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Results on Balanced Products of Distributions in Colombeau Algebra

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Abstract. Various products of distributions with coinciding point singularities are derived when the products are 'balanced' so that their sum is a generalized function which is associated to a distribution. These products follow the idea of a known result on distributional products published by Jan Mikusiński in 1966. The results in the present paper are obtained in the Colombeau algebra of generalized functions, which provides an efficient tool for dealing with nonlinear problems of Schwartz distributions.

Keywords: *Schwartz distributions, singular products, Colombeau generalized functions*

AMS subject classification: 46F10, 46F30

0. Introduction

The Colombeau algebra $\mathcal{G}$ [1] has become very popular lately since it has almost optimal properties, as long as the problem of multiplication of Schwartz distributions is concerned: $\mathcal{G}$ is an associative differential algebra with the distributions linearly embedded in it and the multiplication is compatible with the products of C^∞-differentiable functions. Moreover, the notion of 'association' in $\mathcal{G}$ (denoted by $\approx$), being a faithful generalization of the equality of distributions, enables obtaining results 'on distributional level'. We give in the last section a short review of the fundamentals of Colombeau algebra on the real line.

Recall now the well-known result published by Jan Mikusiński in [7]:

$$x^{-1} \cdot x^{-1} - \pi^2 \delta(x) \cdot \delta(x) = x^{-2}. \tag{1}$$

Though, neither of the products on the left-hand side here exists, their difference still has a correct meaning in the distribution space $\mathcal{D}'(\mathbb{R})$. Formulas including balanced products of distributions with coinciding singularities can be found in the mathematical and physical literature. For balanced products of this kind, we used the name 'products of Mikusiński type' in a previous paper [2], where we derived the next generalization of equation (1) in the Colombeau algebra of tempered generalized functions. If $\widetilde{x^{-p}}$ and

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$\widetilde{\delta}^{(p)}(x)$ are the embeddings in $\mathcal{G}$ of the distributions x^{-p} and $\delta^{(p)}(x)$, where $\sim\, : \mathcal{D}'(\mathbb{R}) \to \mathcal{G}(\mathbb{R})$ denotes the embedding map, then for arbitrary $p, q \in \mathbb{N}$ the balanced product

$$\widetilde{x^{-p}} \cdot \widetilde{x^{-q}} - \pi^2 \frac{(-1)^{p+q}}{(p-1)!\,(q-1)!}\, \widetilde{\delta}^{(p-1)}(x) \cdot \widetilde{\delta}^{(q-1)}(x) \approx x^{-p-q}. \tag{2}$$

holds. Following the pattern of the basic Mikusiński product (1), we prove in this paper further results on balanced products of the basic distributions with singular point support $x_{\pm}^{-1}, x^{-1}, (x \pm i0)^{-1}$ and $\delta(x)$ $(x \in \mathbb{R})$. We evaluate the products as the distributions are embedded in the Colombeau algebra and prove that each of the products admits an associated distribution.

1. Notation and definitions

We first recall some definitions of the distributions in consideration.

Notation 1. If $a \in \mathbb{C}$ and $\operatorname{Re} a > -1$, denote as usual the locally-integrable functions

$$x_+^a = \begin{cases} x^a & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \text{and} \qquad x_-^a = \begin{cases} (-x)^a & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\ln x_+ = \begin{cases} \ln x & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \qquad \ln x_- = \begin{cases} \ln(-x) & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\ln|x| = \ln x_+ + \ln x_- \qquad \qquad \ln|x| \operatorname{sgn} x = \ln x_+ - \ln x_-.$$

If $\mathbb{N}$ stands for the natural numbers, denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We shall use the shorthand notation $\partial_x = \frac{d}{dx}$ also when $x \in \mathbb{R}$. Now we can define the distributions $x_{\pm}^a$ for any $a \in \Omega = \mathbb{R} \setminus \{-\mathbb{N}\}$ by setting

$$x_+^a = \partial_x x_+^{a+r}(x) \qquad \text{and} \qquad x_-^a = (-1)^r \partial_x x_-^{a+r}(x)$$

