo-Anosov elements of *MCG* such that no nontrivial power of f is conjugate to a power of g. Then there are neighborhoods $U^\pm$ of $\Lambda_f^\pm$ such that there is no $h \in MCG$ with $h(\Lambda_g^\pm) \in U^\pm$. There are two proofs of this claim, both modelled on the proof of a similar assertion for two hyperbolic elements in a discrete subgroup of SO$(n, 1)$. Indeed, the existence of h forces the geodesics associated to f and g in the orbit space to be very close, which is impossible since they are distinct closed geodesics (as sets). There are two variants of this argument for *MCG*, one using the Teichmüller metric and the other the Weil–Petersson metric. If $\mathcal{F}_f^\pm$ are the stable and unstable measured foliations associated with f, then the Teichmüller axis A_f of f consists of conformal structures obtained from Euclidean metrics with singularities of the form $ds^2 = e^{2t}d\mu_+^2 + e^{-2t}d\mu_-^2$ where $\mu_\pm$ are the measures on $\mathcal{F}_f^\pm$ and $t \in \mathbb{R}$. If $\mathcal{F}^\pm_{h_i g h_i^{-1}} \to \mathcal{F}_f^\pm$ as measured foliations then clearly $A_{h_i g h_i^{-1}} \to A_f$ uniformly on compact sets and we have a contradiction as before. The Weil–Petersson version uses the fact that pseudo-Anosov elements of *MCG* have unique geodesic axes plus a theorem of Brock–Masur–Minsky [BMM], Corollary 1.6, which says that axes are close when the associated stable and unstable laminations are close in $\mathcal{PML}$.

It follows that whenever we are given $g_1, g_2, \ldots, g_m \in MCG$ such that nontrivial powers of g_j are not conjugate to powers of f_i's then by choosing $D_i^\pm$'s sufficiently small we can arrange that the g_j's have bounded orbits. This is because $(\Lambda_{g_j}^+ | \Lambda_{g_j}^-) = 0$ by construction, so if we take any $\Lambda \in \mathcal{PML} - \{\Lambda_{g_j}^\pm\}$ and set $x = (\Lambda_{g_j}^+, \Lambda_{g_j}^-, \Lambda)$ then $\rho(x, g_j^N(x)) = 0$ for any N.

Now suppose that $\mathcal{X}_1$, $\mathcal{X}_2$, … is a sequence of Bowditch complexes obtained by taking smaller and smaller neighborhoods $D_i^\pm$. All $\mathcal{X}_i$ have the same vertex sets, but, for $i < j$, $\mathcal{X}_j$ in general has more edges than $\mathcal{X}_i$, so we have natural maps $\mathcal{X}_1 \to \mathcal{X}_2 \to \cdots$. One may wonder whether eventually this sequence consists of quasi-isometries (all maps are clearly coarsely onto). The answer to this question is negative. To see this, choose some pseudo-Anosov homeomorphism g whose stable and unstable laminations are very close but not equal to those of f_1. This is possible by the work of Farb and Mosher on Schottky subgroups of *MCG* [FM02]. Furthermore, one can arrange that nontrivial powers of g are not conjugate to powers of f_i's (that's automatic once the (un)stable laminations of g are sufficiently close to those of f_1). It then follows that, for small i, g has positive translation length in $\mathcal{X}_i$ and for large i its orbits are bounded.

3.4. WPD.

For the construction of quasi-homomorphisms on groups acting isometrically on hyperbolic complexes it is important to have Weak Proper Discontinuity of the action [BF02].

Proposition 3.13. *The elements $f_1, \ldots, f_k$ chosen at the start of the construction satisfy WPD: For every $i = 1, 2, \ldots, k$, every $x \in \mathcal{X}$, and every $C > 0$ there is*

$N > 0$ such that

$$\{g \in MCG \mid d(x, g(x)) \le C, d(f_i^N(x), gf_i^N(x)) \le C\}$$

is finite.

We will omit the proof since it is easier than the corresponding statement for $\mathrm{Out}(F_n)$, which we prove in Section 4.5.

4. A hyperbolic complex for $\mathrm{Out}(F_n)$

Recall that $f \in \mathrm{Out}(F_n)$ is *fully irreducible* if for all proper free factors F of F_n and all $k > 0$ we have that $f^k(F)$ is not conjugate to F. For convenience, we restate our main result.

Main Theorem. *For any finite collection $f_1, \dots, f_k$ of fully irreducible elements of* $\mathrm{Out}(F_n)$ *there is a connected δ-hyperbolic graph $\mathcal{X}$ equipped with an* (*isometric*) *action of* $\mathrm{Out}(F_n)$ *such that*

- *the stabilizer in* $\mathrm{Out}(F_n)$ *of a simplicial tree in $\overline{\mathcal{PT}}$ has bounded orbits,*
- *the stabilizer in* $\mathrm{Out}(F_n)$ *of a proper free factor $F \subset F_n$ has bounded orbits, and*
- *$f_1, \dots, f_k$ have nonzero translation lengths.*

We will start with some preliminaries. By $\mathcal{T} = \mathcal{T}_n$ denote the space of free cocompact simplicial metric F_n-trees without vertices of valence 1 and 2. If γ is a conjugacy class in F_n and $T \in \mathcal{T}$, denote by $\langle T, \gamma \rangle$ the translation length of γ in T. The group $\mathrm{Out}(F_n)$ acts on the right on $\mathcal{T}$ by the "change of marking", i.e., by the rule that $\langle Tg, \gamma \rangle = \langle T, g(\gamma) \rangle$. The group $\mathbb{R}^+$ acts on $\mathcal{T}$ by scaling and this action commutes with the action of $\mathrm{Out}(F_n)$. The projectivized space $\mathcal{PT} = \mathcal{PT}_n = \mathcal{T}/\mathbb{R}^+$ is Culler–Vogtmann's Outer space [CV86]. By $\overline{\mathcal{T}}$ denote the closure of $\mathcal{T}$ in the space of minimal F_n-trees. Both $\mathrm{Out}(F_n)$ and $\mathbb{R}^+$ continue to act on $\overline{\mathcal{T}}$; let $\overline{\mathcal{PT}}$ be the projectivization of $\overline{\mathcal{T}}$. This is Culler–Morgan's equivariant compactification of Outer space [CM87].

To every fully irreducible outer automorphism f one associates the stable tree T_f^+ and the unstable tree T_f^-. In $\overline{\mathcal{PT}}$ they are defined as limits $T_f^+ = \lim_{k\to\infty} T_0 f^k$ and $T_f^- = \lim_{k\to\infty} T_0 f^{-k}$ for any tree T_0 in Outer space. In $\overline{\mathcal{T}}$ they are defined only up to scale, but in a similar way after choosing the right scaling factors. More precisely,

$$T_f^+ = \lim_{k\to\infty} T_0 f^k/\lambda^k \quad \text{and} \quad T_f^- = \lim_{k\to\infty} T_0 f^{-k}/\mu^k,$$

where λ is the growth rate of f and μ the growth rate of f^{-1} (see below). These trees satisfy $T_f^+ f = \lambda T_f^+$ and $T_f^- f = T_f^-/\mu$. The following important fact was proved by Levitt and Lustig [LL03].

Proposition 4.1. *The fully irreducible automorphism f acts on $\overline{\mathcal{PT}}$ with north-south dynamics: $T_f^{\pm}$ are the only fixed points, and any compact set that does not contain T_f^- (resp. T_f^+) converges uniformly under iteration by f (resp. f^{-1}) to T_f^+ (resp. T_f^-).*

For convenience, we will say that a tree T is an *irreducible tree* if $T = T_f^+$ for some fully irreducible automorphism f.

4.1. Some train track facts. Recall that a fully irreducible automorphism is *geometric* if it is induced by a homeomorphism of a compact surface with (necessarily connected) boundary; otherwise it is *non-geometric*. A fully irreducible automorphism is geometric if and only if it has a nontrivial periodic conjugacy class (which is necessarily either fixed or sent to its inverse) [BH92]. A fully irreducible automorphism is non-geometric if and only if the associated stable tree is free (i.e., every nontrivial element has nonzero translation length).

In this section we generalize some of the lemmas from [BFH97]. In that paper we proved, for example, that the action of F_n on the product $T_f^+ \times T_f^-$ of the stable and the unstable tree of a fully irreducible automorphism is discrete. The case of a geometric f is classical, and we focused our attention on non-geometric f. Here we are interested in the action of F_n on the product $T_1 \times T_2$ of two irreducible trees, associated with two possibly unrelated automorphisms. The proofs in this more general setting are only slight variations of the original.