where $r \in \mathbb{N}_0$ is such that $a+r > -1$ and the derivatives are in distributional sense. This definition can be extended also for negative integer values of a by a procedure essentially due to M. Riesz (see [5: Section 3.2]). For each $\psi(x) \in \mathcal{D}(\mathbb{R}), a \mapsto \langle x_+^a, \psi \rangle$ is an analytic function of a on the set Ω. The excluded points are simple poles of this function. For any $p \in \mathbb{N}_0$, the residue at $a = -p-1$ is $\lim_{a \to -p-1}(a+p+1)\langle x_+^a, \psi\rangle = \frac{\psi^{(p)}(0)}{p!}$. Subtracting the singular part, one gets for any $p \in \mathbb{N}_0$

$$\lim_{a \to -p-1} \langle x_+^a, \psi \rangle - \frac{1}{p!}\psi^{(p)}(0) = -\frac{1}{p!}\int_0^\infty \ln x\, \psi^{(p)}(x)\, dx + \frac{\psi^{(p)}(0)}{p!} \sum_{k=1}^{p} \frac{1}{k}. \tag{3}$$

The right-hand side of this equation, which is the principal part of the Laurent expansion, was proposed by Hörmander in [5] to define the distribution x_+^{-p-1}, acting here on the test function $\psi(x)$. Note that this definition exactly coincides with that of the distribution $x_+^{-p-1} \equiv F_{-p-1}(x_+, \lambda)$ introduced by Gelfand and Shilov, as regularization of the integral $\int_{\mathbb{R}_+} x^\lambda \psi(x)\, dx$ taken at the points $\lambda = -p-1$ [3: Section 1.4]. In the present paper, however, we shall limit ourselves to the particular case $p = 0$ when, in view of Notation 1, equation (3) amounts to the definitions

$$x_+^{-1} = \partial_x \ln x \qquad \text{and} \qquad x_-^{-1} = -\partial_x \ln x_- . \tag{4}$$

This is in consistency with the expected relations

$$x_+^{-p}|_{x\mapsto -x} = x_-^{-p} \qquad \text{and} \qquad x_+^{-} - x_-^{-1} = x^{-1} \tag{5}$$

where x^{-1} is defined as usual as the distributional derivative $x^{-1} = \partial_x \ln|x|$. Denote also $x_+^{-} + x_-^{-1} = |x|^{-1}$. Recall finally the definition of the distribution

$$(x \pm i0)^{-1} := \lim_{y\to 0_+} (x \pm iy)^{-1} = x^{-1} \mp i\pi\delta(x). \tag{6}$$

2. Results on distributional products in $\mathcal{G}(\mathbb{R})$

We now proceed to particular balanced products of the above distributions with coinciding singular supports obtained in the Colombeau algebra. As usual, in the following H denotes the Heaviside function.

Theorem 1. *The embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions $x_\pm^{-1}, H, \check{H}$, and $\delta(x)$ satisfy*

$$\widetilde{x_-^{-1}} \cdot \widetilde{H} - \widetilde{\ln x_-} \cdot \widetilde{\delta}(x) \approx 0, \qquad \textit{and} \qquad \widetilde{x_+^{-1}} \cdot \widetilde{\check{H}} - \widetilde{\ln x_+} \cdot \widetilde{\delta}(x) \approx 0. \tag{7}$$

Proof. For given $\varphi \in A_0(\mathbb{R})$, suppose without lost of generality that $\operatorname{supp}\varphi(x) \subseteq [-l, l]$ for some $l \in \mathbb{R}_+$. Then equations (4) and (18), embedding rule (17) and the substitution $v = \frac{y-x}{\varepsilon}$ give for the representatives

$$\widetilde{x_-^{-1}}(\varphi_\varepsilon, x) = \frac{1}{\varepsilon^2}\int_{-\varepsilon l + x}^{0} \ln(-y)\varphi'\Big(\frac{y-x}{\varepsilon}\Big)dy = \frac{1}{\varepsilon}\int_{-l}^{-\frac{x}{\varepsilon}} \ln(-\varepsilon v - x)\varphi'(v)\,dv. \tag{8}$$