Recall that a map $\rho\colon H \to H$ on a finite graph without valence 1 vertices is a *train track map* if it sends vertices to vertices and for every $i > 0$ the map ρ^i restricted to any edge is locally injective. Such a map is a *topological representative* of some $f \in \mathrm{Out}(F_n)$ if after a suitable identification (called *marking*) $\pi_1(H) \cong F_n$ the map ρ induces f in π_1. Every fully irreducible automorphism f admits a train track representative ρ [BH92]. Up to scale, there is a unique assignment of lengths to the edges of H and a constant λ (the *growth rate* of f) so that for every edge e we have $\mathrm{length}(\rho(e)) = \lambda\, \mathrm{length}(e)$.

Replace ρ by a power if necessary so that there is a fixed point x in the interior of some edge. Let I be an ε-neighborhood around x so that $\rho(I) \supset I$ (and the orientation is preserved). Choose an isometry $\ell\colon (-\varepsilon, \varepsilon) \to I$ and extend it uniquely to a locally isometric immersion $\ell\colon \mathbb{R} \to H$ such that $\ell(\lambda^N t) = \rho^N(\ell(t))$. A *stable leaf segment* is the restriction of ℓ to a finite segment (possibly reparameterized). The collection of stable leaf segments does not depend on the choice of x and I. One can talk about stable leaf segments with respect to a different graph H' representing F_n: if $\tau\colon H \to H'$ is a given homotopy equivalence, let $[\tau\ell]$ be the induced line in H' pulled tight, and then consider finite subsegments of this line. The collection of these segments does not depend on the choice of the train track representative $\rho\colon H \to H$.

Likewise, *unstable leaf segments* are stable leaf segments for f^{-1}.

An edge-path p in H is *legal* if $\rho^i p$ is locally injective for all $i = 0, 1, \ldots$. For example, edges and stable leaf segments (those that are also edge-paths) are legal.

If an immersed edge-path has the form $p = u \cdot v \cdot w$ where v is a legal segment, then $\rho(p) = \rho(u)\rho(v)\rho(w)$. Now if v is sufficiently long, say $|v| > C$, then after canceling against $\rho(u)$ and $\rho(w)$ what is left of $\rho(v)$ will still be longer than v. Such a constant C is called a *critical constant*.

Let $\rho\colon H \to H$ be a train track map representing a fully irreducible automorphism f. An immersed line ℓ in H will be called *bad* if the legal segments of tightened iterates $[\rho^i(\ell)]$ have uniformly bounded size for $i = 0, 1, 2, 3, \dots$. The uniform bound can be taken to be a critical constant.

Examples of bad lines are: unstable leaves, $\dots\gamma\gamma\gamma\dots$ where γ is periodic, as well as concatenations of unstable half-lines with fixed endpoints (possibly with powers of γ inserted between them).

Lemma 4.2. *A bad line ℓ either contains arbitrarily long unstable leaf segments or for every K there is N such that $\rho^N(\ell)$ contains a segment of length $\geq K$ representing a periodic conjugacy class.*

Proof. Fix $C > 0$. Let $\rho'\colon H' \to H'$ be a train track representative for f^{-1} and let $\tau\colon H \to H'$ be the difference of markings. If f is non-geometric we may apply [BFH97], Lemma 2.10. It says that for any $C' > 0$ there is N_0 such that for any immersed line ℓ' in H, either $\rho^{N_0}(\ell')$ contains a stable leaf segment of length $> C'$ or $\rho'^{N_0}\tau\ell'$ contains a stable (for f^{-1}) leaf segment of length $> C'$. Apply this lemma to the line $\ell' = \rho^{N_0}(\ell)$ to conclude that when $\tau\ell$ is transferred to H' it contains a legal segment of length $> C'$. When C' is big enough, this means that ℓ contains a long unstable leaf segment.

The proof of Lemma 2.10 in [BFH97] holds also for geometric f provided that there is a uniform bound on the length of a Nielsen segment in all iterates $[\rho^i(\ell)]$, $i = 0, 1, 2, \dots$. If there is no such bound, the statement trivially holds. $\square$

Corollary 4.3. *Let ℓ be a bad line and suppose that there is a uniform bound on the length of periodic segments in the iterates of ℓ. Then for every A there is B such that every segment in ℓ of length B contains an unstable leaf segment of length A.*

Proof. If this is false, then there is a sequence of longer and longer segments in ℓ that don't contain unstable leaf segments of length A. Passing to a subsequence and taking a limit produces a line that violates Lemma 4.2. $\square$

Lemma 4.4. *Let $\rho\colon H \to H$ and $\rho'\colon H' \to H'$ be train track representatives of two fully irreducible automorphisms f and f' and let $\tau\colon H \to H'$ be a homotopy equivalence representing the difference of markings. Assume $\tau\rho^k \not\simeq \rho'^l\tau$ for all $k, l > 0$ (equivalently, the (un)stable trees for f, f' are distinct).*

Then for every $C > 0$ there are $N_0, L_0 > 0$ such that if ι is an immersion of a line, a circle of length $\geq L_0$, or a closed interval of length $\geq L_0$ and ι' is obtained from $\tau\iota$ by pulling tight, then either

(A) $[\rho^{N_0}\iota]$ *contains a legal segment of length* $> C$, *or*
(B) $[\rho'^{N_0}\iota']$ *contains a legal segment of length* $> C$, *or*
(C) $[\rho^N\iota]$ *contains a* ρ*-periodic segment of length* $> C$ *for some* $N > 0$, *or*
(D) $[\rho'^N\iota']$ *contains a* ρ'*-periodic segment of length* $> C$ *for some* $N > 0$.

Proof. It suffices to prove the statement for lines; the other cases follow by taking limits.

Suppose the statement is false. Then we have a sequence of immersed lines that fail (A)–(D) with $N_0 \to \infty$. Denote by ι a limiting line of this sequence. Then ι is a bad line with respect to both ρ and ρ' and satisfies Corollary 4.3 for both. We conclude that long unstable leaf segments for ρ contain long unstable leaf segments for ρ' and vice versa. Then [BFH97], Theorem 2.14, implies that f and f' have common positive powers, contradiction. (In the language of [BFH97] we have shown that f and f' have the same unstable laminations.) $\square$

Recall that the *legality* of an immersed loop α in H is the ratio

$$\mathrm{LEG}_H(\alpha) = \frac{\text{sum of the lengths of maximal legal leaf segments of} \alpha \text{ of length} \geq C}{\mathrm{length}(\alpha)}$$

where $C > 0$ is a sufficiently big constant (for example, bigger than a critical constant).

The next lemma is a variation of [BFH97], Lemma 5.5, and we omit the proof.

Lemma 4.5. (1) *For every* $\varepsilon > 0$ *and* $A > 0$ *there is* $N_3 = N_3(\varepsilon, A)$ *such that if* $\mathrm{LEG}_H(\alpha) \geq \varepsilon$ *then*

$$\mathrm{length}[\rho^N(\alpha)] \geq A\,\mathrm{length}(\alpha)$$

for all $N \geq N_3$.

(2) *For every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* $\mathrm{LEG}_H(\alpha) \geq \varepsilon$ *then* $\langle T_f^+, \alpha\rangle \geq \delta\ \mathrm{length}_H(\alpha)$.

(3) *For every* $\varepsilon > 0$ *and every* $L > 0$ *there is* $N_4 = N_4(\varepsilon, L) > 0$ *such that if* $\mathrm{LEG}_H(\alpha) \geq \varepsilon$ *then for all* $N \geq N_4$ *the set of points of* $[\rho^N(\alpha)]$ *whose* L*-neighborhood is a stable leaf segment has total length* $\geq (1-\varepsilon)\,\mathrm{length}[\rho^N(\alpha)]$.

The following generalizes [BFH97], Lemma 5.6. We say that a conjugacy class α is *primitive* if any of its elements can be extended to a basis of F_n.

Lemma 4.6. *Let* f, f' *be fully irreducible automorphisms with* $T_f^+ \neq T_{f'}^+$ *(projectively), and therefore* $T_f^- \neq T_{f'}^-$. *Let* $\rho\colon H \to H$, $\rho'\colon H' \to H'$ *and* N_0 *be as in Lemma* 4.4 *and let* α *be a conjugacy class. Assume either that* f, f' *are nongeometric or that* α *is a primitive conjugacy class. Then there is* $\varepsilon > 0$ *such that for every* $N \geq N_0$ *either* $\mathrm{LEG}_H(\rho^N(\alpha)_H) \geq \varepsilon$ *or* $\mathrm{LEG}_{H'}(\rho'^N(\alpha)_{H'}) \geq \varepsilon$.