Similarly, one has for the representatives of the Heaviside and δ-functions

$$\begin{aligned} \widetilde{H}(\varphi_\varepsilon, x) &= \frac{1}{\varepsilon}\int_0^{\varepsilon l + x} \varphi\Big(\frac{y-x}{\varepsilon}\Big)dy = \int_{-\frac{x}{\varepsilon}}^{l} \varphi(u)\,du \\ \widetilde{\delta}(\varphi_\varepsilon, x) &= \frac{1}{\varepsilon}\varphi\Big(-\frac{x}{\varepsilon}\Big). \end{aligned} \tag{9}$$

For an arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ consider now

$$F = \int_{-\infty}^{\infty} \psi(x)\widetilde{x_-^{-1}}(\varphi_\varepsilon, x)\widetilde{H}(\varphi_\varepsilon, x)\,dx.$$

The substitution $w=-\frac{x}{\varepsilon}$, the Taylor theorem, and the change of the order of integration, which is permissible here, together yield

$$\begin{aligned} F &= \int_{-l}^{l} dw\, \psi(-\varepsilon w) \int_{w}^{l} du\, \varphi(u) \int_{-l}^{w} \ln(\varepsilon w - \varepsilon v)\varphi'(v)\, dv \\ &= [\psi(0)+O(\varepsilon)] \int_{-l}^{l} dw \int_{w}^{l} du\, \varphi(u) \int_{-l}^{w} \ln(\varepsilon w - \varepsilon v)\varphi'(v)\, dv \\ &= \psi(0) \int_{-l}^{l} du\, \varphi(u) \int_{-l}^{u} dv\, \varphi'(v) \int_{v}^{u} \ln(\varepsilon w - \varepsilon v)\, dw + O(\varepsilon). \end{aligned}$$

Note that to obtain the latter asymptotic evaluation we have taken into account that the second term in the Taylor expansion is multiplied by definite integrals that are majorizable by constants. The substitution $w \to t = \frac{w-v}{u-v}$, together with the relation $w-v=(u-v)t$, yields further

$$\begin{aligned} F &= \psi(0) \int_{-l}^{l} du\, \varphi(u) \int_{-l}^{u} dv\, \varphi'(v)(u-v) \int_{0}^{1} \ln(t\varepsilon u - t\varepsilon v)\, dt + O(\varepsilon) \\ &= \psi(0) \int_{-l}^{l} du\, \varphi(u) \int_{-l}^{u} \varphi'(v)(u-v)\big[\ln(\varepsilon u - \varepsilon v) - 1\big]\, dv + O(\varepsilon). \end{aligned}$$

We have used that $\int_0^1 \ln t dt = (t\ln t - t)\big|_0^1 = -1$. Integration by parts in the variable v now gives

$$\begin{aligned} &\int_{-l}^{u} (u-v)\big[\ln(\varepsilon u - \varepsilon v) - 1\big]\varphi'(v)\, dv \\ &\qquad = 0 + \int_{-l}^{u} \big[\ln(\varepsilon u - \varepsilon v) - 1\big]\varphi(v)\, dv + \int_{-l}^{u} \varphi(v)\, dv \\ &\qquad = \int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi(v)\, dv. \end{aligned}$$

Thus

$$F = \psi(0) \int_{-l}^{l} du\, \varphi(u) \int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi(v)\, dv + O(\varepsilon). \tag{10}$$

On the other hand, representation (9), substitution $u=-\frac{x}{\varepsilon}$, and the Taylor theorem yield

$$\begin{aligned} G :&= \int_{-\infty}^{\infty} \psi(x)\widetilde{\ln x_-}(\varphi_\varepsilon, x)\widetilde{\delta}(\varphi_\varepsilon, x)\, dx \\ &= \int_{-l}^{l} du\, \psi(-\varepsilon u)\varphi(u) \int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi(v)\, dv \qquad (11) \\ &= \psi(0) \int_{-l}^{l} du\, \varphi(u) \int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi(v)\, dv + O(\varepsilon). \end{aligned}$$

Equations (10) and (11) therefore give $F-G=O(\varepsilon)$. By linearity, this implies

$$\lim_{\varepsilon\to 0_+} \int_{-\infty}^{\infty} \psi(x) \left[\widetilde{x_-^{-1}}(\varphi_\varepsilon, x)\widetilde{H}(\varphi_\varepsilon, x) - \widetilde{\ln x_-}(\varphi_\varepsilon, x)\widetilde{\delta}(\varphi_\varepsilon, x)\right] dx = 0.$$

According to Definition A2 (see Section 4: Appendix), this proves the first equation in (7). Then the replacement $x \mapsto -x$ in the latter equation proves the second one and the theorem ∎

The balanced products obtained for the components supported on the real half-line can be now used for obtaining results for the distribution x^{-1}. Indeed, the next assertion can be deduced from the result of Theorem 1.