Proof. We first argue that (C) and (D) of Lemma 4.4 cannot occur when applied to α. This is clear if f and f' are non-geometric, so assume that α is primitive. We now use an argument of Yael Algom-Kfir [AK]. The loop $\rho^N(\alpha)$ also represents a primitive element, while the loop representing the indivisible fixed class γ crosses every edge of H twice. This is true after collapsing a maximal forest in H as well, and the Whitehead graph of γ in the resulting rose is a circle that passes through every vertex. It follows that any loop that contains two consecutive copies of γ will have Whitehead graph that contains this circle, and hence it does not have a cut point. But it is a classical theorem of Whitehead [Whi36] that the Whitehead graph of a primitive class is either disconnected or contains a cut vertex. Thus $[f^N(\alpha)]$ cannot contain two consecutive copies of γ. This finishes the proof that (C) and (D) cannot occur.

The rest of the argument is identical to the proof of Lemma 5.6 in [BFH97], using Lemma 4.4 in place of Lemma 2.10 of [BFH97]. □

By $|\alpha|$ denote the length of the conjugacy class α with respect to a fixed graph.

Corollary 4.7. *Let f, g be fully irreducible automorphisms and assume that $T_f^+ \neq T_g^+$ (equivalently, $T_f^- \neq T_g^-$). There is $\delta > 0$ such that for all primitive conjugacy classes α we have either $\langle T_f^+, \alpha\rangle \geq \delta|\alpha|$ or $\langle T_g^+, \alpha\rangle \geq \delta|\alpha|$. In particular, for any $C > 0$ there are only finitely many primitive conjugacy classes α with both $\langle T_f^+, \alpha\rangle < C$ and $\langle T_g^+, \alpha\rangle < C$.*

If f and g are non-geometric then these statements hold for all conjugacy classes.

Proof. This follows from Lemma 4.5(2) and Lemma 4.6. □

Remark 4.8. Note that there is $M > 0$ such that $\langle T_1, \alpha\rangle + \langle T_2, \alpha\rangle \leq M|\alpha|$ for any conjugacy class α. This is because there is an equivariant Lipschitz map $T \to T_i$, $i = 1, 2$, from any tree T in Outer space. Therefore,

$$\langle T_f, \alpha\rangle + \langle T_g, \alpha\rangle \sim |\alpha|$$

for primitive α (or all α if f, g are non-geometric) in the sense that the ratio is bounded away from 0 and ∞.

If a sequence T_i in $\overline{\mathcal{T}}$ converges projectively then we say that a sequence $\lambda_i > 0$ is a *scaling sequence for* T_i if T_i/λ_i converges in $\overline{\mathcal{T}_i}$ (without further scaling). If one scaling sequence for T_i converges to infinity then they all do. For a sequence of distinct g_i, one expects a scaling sequence for Tg_i to converge to infinity, but note that $\mu^i T_f^- f^i = T_f^-$, so the scaling sequence is $1/\mu^i$. The following proposition is a weak converse to this.

Proposition 4.9. *Assume that $p_0 \neq q_0$ are irreducible trees, $p_i = p_0 \cdot g_i$, $q_i = q_0 \cdot g_i$ for a sequence of distinct $g_i \in \mathrm{Out}(F_n)$. Also assume that p_i and q_i converge projectively. Then a scaling sequence for either p_i or q_i converges to infinity.*

Proof. Suppose $p_i/\lambda_i \to p$ and $q_i/\mu_i \to q$. Let α be a primitive conjugacy class in F_n. Therefore,

$$\langle p_0 g_i/\lambda_i, \alpha\rangle \to \langle p, \alpha\rangle$$

and hence

$$\langle p_0, g_i(\alpha)\rangle \lesssim \lambda_i,$$

and similarly $\langle q_0, g_i(\alpha)\rangle \lesssim \mu_i$.

Now suppose that both λ_i and μ_i are bounded. By Corollary 4.7 there are only finitely many possibilities for $g_i(\alpha)$. Now apply this to the primitive conjugacy classes of elements in F_n of word length ≤ 2. Since an automorphism that fixes these conjugacy classes is necessarily inner (a standard fact), it follows that there are only finitely many choices for g_i, contradiction. □

4.2. Measured geodesic currents. Measured geodesic currents (or just currents in the sequel) were introduced by Francis Bonahon, first on hyperbolic surfaces [Bon88] in order to study the geometry of Teichmüller space, and later in the setting of any word-hyperbolic group [Bon91]. Of interest for this paper is the case of free groups, further studied by Reiner Martin in his thesis [Mar95], and more recently by Ilya Kapovich, Martin Lustig and others (see [KL09] and references therein). Martin's thesis has never been published, but most of his results are available in [Kap06].

Let ∂F_n denote the Cantor set of ends of F_n and let $\partial^2 F_n = (\partial F_n \times \partial F_n - \Delta)/\mathbb{Z}_2$ be the space of unordered pairs of distinct points of ∂F_n (thought of as the space of unoriented bi-infinite geodesics in F_n). By $\mathcal{C}(F_n)$ denote the collection of compact open subsets of $\partial^2 F_n$. A *current* η is an additive function $\mathcal{C}(F_n) \to [0,\infty)$ which is invariant under the (diagonal) action of F_n ("additive" means that $\eta(C_1 \sqcup C_2) = \eta(C_1) + \eta(C_2)$). The space $\mathcal{MC}(F_n)$ of currents has the structure of the cone (positive linear combinations of currents are currents) and a natural topology, as a subset of $[0,\infty)^{\mathcal{C}(F_n)}$. Projectivizing gives a compact space $\mathcal{PMC}(F_n)$ of projectivized (measured geodesic) currents.

For each indivisible conjugacy class γ in F_n we define a current η_γ induced by γ: if $C \in \mathcal{C}(F_n)$ then $\eta_\gamma(C)$ is the number of lifts of γ which are in C. If $\gamma = \beta^k$, with β indivisible and $k > 0$, define $\eta_\gamma = k\eta_\beta$. Thus the set of conjugacy classes in F_n can be viewed as a subset of $\mathcal{MC}(F_n)$ and their image in $\mathcal{PMC}(F_n)$ is dense [Mar95]. The group $\mathrm{Out}(F_n)$ acts on the space of currents via

$$g(\eta)(C) = \eta(g^{-1}(C)).$$

This action extends the action on conjugacy classes in the sense that $g(\eta_\gamma) = \eta_{g(\gamma)}$.

For a fully irreducible automorphism f one can define the *stable current* Υ_f^+ and the *unstable current* Υ_f^-. Projectively they can be defined as $\Upsilon_f^+ = \lim_{k\to\infty} f^k(\eta_\gamma)$ and $\Upsilon_f^- = \lim_{k\to\infty} f^{-k}(\eta_\gamma)$ for any primitive conjugacy class γ in F_n (or indeed any non-periodic conjugacy class). In $\mathcal{MC}(F_n)$ the stable and unstable currents are defined only up to scale,

$$\Upsilon_f^+ = \lim_{k\to\infty} f^k(\eta_\gamma)/\lambda^k \quad \text{and} \quad \Upsilon_f^- = \lim_{k\to\infty} f^{-k}(\eta_\gamma)/\mu^k,$$

where λ and μ are the growth rates of f and f^{-1}.

The following important fact was proved by Martin [Mar95].

Proposition 4.10. *Every non-geometric fully irreducible automorphism f acts on $\mathcal{PMC}(F_n)$ with north-south dynamics: $\Upsilon_f^\pm$ are the only two fixed points and every compact set that does not contain Υ_f^- (resp. Υ_f^+) uniformly converges to Υ_f^+ (resp. Υ_f^-) under iteration by f (resp. f^{-1}).*

Of course, this result is false for geometric automorphisms since the current representing the boundary is fixed as well. However, Martin also observed that the above theorem holds for geometric automorphisms as well, provided that one restricts to a certain closed invariant subset $\mathcal{M}(F_n)$ in $\mathcal{PMC}(F_n)$. This set is defined as the closure of the set of projectivized currents of the form η_γ where γ is a primitive conjugacy class. Thus $\mathcal{M}(F_n)$ contains all currents of the form $\Upsilon_f^\pm$. It is also known that, for $n \geq 3$, $\mathcal{M}(F_n)$ is the unique minimal nonempty closed $\mathrm{Out}(F_n)$-invariant subset of $\mathcal{PMC}(F_n)$ [KL07].

Proposition 4.11. *Every fully irreducible automorphism f acts on $\mathcal{M}(F_n)$ with north-south dynamics: $\Upsilon_f^\pm$ are the only fixed points and every compact subset of $\mathcal{M}(F_n)$ that does not contain Υ_f^- (resp. Υ_f^+) converges uniformly to Υ_f^+ (resp. Υ_f^-) under iteration by f (resp. f^{-1}).*

Proofs of Propositions 4.10 and 4.11. Let $\rho\colon H \to H$ be a train track representative for f and let ℓ be a stable leaf. A typical compact and open set $C \subset \partial^2 F_n$ is determined by a finite edge path in the universal cover $\widetilde{H}$ – it consists of all lines that contain this path. So one can view a current η as assigning a number to such an edge path. Equivariance dictates that translates be assigned the same number, thus η assigns numbers to edge paths in H. Additivity then translates to saying that $\eta(\pi) = \sum \eta(\pi_i)$ as π_i range over all 1-edge extensions of π. The current Υ_f^+ assigns 0 to edge paths that are not crossed by ℓ, and more generally it assigns the frequency of occurrence of this path in ℓ. Lemma 4.5 (3) implies that all conjugacy classes α with $\mathrm{LEG}_H(\alpha) \geq \varepsilon$ converge to Υ_f^+ uniformly. The same statement holds when f is replaced by f^{-1} and H by a train track graph for f^{-1}. Lemma 4.6 now implies that every primitive conjugacy class α (or any non-trivial class if f is non-geometric) either converges uniformly to Υ_f^+ under forward iteration, or to Υ_f^- under backward iteration. Since conjugacy classes are dense in $\mathcal{PMC}(F_n)$ and primitive conjugacy classes are dense in $\mathcal{M}(F_n)$, both propositions follow. □

In [KL09] I. Kapovich and Lustig extended the length pairing between trees and conjugacy classes to trees and currents. More precisely, they proved the following.

Proposition 4.12. *There is a length pairing* $\langle\cdot,\cdot\rangle : \overline{\mathcal{T}} \times \mathcal{MC}(F_n) \to [0,\infty)$ *satisfying:*

- *it extends the usual length pairing, i.e.,* $\langle T, \eta_\gamma\rangle = \langle T, \gamma\rangle$ *for any conjugacy class* γ,
- $\langle Tg, \eta\rangle = \langle T, g(\eta)\rangle$,
- *it is homogeneous in the first coordinate, i.e.,*
$$\langle \lambda T, \eta\rangle = \lambda\langle T, \eta\rangle$$
for $\lambda > 0$,
- *it is linear in the second coordinate, i.e.,*
$$\langle T, \lambda_1\eta_1 + \lambda_2\eta_2\rangle = \lambda_1\langle T, \eta_1\rangle + \lambda_2\langle T, \eta_2\rangle$$
for $\lambda_1, \lambda_2 \geq 0$, *and*
- *it is continuous.*

The following statements are easy consequences of the above.

Corollary 4.13. *Let* f *be any fully irreducible automorphism,* $T \in \overline{\mathcal{T}}$, $\Upsilon \in \mathcal{MC}(F_n)$. *Then*

(1) $\langle T_f^{\pm}, \Upsilon_f^{\mp}\rangle = 0$.

(2) *Assume either that* f *is non-geometric or that* $\Upsilon \in \mathcal{M}(F_n)$. *If* $\langle T_f^{\pm}, \Upsilon\rangle = 0$ *then* $\Upsilon = \Upsilon_f^{\mp}$ *(projectively).*

(3) *If* $\langle T, \Upsilon_f^{\pm}\rangle = 0$ *then* $T = T_f^{\mp}$ *(projectively).*

Proof. (1) $\langle T_f^+, \Upsilon_f^-\rangle = \langle \lim T_0 f^i/\lambda^i, \lim f^{-i}(\eta_\gamma)/\mu^i\rangle = \lim\langle T_0, \eta_\gamma\rangle/(\lambda^i\mu^i) = 0$.

(2) Let γ be a primitive conjugacy class. We start by observing that $\langle T_f^+, \Upsilon_f^+\rangle = \lim\langle T_0 f^i/\lambda^i, f^i(\eta_\gamma)/\lambda^i\rangle = \lim\langle T_0 f^{2i}/\lambda^{2i}, \gamma\rangle = \langle T_f^+, \gamma\rangle > 0$. If we had $\langle T_f^+, \Upsilon\rangle = 0$ for some $\Upsilon \neq \Upsilon_f^-$, then $f^i(\Upsilon)/\lambda_i \to \Upsilon_f^+$ for a suitable scaling sequence λ_i (actually, one can take $\lambda_i = \lambda^i$), and by continuity we would conclude $\langle T_f^+, \Upsilon_f^+\rangle = 0$, contradiction.

(3) The proof is similar to (2). □

When T is an irreducible tree, denote by T^* the dual current, i.e., if $T = T_f^+$ then $T^* = \Upsilon_f^-$. (This is well defined since if $T_f^+ = T_g^+$, then $f^m = g^k$ for some $m, k > 0$ and therefore $\Upsilon_f^{\pm} = \Upsilon_g^{\pm}$.) Thus $\langle T, T^*\rangle = 0$. The current T^* is defined only up to scale. Also note that $(Tf)^* = f^{-1}(T^*)$ and that T^* is the only current in $\mathcal{M}(F_n)$ whose length in T is 0.

Lemma 4.14. *Let T_i, T be irreducible trees. Then $T_i \to T$ if and only if $T_i^* \to T^*$ (projectively).*

Proof. Say $T_i/\lambda_i \to S$ and $T_i^*/\mu_i \to \Upsilon$ (without scaling). Then $\langle S, \Upsilon\rangle = \langle \lim T_i/\lambda_i, \lim T_i^*/\mu_i\rangle = 0$, so if $S = T$ then $\Upsilon = T^*$ (note that $\Upsilon \in \mathcal{M}(F_n)$), and if $\Upsilon = T^*$ then $S = T$. □

Lemma 4.15. *Let T be an irreducible tree. Suppose trees Tg_i converge projectively to a tree $\neq T$. Suppose also that $g_i(T^*)$ converges projectively to a current $\neq T^*$ (or, equivalently by Lemma 4.14, Tg_i^{-1} converges projectively to a tree $\neq T$). Then a scaling sequence for Tg_i is also a scaling sequence for $g_i(T^*)$.*

Proof. Suppose that $Tg_i/\lambda_i \to T'$. We have $\langle T, g_i(T^*)/\lambda_i\rangle = \langle Tg_i/\lambda_i, T^*\rangle \to \langle T', T^*\rangle > 0$, so λ_i is a scaling sequence for $g_i(T^*)$. □

Lemma 4.16. *Suppose $a \neq b$ are two irreducible trees and that ag_i and bg_i converge projectively to a and b respectively. Also assume that $g_i(a^*)$ converges projectively to a current $\neq b^*$ and $g_i(b^*)$ converges projectively to a current $\neq a^*$ (equivalently, ag_i^{-1} converges projectively to a tree $\neq b$ and bg_i^{-1} to a tree $\neq a$). Then a scaling sequence for ag_i is a scaling sequence for $g_i(b^*)$ and a scaling sequence for bg_i is a scaling sequence for $g_i(a^*)$.*

Proof. Suppose $ag_i/\lambda_i \to a$. We have

$$\langle a, g_i(b^*)/\lambda_i\rangle = \langle ag_i/\lambda_i, b^*\rangle \to \langle a, b^*\rangle > 0$$

which means that λ_i is a scaling sequence for $g_i(b^*)$. The claim about $g_i(a^*)$ is similar. □

4.3. Verification of (A1) and (A2). Fix a finite collection of fully irreducible automorphisms $f_1, \dots, f_k$. Choose small closed neighborhoods $D_i^\pm$ of $T_{f_i}^\pm$ determining annuli $A_i = (D_i^-, D_i^+)$ and consider the corresponding annulus system $\mathcal{A} = \{\pm A_i g \mid g \in \mathrm{Out}(F_n),\ i = 1, \dots, k\}$ consisting of all translates of these. For notational simplicity we will assume $k = 1$, $f = f_1$ and $D^\pm = D_1^\pm$.