Theorem 2. *The balanced products*

$$\widetilde{x^{-1}} \cdot \widetilde{H} + \widetilde{\ln|x|} \cdot \widetilde{\delta}(x) \approx x_+^{-1} \tag{12}$$

$$-\widetilde{x^{-1}} \cdot \widetilde{\check{H}} + \widetilde{\ln|x|} \cdot \widetilde{\delta}(x) \approx x_-^{-1} \tag{13}$$

hold in $\mathcal{G}(\mathbb{R})$.

Proof. Consider first the following chain of identities and associations in $\mathcal{G}(\mathbb{R})$, taking into account the first equation in (7):

$$\widetilde{x_+^{-1}} \cdot \widetilde{H} = \widetilde{x_+^{-1}} \cdot (1 - \widetilde{\check{H}}) = \widetilde{x_+^{-1}} - \widetilde{x_+^{-1}} \cdot \widetilde{\check{H}} \approx \widetilde{x_+^{-1}} - \widetilde{\ln x_+} \cdot \widetilde{\delta}(x).$$

Thus

$$\widetilde{x_+^{-1}} \cdot \widetilde{H} + \widetilde{\ln x_+} \cdot \widetilde{\delta}(x) \approx \widetilde{x_+^{-1}}$$

which, allowing for equivalence relation (19) for distributional embedding in $\mathcal{G}$, proves the balanced product

$$\widetilde{x_+^{-1}} \cdot \widetilde{H} + \widetilde{\ln x_+} \cdot \widetilde{\delta}(x) \approx x_+^{-1}. \tag{14}$$

Employing now equations (5) and (7) and the latter equation, we get the chain of identities and associations

$$\begin{aligned}
\widetilde{x^{-1}} \cdot \widetilde{H} &= \widetilde{x_+^{-1}} \cdot \widetilde{H} - \widetilde{x_-^{-1}} \cdot \widetilde{H} \\
&\approx -\widetilde{\ln x_+} \cdot \widetilde{\delta}(x) + \widetilde{x_+^{-1}} + \widetilde{\ln x_-} \cdot \widetilde{\delta}(x) \\
&= -\widetilde{\ln|x|} \cdot \widetilde{\delta}(x) + \widetilde{x_+^{-1}}
\end{aligned}$$

or else

$$\widetilde{x^{-1}} \cdot \widetilde{H} + \widetilde{\ln|x|} \cdot \widetilde{\delta}(x) \approx \widetilde{x_+^{-1}}.$$

In view of equivalence relation (19), this proves (12). Equation (13) can be obtained from the latter equation on the replacement $x \mapsto -x$ ∎

Remark. Proceeding similarly to the above proof, one easily gets on the strength of equations (5), (7) and (14)

$$\widetilde{|x|^{-1}} \cdot \widetilde{H} = \left(\widetilde{x_+^{-1}} + \widetilde{x_-^{-1}}\right) \cdot \widetilde{H} = \widetilde{x_+^{-1}} \cdot \widetilde{H} + \widetilde{x_-^{-1}} \cdot \widetilde{H} \approx x_+^{-1}.$$

This product coincides with a known result obtained as a model product in distribution theory (cf. [8: Chapter 7]).

A direct consequence from the result of Theorem 2 is given by

Corollary 1. *For the embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions $(x \pm i0)^{-1}$ and $\delta(x)$, the balanced product*

$$\widetilde{(x \pm i0)^{-1}} \cdot \widetilde{H} + \widetilde{\ln|x|} \cdot \widetilde{\delta}(x) \approx x_+^{-1} \mp i\frac{\pi}{2}\delta(x) \tag{15}$$

holds.