Lemma 4.17. *If $D^\pm$ are chosen small enough, the following holds. If a, b, c, d are irreducible trees and $a \neq b$, $c \neq d$, then $(ab|cd) < \infty$.*

Proof. By translating, we may assume that a, b, c, d are outside $D^\pm$. If $(ab|cd) = \infty$, then there are infinitely many distinct $g_i \in \mathrm{Out}(F_n)$ so that $ag_i^{-1}, bg_i^{-1} \in D^-$ and $cg_i^{-1}, dg_i^{-1} \in D^+$ (or switch D^- and D^+). We may assume that these sequences converge projectively. Let $\alpha_i, \beta_i, \gamma_i, \delta_i$ be scaling sequences for ag_i^{-1}, bg_i^{-1}, cg_i^{-1}, dg_i^{-1}, respectively, so, e.g., ag_i^{-1}/α_i converges in $\overline{\mathcal{T}}$. Likewise, let α_i', β_i', γ_i', δ_i' be scaling sequences for ag_i, bg_i, cg_i, dg_i, respectively. By Proposition 4.9 at least three of $\alpha_i, \beta_i, \gamma_i, \delta_i$ go to ∞, and we assume that $\alpha_i, \beta_i, \gamma_i \to \infty$. Likewise three of $\alpha'_i, \beta'_i, \gamma'_i, \delta'_i$ go to ∞, say $\alpha'_i, \beta'_i, \delta'_i$ (the other possibilities are similar).

Now we have the following cases.

Case 1. $bg_i \to T_b \neq b$ (projectively). Then by Lemma 4.15, β'_i is a scaling sequence for $g_i(b^*_i)$ (note that $bg_i^{-1} \in D^-$, so cannot converge to b). If we let $T_a = \lim ag_i^{-1}/\alpha_i$ and $\Upsilon_b = \lim g_i(b^*)/\beta'_i$ then

$$\langle ag_i^{-1}/\alpha_i, g_i(b^*)/\beta'_i\rangle = \langle a, b^*\rangle/(\alpha_i\beta'_i) \to 0,$$

so $\langle T_a, \Upsilon_b\rangle = 0$. Likewise $\langle T_c, \Upsilon_b\rangle = 0$ where $T_c = \lim cg_i^{-1}/\gamma_i$. But that is a contradiction – there is no current in $\mathcal{M}(F_n)$ that has length 0 in trees close to both T_f^+ and T_f^-. (Note that T_a is close to T_f^- and T_c to T_f^+.) Indeed, a limiting argument would produce a current in $\mathcal{M}(F_n)$ whose length is 0 in both T_f^+ and T_f^-, violating Corollary 4.13.

Case 2. $ag_i \to T_a \neq a$. This is the same as Case 1 after exchanging the roles of a and b.

Case 3. $ag_i \to a$ and $bg_i \to b$. Then by Lemma 4.16 scaling sequences for $g_i(a^*)$ and $g_i(b^*)$ are β'_i and α'_i (note that $ag_i^{-1}, bg_i^{-1} \in D^-$, so neither can converge to a or b) and they also go to ∞, so the same argument as in Case 1 holds. □

Lemma 4.18. *If $D^\pm$ are small enough then for any irreducible trees a, b, c, d we have either $(ac|bd) = 0$ or $(ad|bc) = 0$.*

Proof. Suppose that $(ac|bd) > 0$ and $(ad|bc) > 0$. After replacing a, b, c, d by (simultaneous) translates if necessary, we may assume that $a, c \in D^-$ and $b, d \in D^+$ (or interchange D^- and D^+). Now there are two cases.

Case 1. There is $g \in \mathrm{Out}(F_n)$ such that $ag, dg \in D^-$, $bg, cg \in D^+$.

Consider the expression

$$\frac{\langle a, c^*\rangle\langle b, d^*\rangle}{\langle a, d^*\rangle\langle b, c^*\rangle}.$$

This expression does not change after scaling a, b, c^*, d^*, and it does not change after applying g, i.e., replacing a, b by ag, bg and c^*, d^* by $g^{-1}(c^*)$, $g^{-1}(d^*)$.

Also note that when $c_i \to T$ (an irreducible tree) then $c^*_i \to T^*$ so we may assume that a is close to T_f^-, c^* to $(T_f^-)^*$, b to T_f^+, and d^* to $(T_f^+)^*$. By the continuity of the pairing, the expression above is small (both numbers in the numerator are close to 0, the numbers in the denominator are close to $\langle T_f^-, (T_f^+)^*\rangle > 0$ and $\langle T_f^+, (T_f^-)^*\rangle > 0$). After applying g, the expression is close to ∞ (the numbers in the numerator are close to $\langle T_f^-, (T_f^+)^*\rangle > 0$ and $\langle T_f^+, (T_f^-)^*\rangle > 0$, and both numbers in the denominator are close to 0), contradiction.

Case 2. There is $g \in \mathrm{Out}(F_n)$ such that $ag, dg \in D^+$, $bg, cg \in D^-$. The argument is similar. □

Proof of the Main Theorem. Let $M \subset \overline{\mathcal{PT}}$ consisting of all irreducible trees. For the annulus system take the restriction to M of the annulus system considered above.

Thus (A1) follows from Lemma 4.17 and (A2) from Lemma 4.18. The resulting Bowditch complex $\mathcal{X}$ is hyperbolic. The statements about orbits and translation lengths are verified in the next section. $\square$

4.4. Orbits

Proposition 4.19. *The elements $f_1, \ldots, f_k$ chosen at the start of the construction have nonzero translation lengths.*

Proof. By the north-south dynamics we see that $d(x, xf_i^N) \to \infty$; in fact we have $\liminf \frac{d(x, xf_i^N)}{N} > 0$ for any $x \in \mathcal{X}$. $\square$

Lemma 4.20. *Let $S \in \overline{\mathcal{T}}$ be a simplicial tree and $g_i \in \mathrm{Out}(F_n)$ an infinite sequence such that $Sg_i \to T$ projectively. If λ_i is a scaling sequence such that $Sg_i/\lambda_i \to T$, then λ_i is bounded from below. Furthermore, if for every $\varepsilon > 0$ there are conjugacy classes with length in T in the interval $(0, \varepsilon)$ then $\lambda_i \to \infty$.*

Proof. Let γ be a conjugacy class with $\langle T, \gamma \rangle = L > 0$. Then

$$\langle S, g_i(\gamma) \rangle / \lambda_i = \langle Sg_i/\lambda_i, \gamma \rangle \to L,$$

and since eventually $\langle S, g_i(\gamma) \rangle \geq \eta > 0$ (nonzero translation lengths in S are always bounded away from 0) we see that $\liminf \lambda_i \geq \frac{\eta}{L}$. $\square$

Lemma 4.21. *If S is a simplicial tree and p, q are irreducible trees, $p \neq q$, then $(S|pq) < \infty$.*

Proof. To simplify notation we assume that $\{f_1, \ldots, f_k\} = \{f\}$. We may assume that $p, q \notin D^+ \cup D^-$. If $(S|pq) = \infty$ then there are infinitely many $g_i \in \mathrm{Out}(F_n)$ such that $Sg_i^{-1} \in D^-$ and $pg_i^{-1}, qg_i^{-1} \in D^+$ (or interchange D^- and D^+). We may assume that sequences Sg_i^{-1}, $g_i(p^*)$, $g_i(q^*)$ converge projectively, say to T, Υ_p, Υ_q, respectively. A scaling sequence for one of pg_i or qg_i must go to infinity by Proposition 4.9, say for the former. Then Lemma 4.15 implies that a scaling sequence μ_i for $g_i(p^*)$ also goes to infinity ($pg_i^{-1} \in D^-$ cannot converge to $p \notin D^-$). Let λ_i be a scaling sequence for Sg_i^{-1}. Since $\lambda_i \mu_i \to \infty$ by Lemma 4.20, it follows that

$$\langle T, \Upsilon_p \rangle = \langle \lim Sg_i^{-1}/\lambda_i, \lim g_i(p^*)/\mu_i \rangle = \lim \langle S, p^* \rangle / (\lambda_i \mu_i) = 0,$$

which is a contradiction since T is close to T^- and Υ_p is close to $(T^+)^*$. $\square$

Proposition 4.22. *Let S be a simplicial tree. Then the stabilizer* $\mathrm{Stab}(S) \subset \mathrm{Out}(F_n)$ *acts on $\mathcal{X}$ with bounded orbits.*

Proof. Identical to the proof of Proposition 3.7. $\square$

Lemma 4.23. *If S, S' are simplicial trees in $\overline{\mathcal{T}}$ and if there is a nontrivial conjugacy class γ that is elliptic in both S and S' and that is contained in a proper free factor of F_n, then $(S|S') = 0$.*

Proof. Martin [Mar95] proved that $\eta_\gamma \in \mathcal{M}$ (and he proved the converse as well). If $(S|S') > 0$ then there is $g \in \mathrm{Out}(F_n)$ with $Sg^{-1} \in D^-$ and $S'g^{-1} \in D^+$ (or interchange D^+ and D^-). Then $\langle Sg^{-1}, g(\gamma)\rangle = \langle S'g^{-1}, g(\gamma)\rangle = 0$, so the current $\eta_{g(\gamma)}$ has length 0 in a tree close to T^+ and in a tree close to T^-, contradiction. □

Proposition 4.24. *The stabilizer of the conjugacy class of a proper free factor has bounded orbits in $\mathcal{X}$.*

Proof. The proof is a variation of the proof of Proposition 3.7. Let A be a proper free factor of F_n. Fix a simplicial F_n-tree S with A fixing a vertex. Let $(p_1, p_2, p_3) \in \mathcal{X}$ and let $g \in \mathrm{Out}(F_n)$ fix A. We will argue that $d((p_1, p_2, p_3), (p_1 g, p_2 g, p_3 g))$ is bounded independently of g. By Lemma 4.21 we have $N = \max_{i,j}(p_i p_j|S) < \infty$.

Suppose that there are D disjoint annuli separating p_i, p_j from $p_u g$, $p_v g$ for some $i \neq j$ and $u \neq v$. Now consider S and $S' = Sg$. By Lemma 4.23 no annulus separates S from S'. Moreover, at most N annuli separate S from p_i, p_j and at most N annuli separate S' from $p_u g$, $p_v g$. We deduce that $D \leq 2N + 2$. □

4.4.1. The complex of simplicial trees. Lemma 4.23 suggests the definition of another $\mathrm{Out}(F_n)$-complex, namely the *complex of simplicial trees* $\mathcal{ST}(F_n)$. A vertex is represented by a minimal, non-free, simplicial F_n-tree in $\overline{\mathcal{T}}$ without valence 2 vertices and all edge lengths 1. (Recall [CL95] that a minimal nontrivial simplicial F_n-tree is in $\overline{\mathcal{T}}$ if and only if it is *very small*, i.e., the edge stabilizers are cyclic, and for $g \neq 1$ we have that $\mathrm{Fix}(g)$ does not contain a tripod and $\mathrm{Fix}(g^m) = \mathrm{Fix}(g)$ for all $m \neq 0$.) Two such trees span an edge if there is a nontrivial conjugacy class γ that is elliptic in both trees and such that γ is contained in a proper free factor.

When $n = 2$ this graph is quasi-isometric to the Farey graph.

Now define $\Phi \colon \mathcal{ST}(F_n) \to \mathcal{X}$ by the rule that $\Phi(T)$ is a triple that belongs to an orbit of uniformly bounded size (see the discussion in Section 3.3).

Proposition 4.25. *Φ is coarsely well defined and it is Lipschitz.*

Proof. Identical to the proof of Proposition 3.10. □

Question. Is $\Phi \colon \mathcal{ST}(F_n) \to \mathcal{X}$ coarsely onto?

Question. How dependent is $\mathcal{X}$ on the choice of $D^\pm$? For example, when $D^\pm$ keeps getting smaller, we expect that the natural maps between $\mathcal{X}$'s do not eventually become quasi-isometries. Does Remark 3.12 hold in the $\mathrm{Out}(F_n)$ world?

Question. Is $\mathcal{X}$ quasi-isometric to a tree (provided that $D^\pm$ is sufficiently small)?

We finish this section by comparing $\mathcal{X}$ to two other $\mathrm{Out}(F_n)$-complexes.

4.4.2. The complex of free factors. Let $\mathcal{F}(F_n)$ denote *complex of free factors*: its vertices are conjugacy classes of proper free factors, and its simplices are conjugacy classes of chains (ordered by inclusion) of proper free factors. This complex has been introduced and studied by Hatcher and Vogtmann [HV98]. It is a discrete set when $n = 2$ and it is connected when the rank $n > 2$.

There is a map $\Psi\colon \mathcal{F}(F_n) \to \mathcal{ST}(F_n)$ given by the rule that $\Psi(F)$ is the Bass–Serre tree associated with a splitting $F_n = F * F'$. This map is coarsely well defined and Lipschitz.

4.4.3. The splitting complex. A tree $S \in \overline{\mathcal{T}}$ is a *splitting tree* if it is the Bass–Serre tree of a nontrivial splitting $F_n = A * B$. The *splitting complex* is the simplicial complex $\mathcal{S}(F_n)$ whose vertices are splitting trees, and a collection S_i of such trees spans a simplex if there is a simplicial F_n-tree S with trivial edge stabilizers such that each S_i can be obtained from S by equivariantly collapsing collections of edges. When $n = 2$ this complex is a discrete set and when $n > 2$ it is connected.

There is a map $\Sigma\colon \mathcal{S}(F_n) \to \mathcal{F}(F_n)$ that to a splitting $A * B$ assigns A. When $n > 2$ this map is coarsely well defined and Lipschitz.

To summarize, for $n > 2$ we have maps

$$\mathcal{S}(F_n) \xrightarrow{\Sigma} \mathcal{F}(F_n) \xrightarrow{\Psi} \mathcal{ST}(F_n) \xrightarrow{\Phi} \mathcal{X}.$$

The map Σ is coarsely onto by construction. The composition $\Psi\Sigma\colon \mathcal{S}(F_n) \to \mathcal{ST}(F_n)$ (and hence also Ψ) is coarsely onto, because a finite graph of groups with cyclic edge groups representing F_n can be converted to a finite graph of groups with trivial edge groups by a (bounded) sequence of elementary moves; see [She55], [Swa86].

Remark 4.26. If one takes $\Phi(T)$ to be the subset of $\mathcal{X}$ consisting of points whose orbit under $\mathrm{Stab}(T)$ has diameter bounded by K_d (see Section 3.3), then Φ becomes an equivariant coarse map. It follows immediately that translation lengths in $\mathcal{ST}(F_n)$ of fully irreducible automorphisms are positive. The same statement holds for $\mathcal{S}(F_n)$ and $\mathcal{F}(F_n)$. This fact was proved for non-geometric automorphisms in [KL09].

Question. Are Σ and Ψ quasi-isometries? We expect that Σ is not. More precisely, take a pseudo-Anosov homeomorphism f on a surface with two boundary components and view it as an element of $\mathrm{Out}(F_n)$ (this is possible when $n > 2$). Then f acts with bounded orbits on $\mathcal{F}(F_n)$, but we expect that f has positive translation length in $\mathcal{S}(F_n)$.

4.5. WPD

Proposition 4.27. *The elements $f_1, \dots, f_k$ chosen at the start of the construction satisfy WPD: For every $i = 1, 2, \dots, k$, every $x \in \mathcal{X}$, and every $C > 0$ there is*

$N > 0$ *such that*

$$\{g \in \mathrm{Out}(F_n) \mid d(x, xg) \le C,\ d(xf_i^N, xf_i^N g) \le C\}$$

is finite.

The proof requires two lemmas.

Lemma 4.28. *Let U and U' be disjoint closed sets in $\overline{\mathcal{PT}}$ and let $P \notin U \cup U'$ be irreducible. Suppose that for $s = 1, 2$:*

- *there are $f_s, f_s' \in \mathrm{Out}(F_n)$ such that $S_s = Pf_s \in U$ and $S_s' = Pf_s' \in U'$,*
- *$S_1 \neq S_2$ and $S_1' \neq S_2'$,*
- *there are infinitely many distinct $g_i \in \mathrm{Out}(F_n)$ such that $S_s g_i \in U$ and $S_s' g_i \in U'$ for all i, and*
- *the sequences $S_s g_i$, $S_s' g_i$, $g_i^{-1} S_s^*$, and $g_i^{-1} S_s'^*$ converge projectively.*

Then

(1) *three of four given scaling sequences for $S_s g_i$ and $S_s' g_i$, $s = 1, 2$, converge to infinity, and*

(2) *three of four given scaling sequences for $g_i^{-1} S_s^*$ and $g_i^{-1} S_s'^*$, $s = 1, 2$, converge to infinity.*

Proof. Item 1 follows from Proposition 4.9.

By taking a subsequence, we may assume that Pg_i^{-1} converges projectively. Using Proposition 4.9 again, we may assume that the scaling sequence μ_i for Pg_i^{-1} converges to infinity. (Otherwise replace P by an element in the same orbit that is also not in $U \cup U'$.)

For Item 2, we claim that if $Pg_i^{-1} \not\to S_1$ projectively then μ_i is a scaling sequence for $g_i^{-1} S_1^*$. Note that μ_i is also a scaling sequence for the translated sequence $Pg_i^{-1} f_1^{-1}$. By Lemma 4.15 μ_i is a scaling sequence for $g_i^{-1} S_1^* = g_i^{-1} f_1^{-1} P^*$ if two conditions hold: projectively $Pg_i^{-1} f_1^{-1} \not\to P$ and $Pf_1 g_i \not\to P$. By hypothesis, $Pf_1 g_i = S_1 g_i \not\to P$ and so the second condition holds. If the first condition fails, then $Pg_i^{-1} \to Pf_1 = S_1$.