Proof. Employing equations (6) and (12), one gets

$$\widetilde{(x \pm i0)^{-1}} \cdot \widetilde{H} = \widetilde{x}^{-1} \cdot \widetilde{H} \mp i\pi\widetilde{\delta}(x) \cdot \widetilde{H} \approx -\widetilde{\ln|x|} \cdot \widetilde{\delta}(x) + \widetilde{x_+^{-1}} \mp \frac{i\pi}{2}\delta(x)$$

which in view of equivalence relation (19) proves equation (15) ∎

Finally, we evaluate the products of the distributions $x_\mp^{-1}$ with the δ function. They exist only as balanced products, as demonstrated by the following

Theorem 3. *The embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions $x_\pm^{-1}$ and $\delta(x)$ satisfy*

$$2\widetilde{x_\mp^{-1}} \cdot \widetilde{\delta}(x) + \widetilde{\ln|x|\mathrm{sgn}\, x} \cdot \widetilde{\delta}'(x) \approx \pm\frac{1}{2}\,\delta'(x). \tag{16}$$

Proof. For an arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$, evaluate first

$$F_1 := \int_{-\infty}^{\infty} \psi(x)\widetilde{x_-^{-1}}(\varphi_\varepsilon, x)\widetilde{\delta}(\varphi_\varepsilon, x)\, dx.$$

For a given $\varphi \in A_0(\mathbb{R})$, equations (8) and (9) for the representatives, the substitution $u = -\frac{x}{\varepsilon}$, and the Taylor theorem together will give

$$\begin{aligned} F_1 &= \frac{1}{\varepsilon^2}\int_{-\varepsilon l}^{\varepsilon l} dx\, \psi(x)\varphi\Big(-\frac{x}{\varepsilon}\Big)\int_{-l}^{-\frac{x}{\varepsilon}} \ln(-\varepsilon v - x)\varphi'(v)\, dv \\ &= \frac{1}{\varepsilon}\int_{-l}^{l} du\, \psi(-\varepsilon u)\varphi(u)\int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi'(v)\, dv \\ &= \frac{\psi(0)}{\varepsilon}\int_{-l}^{l} dv\, \varphi'(v)\int_{v}^{l} \ln(\varepsilon u - \varepsilon v)\varphi(u)\, du \\ &\quad - \psi'(0)\int_{-l}^{l} du\, \varphi(u)u\int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)\varphi'(v)\, dv + O(\varepsilon). \end{aligned}$$

Here a change of the order of integration in the first summand is made. Denoting next

$$F_2 = \int_{-\infty}^{\infty} \psi(x)\widetilde{\ln x_+}(\varphi_\varepsilon, x)\widetilde{\delta}'(\varphi_\varepsilon, x)\, dx$$

we obtain, making the substitution $v = -\frac{x}{\varepsilon}$ and applying the Taylor theorem,

$$\begin{aligned} F_2 &= -\frac{1}{\varepsilon^2}\int_{-\varepsilon l}^{\varepsilon l} dx\, \psi(x)\varphi'\Big(-\frac{x}{\varepsilon}\Big)\int_{-\frac{x}{\varepsilon}}^{l} \ln(\varepsilon u + x)\varphi(u)\, du \\ &= -\frac{1}{\varepsilon}\int_{-l}^{l} dv\, \psi(-\varepsilon v)\varphi'(v)\int_{v}^{l} \ln(\varepsilon u - \varepsilon v)\varphi(u)\, du \\ &= -\frac{\psi(0)}{\varepsilon}\int_{-l}^{l} dv\, \varphi'(v)\int_{v}^{l} \ln(\varepsilon u - \varepsilon v)\varphi(u)\, du \\ &\quad + \psi'(0)\int_{-l}^{l} du\, \varphi(u)\int_{-l}^{u} \ln(\varepsilon u - \varepsilon v)v\varphi'(v)\, dv + O(\varepsilon) \end{aligned}$$

where a change of the order of integration in the second summand is done. Observing that the first terms in the two expansions coincide up to sign, we can write

$$F_1 + F_2 = -\psi'(0)\int_{-l}^{l} du\,\varphi(u)\int_{-l}^{u}\ln(\varepsilon u-\varepsilon v)(u-v)\varphi'(v)\,dv + O(\varepsilon).$$

Integration by parts in the variable v and equation (4) in the case $p=1$ yield further

$$\begin{aligned}
F_1 + F_2 &= -\psi'(0)\int_{-l}^{l} du\,\varphi(u)\left[\int_{-l}^{u}\ln(\varepsilon u-\varepsilon v)\varphi(v)\,dv + \int_{-l}^{u}\varphi(v)\,dv\right] + O(\varepsilon)\\
&= -\int_{-\infty}^{\infty}\psi'(x)\widetilde{\ln x_-}(\varphi_\varepsilon,x)\widetilde{\delta}(\varphi_\varepsilon,x)dx + O(\varepsilon) - \psi'(0)\frac{1}{2}\left(\int_{-l}^{u}\varphi(v)\,dv\right)^2\Big|_{-l}^{l}\\
&= \int_{-\infty}^{\infty}\psi(x)\partial_x dx\left[\widetilde{\ln x_-}(\varphi_\varepsilon,x)\widetilde{\delta}(\varphi_\varepsilon,x)\right]dx + O(\varepsilon) - \frac{1}{2}\psi'(0)\\
&= F_1 + \int_{-\infty}^{\infty}\psi(x)\widetilde{\ln x_-}(\varphi_\varepsilon,x)\widetilde{\delta}'(\varphi_\varepsilon,x)\,dx + O(\varepsilon) + \frac{1}{2}\langle\psi,\delta'\rangle.
\end{aligned}$$

By linearity and Notation 1, we therefore obtain

$$\int_{-\infty}^{\infty}\psi(x)\left[2\widetilde{x_-^{-1}}(\varphi_\varepsilon,x)\widetilde{\delta}(\varphi_\varepsilon,x) + \ln|x|\mathrm{sgn}\,x(\varphi_\varepsilon,x)\widetilde{\delta}'(\varphi_\varepsilon,x)\right]dx + O(\varepsilon) = \frac{1}{2}\langle\psi,\delta'\rangle.$$

Taking the limit as $\lim_{\varepsilon\to 0_+}$ in the latter equation, we derive, according to Definition A2 (see Section 4: Appendix), the first equation in (16) (the product of x_-^{-1}). Then, the replacement $x\mapsto -x$ proves the second equation in (16) and the theorem ∎

Combining next the two balanced products in (16), we obtain

Corollary 2. *The completely balanced product*

$$\widetilde{|x|^{-1}}\cdot\widetilde{\delta}(x) + \widetilde{\ln|x|\mathrm{sgn}\,x}\cdot\widetilde{\delta}'(x)\approx 0$$

holds in $\mathcal{G}(\mathbb{R})$.

Finally, proceeding similarly to the proof of Corollary 1, one gets from equations (6) and (16) the following

Corollary 3. *The embeddings in $\mathcal{G}(\mathbb{R})$ of the distributions $(x\pm i0)^{-1}$ and $\delta(x)$ satisfy*

$$\widetilde{(x\pm i0)^{-1}}\cdot\widetilde{\delta}(x) \pm i\pi\widetilde{\delta}(x)\cdot\widetilde{\delta}(x)\approx -\frac{1}{2}\delta'(x).$$

4. Appendix

We recall here the fundamentals of the Colombeau algebra, confining ourselves to the algebra $\mathcal{G}(\mathbb{R})$ on the real line $\mathbb{R}$ that is only used in the paper.

Notation A1. Denote $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ $(i, j \in \mathbb{N}_0)$ the usual Kronecker symbol. Then we put for arbitrary $q \in \mathbb{N}_0$

$$A_q(\mathbb{R}) = \left\{ \varphi(x) \in \mathcal{D}(\mathbb{R}) : \int_{\mathbb{R}} x^j \varphi(x)\, dx = \delta_{0j} \quad (j = 0, 1, ..., q) \right\}.$$

Denote also $\varphi_\varepsilon = \varepsilon^{-1}\varphi(\varepsilon^{-1}x)$ for $\varphi \in A_q(\mathbb{R})$ $(\varepsilon > 0)$ and $\check{g}(x) = g(-x)$ for $x \in \mathbb{R}$.