Item 2 now follows because the sequence Pg_i^{-1} can converge to at most one of S_1, S_2, S_1', S_2'. □

Let $\mathcal{X}'$ be the Bowditch space obtained by using the Out(F_n)-orbit of a single irreducible tree P instead of all irreducible trees.

Lemma 4.29. *$\mathcal{X}'$ and $\mathcal{X}$ are quasi-isometric.*

Proof. Since the metric on $\mathcal{X}'$ is the restriction of the metric on $\mathcal{X}$, it is enough to show that $\mathcal{X}' \subset \mathcal{X}$ is co-bounded. Let $t = (T_1, T_2, T_3) \in \mathcal{X}$. Using north-south dynamics, since $(T_i|T_l T_m) > 0$ (in fact infinite) if $l \neq i \neq m$, there is $p = (P_1, P_2, P_3) \in \mathcal{X}'$

such that $(P_i T_i | P_l T_m) > 0$ (in fact arbitrarily large) if $l \neq i \neq m$. Recalling that $k = 0$, Property (C1) in Section 2.2 (see also the comment on (C1) in that section) implies that $(P_i P_l | T_i T_m) = 0$ for $l \neq i \neq m$, i.e. $\rho(p, t) = 0$. □

Proof of Proposition 4.27. By Lemma 4.29, it is enough to show that the proposition holds for $\mathcal{X}'$. Denote $f = f_i$. By definition of distance in $\mathcal{X}'$, the condition $d(x, y) \leq C$ is equivalent to $\rho(x, y) = \max(X|Y) \leq C'$ for a suitable $C' > 0$, where X, Y range over 2-element subsets of x, y respectively. Construct closed neighborhoods $U_j^{\pm}$ of $T_f^{\pm}$, $j = 0, 1$ and a closed neighborhood Ω^- of Υ_f^- so that:

- $U_0^{\pm} \supset U_1^{\pm}$.
- $\langle T, \Upsilon \rangle > 0$ if $T \in U_0^-$ and $\Upsilon \in \Omega^-$; moreover, a current dual to an irreducible tree $T' \in U_0^+$ belongs to Ω^-.
- $(U_1^{\pm} | M - U_0^{\pm}) > C'$.

This is possible by Proposition 4.1 and the facts listed in Section 4.2. The statement is invariant under replacing x by xf^m for any m, so we may assume that if $x = (S_1, S_2, S_3)$ then, for two values of s say $s = 1, 2$, $S_s \in U_1^-$. In this case, we may also assume that $S_s' = S_s f^N \in U_1^+$ for $s = 2, 3$ and some $N > 0$.

Suppose that $g_1, g_2, \ldots$ is an infinite collection in $\mathrm{Out}(F_n)$ so that $\rho(x, xg_i) \leq C'$ and $\rho(xf^N, xf^N g_i) \leq C'$ for all i. It follows that, for each i, $S_1 g_i$ and $S_2 g_i$ belong to U_0^-. Similarly, for each i, $S_2' g_i$ and $S_3' g_i$ belong to U_0^+. Passing to a subsequence, $S_s g_i$ and $g_i^{-1} S_s^*$, for $s = 1, 2$, and $S_{s'}' g_i$ and $g_i^{-1} S_{s'}'^*$, for $s' = 2, 3$, all converge projectively.

By Lemma 4.28, for either $s = 1$ or 2 a scaling sequence for $S_s g_i$ goes to infinity and for either $s' = 2$ or 3 a scaling sequence for $g_i^{-1} S_{s'}'^*$ goes to infinity. For convenience assume these values are $s = 1$ and $s' = 3$. The other cases differ only in notation.

Let $S_1 g_i / \lambda_i \to T$ and $g_i^{-1} S_3'^* / \mu_i \to \Upsilon$. We have

$$\langle T, \Upsilon \rangle = \langle \lim S_1 g_i / \lambda_i, \lim g_i^{-1} S_3'^* / \mu_i \rangle = \lim \langle S_1, S_3'^* \rangle / (\lambda_i \mu_i) = 0.$$

But $g_i^{-1}(S_3'^*) = (S_3' g_i)^* \in \Omega^-$, so $\Upsilon \in \Omega^-$ and we have a contradiction to the second bullet in the proof. □

4.6. Application to quasi-homomorphisms. Recall that a *quasi-homomorphism* on a group Γ is a function $\phi\colon \Gamma \to \mathbb{R}$ with $\Delta(\phi) := \sup_{\gamma, \gamma' \in \Gamma} |\phi(\gamma\gamma') - \phi(\gamma) - \phi(\gamma')| < \infty$. The collection of all quasi-homomorphisms on Γ is a vector space $QH(\Gamma)$ that contains bounded functions as well as homomorphisms $\Gamma \to \mathbb{R}$. We denote by $\widetilde{QH}(\Gamma)$ the quotient of $QH(\Gamma)$ by the subspace spanned by bounded functions and homomorphisms. Then $\widetilde{QH}(\Gamma)$ can be identified with the kernel of the natural homomorphism $H_b^2(\Gamma; \mathbb{R}) \to H^2(\Gamma; \mathbb{R})$ from the second bounded cohomology of Γ to the standard cohomology.

The following result was announced by U. Hamenstädt, who uses methods of [Ham08]. We say that two fully irreducible automorphisms $f, g \in \mathrm{Out}(F_n)$ are *independent* if $\{T_f^+, T_f^-\} \cap \{T_g^+, T_g^-\} = \emptyset$.

Corollary 4.30. *We have* $\dim \widetilde{QH}(\mathrm{Out}(F_n)) = \infty$. *Moreover, if* $\Gamma < \mathrm{Out}(F_n)$ *is any subgroup containing two independent fully irreducible automorphisms, then* $\dim \widetilde{QH}(\Gamma) = \infty$.

Proof. This follows from the arguments of [BF02]. In that paper the main theorem was proved under the assumption that every hyperbolic element satisfies WPD (and then the action is said to satisfy WPD). In fact, it suffices to know that one of the hyperbolic elements in the Schottky subgroup (the one generated by high powers of independent fully irreducible elements) satisfies WPD. Then [BF02], Proposition 6 (5), shows that there exist hyperbolic elements g_1, g_2 with $g_1 \not\sim g_2$ and then [BF02], Theorem 1, implies the result. □

Corollary 4.31. *Let* Γ *be an irreducible lattice in a semisimple Lie group of rank* ≥ 2. *If* $\Gamma \to \mathrm{Out}(F_n)$ *is an embedding, the image does not contain any fully irreducible automorphisms.*

Proof. Burger and Monod proved that $\widetilde{QH}(\Gamma) = 0$ [BM99]. By Corollary 4.30 the image $H \subset \mathrm{Out}(F_n)$ does not contain two independent fully irreducible automorphisms. Now suppose that $f \in H$ is fully irreducible. If H leaves $T_f^{\pm}$ invariant then H and Γ are virtually cyclic (see [BFH97]), which is impossible. If $h \in H$ does not preserve $T_f^{\pm}$ then f and hfh^{-1} are independent fully irreducible automorphisms in H, contradiction. □

Corollary 4.32. *The Cayley graph of* $\mathrm{Out}(F_n)$ *with respect to a finite generating set contains arbitrarily large balls consisting entirely of fully irreducible automorphisms.*

Proof. Fix a quasi-homomorphism $\phi\colon \mathrm{Out}(F_n) \to \mathbb{R}$ which is unbounded and which arises from our construction. Then there is a constant $C > 0$ such that whenever $g \in \mathrm{Out}(F_n)$ is not fully irreducible, then $|\phi(g)| < C$. This is because g has bounded orbits on $\mathcal{X}$ by Proposition 4.24 and hence there is a uniformly bounded orbit. On such elements ϕ is uniformly bounded by construction of [BF02].

Now fix $R > 0$ and let $C' = \max_{x \in B(1,R)} |\phi(x)|$. Choose some $f \in \mathrm{Out}(F_n)$ such that $\phi(f) > C + C' + \Delta(\phi)$. Then $\phi(g) > C$ for every $g \in B(f, R)$, so this ball consists of fully irreducible automorphisms. □

Remark 4.33. A similar argument shows that for every R there is R' so that every R'-ball contains an R-ball that consists entirely of fully irreducible automorphisms.