Definition A1. Let $\mathcal{E}[\mathbb{R}]$ be the algebra of functions $f(\varphi, x) : A_0(\mathbb{R}) \times \mathbb{R} \to \mathbb{C}$ that are infinitely differentiable, by a fixed 'parameter' φ. Then, the generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbb{R}) = \mathcal{E}_M[\mathbb{R}]/\mathcal{I}[\mathbb{R}].$$

Here $\mathcal{E}_M[\mathbb{R}]$ is the subalgebra of 'moderate' functions such that for each compact subset K of $\mathbb{R}$ and $p \in \mathbb{N}$ there is a $q \in \mathbb{N}$ such that, for each $\varphi \in A_q(\mathbb{R})$,

$$\sup_{x \in K} |\partial_x^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{-q}) \qquad (\varepsilon \to 0_+).$$

The ideal $\mathcal{I}[\mathbb{R}]$ of $\mathcal{E}_M[\mathbb{R}]$ consists of all functions such that for each compact $K \subset \mathbb{R}$ and any $p \in \mathbb{N}$ there is a $q \in \mathbb{N}$ such that, for every $r \geq q$ and $\varphi \in A_r(\mathbb{R})$,

$$\sup_{x \in K} |\partial_x^p f(\varphi_\varepsilon, x)| = O(\varepsilon^{r-q}) \qquad (\varepsilon \to 0_+).$$

The algebra $\mathcal{G}(\mathbb{R})$ contains the distributions on $\mathbb{R}$, canonically embedded as a $\mathbb{C}$-vector subspace by the map

$$i : \mathcal{D}'(\mathbb{R}) \to \mathcal{G} : \qquad u \mapsto \widetilde{u} = \left\{ \widetilde{u}(\varphi, x) := (u * \check{\varphi})(x) : \varphi \in A_q(\mathbb{R}) \right\}. \tag{17}$$

The derivative in the Colombeau algebra is in consistency with this embedding of distributions into the algebra [1: Chapter 3]:

$$\partial_x \widetilde{u} = \widetilde{\partial_x u} \qquad (u \in \mathcal{D}'(\mathbb{R})). \tag{18}$$

The equality of generalized functions in $\mathcal{G}(\mathbb{R})$ is very strict and a weaker form of equality in the sense of association is introduced that plays a fundamental role in the Colombeau theory.

Definition A2. A generalized function $f \in \mathcal{G}(\mathbb{R})$ is said to be 'associated' with

(a) another function $g \in \mathcal{G}$, denoted $f \approx g$

or

(b) a distribution $u \in \mathcal{D}'(\mathbb{R})$, denoted $f \approx u$

if for some representatives $f(\varphi_\varepsilon, x)$ and $g(\varphi_\varepsilon, x)$ and arbitrary $\psi(x) \in \mathcal{D}(\mathbb{R})$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in A_q(\mathbb{R})$,

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} \big[f(\varphi_\varepsilon, x) - g(\varphi_\varepsilon, x)\big]\psi(x)\, dx = 0$$

or

$$\lim_{\varepsilon \to 0_+} \int_{\mathbb{R}} f(\varphi_\varepsilon, x)\psi(x)\, dx = \langle u, \psi \rangle,$$

respectively.

These definitions are independent of the representatives chosen, and the association is a faithful generalization of the equality of distributions [1]. This fact also implies the following equivalence relation on the embedding of distributions:

$$f \approx \widetilde{u} \iff f \approx u \qquad \text{for each } f \in \mathcal{G} \text{ and } u \in \mathcal{D}'(\mathbb{R}). \tag{19}$$

Now, by the Colombeau product of distributions it is meant the distribution associated to the product of their embeddings in $\mathcal{G}(\mathbb{R})$.

The following coherence result holds [8: Proposition 10.3]

If the regularized model product (*in the terminology of Kamiński*) *of two distributions exists, then their Colombeau product also exists and coincides with the former.*

On the other hand, in the general setting of Colombeau algebra $\mathcal{G}(\mathbb{R}^m)$ [1] (when the parameter functions φ are not defined as tensor products), as well as in the algebra $\mathcal{G}(\mathbb{R})$ on the real line, this assertion turns into an equivalence, according to a result by Jelínek [6]; cf. also a recent study on Colombeau algebras in [4].

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