4.7. Dictionary. The table below provides a correspondence between some objects associated with $\mathrm{Out}(F_n)$ and others associated with MCG.

Dictionary	
F_n	S a compact surface
$\mathrm{Out}(F_n)$	MCG
primitive element	non-∂-parallel scc
free factor	connected subsurface
fully irreducible f	pseudo-Anosov f
simplicial tree	multicurve
$\bar{\mathcal{T}}$	$\mathcal{ML}$
$\mathcal{MC}(F_n)$	$\mathcal{MC}$
T_f^+	Λ_f^+
$\mathcal{M}(F_n) = \overline{\{\eta_\gamma \mid \gamma \text{ primitive}\}}$	$\overline{\{\eta_\gamma \mid \gamma \text{ a non-}\partial\text{-parallel scc}\}}$
$\bar{\mathcal{T}} \times \mathcal{MC}(F_n) \xrightarrow{\langle \cdot,\cdot \rangle} [0,\infty)$	$\mathcal{ML} \times \mathcal{MC} \xrightarrow{\langle \cdot,\cdot \rangle} [0,\infty)$

Remark 4.34. The space $\mathcal{MC}$ of measured currents on a surface and an intersection pairing $\langle \cdot, \cdot \rangle : \mathcal{MC} \times \mathcal{MC} \to [0,\infty)$ was introduced by Bonahon [Bon88]. He also produces an embedding $\mathcal{ML} \to \mathcal{MC}$ (whose image in $\mathcal{PMC}$ is the closure of the set of η_γ's) and the pairing in the table is obtained by restriction. A *multicurve* is a measured lamination with support a collection of disjoint simple closed curves (scc's). According to Skora [Sko96], $\mathcal{PML}$ can be identified with projectivized space of small $\pi_1(S)$-trees in which boundary curves are elliptic. The subspaces of simplicial trees in $\mathcal{PT}$ and $\mathcal{PML}$ are dense.

A version of Corollary 4.7 holds for surfaces. Fix a complete hyperbolic structure on the interior of S and by $|\alpha|$ denote the hyperbolic length of the closed geodesic homotopic to α.

Theorem 4.35. *Let f and g be pseudo-Anosov and assume $\Lambda_f^+ \neq \Lambda_g^+$. There is $\delta > 0$ such that for all non-boundary-parallel simple closed curves α we have either $\langle \Lambda_f^+, \alpha \rangle \geq \delta|\alpha|$ or $\langle \Lambda_g^+, \alpha \rangle \geq \delta|\alpha|$.*

The ingredients of the proof are that simple closed geodesics never enter a neighborhood of any cusp and that on the complement of these neighborhoods the hyperbolic metric is comparable to the Euclidean metric with cone singularities determined by Λ_f^+ and Λ_g^+.

Given Theorem 4.35, a proof of Proposition 3.13 and an alternate proof of Theorem 3.1 can be obtained by using the dictionary to translate the proofs in the $\mathrm{Out}(F_n)$ case.

References

[AK] Y. Algom-Kfir, Strongly contracting geodesics in Outer space. Preprint 2008. arXiv:0812.1555

[BF02] M. Bestvina and K. Fujiwara, Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 69–89. Zbl 1021.57001 MR 1914565

[BF07] M. Bestvina and K. Fujiwara, Quasi-homomorphisms on mapping class groups. *Glas. Mat. Ser. III* **42(62)** (2007), 213–236. Zbl 1155.57301 MR 2332668

[BFH97] M. Bestvina, M. Feighn, and M. Handel, Laminations, trees, and irreducible automorphisms of free groups. *Geom. Funct. Anal.* **7** (1997), 215–244. Zbl 0884.57002 MR 1445386

[BH92] M. Bestvina and M. Handel, Train tracks and automorphisms of free groups. *Ann. of Math.* (2) **135** (1992), 1–51. Zbl 0757.57004 MR 1147956

[Bon88] F. Bonahon, The geometry of Teichmüller space via geodesic currents. *Invent. Math.* **92** (1988), 139–162. Zbl 0653.32022 MR 931208

[Bon91] F. Bonahon, Geodesic currents on negatively curved groups. In *Arboreal group theory* (Berkeley, CA, 1988), Math. Sci. Res. Inst. Publ. 19, Springer, New York 1991, 143–168. Zbl 0772.57004 MR 1105332

[Bow98] B. H. Bowditch, A topological characterisation of hyperbolic groups. *J. Amer. Math. Soc.* **11** (1998), 643–667. Zbl 0906.20022 MR 1602069

[Bow06] B. H. Bowditch, Intersection numbers and the hyperbolicity of the curve complex. *J. Reine Angew. Math.* **598** (2006), 105–129. Zbl 1119.32006 MR 2270568

[BMM] J. Brock, H. Masur, and Y. Minsky. Asymptotics of Weil-Petersson geodesics I: ending laminations, recurrence, and flows. Preprint 2008. arXiv:0802.1370

[BM99] M. Burger and N. Monod, Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc.* (*JEMS*) **1** (1999), 199–235. Zbl 0932.22008 MR 1694584

[CL95] M. M. Cohen and M. Lustig, Very small group actions on **R**-trees and Dehn twist automorphisms. *Topology* **34** (1995), 575–617. Zbl 0844.20018 MR 1341810

[CM87] M. Culler and J. W. Morgan, Group actions on $\mathbb{R}$-trees. *Proc. London Math. Soc.* (3) **55** (1987), 571–604. Zbl 0658.20021 MR 907233

[CV86] M. Culler and K. Vogtmann, Moduli of graphs and automorphisms of free groups. *Invent. Math.* **84** (1986), 91–119. Zbl 0589.20022 MR 830040

[FLP91] *Travaux de Thurston sur les surfaces*. 2nd ed., Séminaire Orsay 1979, Astérisque **66–67**, Soc. Math. France, Paris 1991. Zbl 0731.57001 MR 0568308

[FM02] B. Farb and L. Mosher, Convex cocompact subgroups of mapping class groups. *Geom. Topol.* **6** (2002), 91–152. Zbl 1021.20034 MR 1914566

[Ham08] U. Hamenstädt, Bounded cohomology and isometry groups of hyperbolic spaces. *J. Eur. Math. Soc.* (*JEMS*) **10** (2008), 315–349. Zbl 1139.22006 MR 2390326

[HV98] A. Hatcher and K. Vogtmann, The complex of free factors of a free group. *Quart. J. Math. Oxford Ser.* (2) **49** (1998), 459–468. Zbl 0935.20015 MR 1660045

[Kap06] I. Kapovich, Currents on free groups. In *Topological and asymptotic aspects of group theory*, Contemp. Math. 394, Amer. Math. Soc., Providence, RI, 2006, 149–176. Zbl 1110.20034 MR 2216713

[KL07] I. Kapovich and M. Lustig, The actions of $\mathrm{Out}(F_k)$ on the boundary of outer space and on the space of currents: minimal sets and equivariant incompatibility. *Ergodic Theory Dynam. Systems* **27** (2007), 827–847. Zbl 1127.20025 MR 2322181

[KL09] I. Kapovich and M. Lustig, Geometric intersection number and analogues of the curve complex for free groups. *Geom. Topol.* **13** (2009), 1805–1833. Zbl 05553981 MR 2496058

[LL03] G. Levitt and M. Lustig, Irreducible automorphisms of F_n have north-south dynamics on compactified outer space. *J. Inst. Math. Jussieu* **2** (2003), 59–72. Zbl 1034.20038 MR 1955207

[Mar95] R. Martin, Non-uniquely ergodic foliations of thin type, measured currents, and automorphisms of free groups. PhD thesis, UCLA, Los Angeles 1995.

[MM99] H. A. Masur and Y. N. Minsky, Geometry of the complex of curves I: Hyperbolicity. *Invent. Math.* **138** (1999), 103–149. Zbl 0941.32012 MR 1714338

[She55] A. Shenitzer, Decomposition of a group with a single defining relation into a free product. *Proc. Amer. Math. Soc.* **6** (1955), 273–279. Zbl 0064.02204 MR 0069174

[Sko96] R. K. Skora, Splittings of surfaces. *J. Amer. Math. Soc.* **9** (1996), 605–616. Zbl 0877.57002 MR 1339846

[Swa86] G. A. Swarup, Decompositions of free groups. *J. Pure Appl. Algebra* **40** (1986), 99–102. Zbl 0579.20021 MR 825183

[Whi36] J. H. C. Whitehead, On certain sets of elements in a free group. *Proc. London Math. Soc.* (2) **41** (1936), 48–56. JFM 62.0079.04 Zbl 0013.24801

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