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Representation Formulas for the General Derivatives of the Fundamental Solution to the Cauchy-Riemann Operator in Clifford Analysis and Applications

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Abstract. In this paper, we discuss several essentially different formulas for the general derivatives $q_{\mathbf{n}}(z)$ of the fundamental solution of the Cauchy-Riemann operator in Clifford Analysis, upon which – among other important applications – the theory of monogenic Eisenstein series is based. Using Fourier and plane wave decomposition methods, we obtain a compact integral representation formula over a half-space, which also lends itself to establish upper bounds on the values $\|q_{\mathbf{n}}(z)\|$. A second formula that we discuss is a recurrence formula involving permutational products of hypercomplex variables by which these estimates can be obtained immediately. We further prove several formulas for $q_{\mathbf{n}}(z)$ in terms of explicit, non-recurrent finite sums, leading themselves to further representations in terms of permutational products but using different and fewer hypercomplex variables than used in the recurrence relations. Summing up a fixed $q_{\mathbf{n}}$ over a given discrete lattice leads to a variant of the Riemann zeta function. We apply one of the closed representation formulas for $q_{\mathbf{n}}(z)$ to express this variant of the Riemann zeta function as a finite sum of real-valued Dirichlet series.

Keywords: *Cauchy-Riemann operator, fundamental solution, permutational products, hypercomplex variables, Dirichlet series*

AMS subject classification: 30G35, 11M36

1. Introduction

One possibility to generalize complex function theory to hypercomplex analysis is the Cauchy-Riemann approach considering real differentiable functions in the kernel of the generalized Cauchy-Riemann operator

$$D_z = \frac{\partial}{\partial z_0} + \sum_{q=1}^{k} \frac{\partial}{\partial z_q} e_q$$

in $\mathcal{A}_{k+1} = \mathbb{R} \oplus \mathbb{R}^k$. Because of the non-commutativity in Clifford algebras one has to distinguish between functions satisfying $D_z f = 0$ called left monogenic and functions

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satisfying $fD_z = 0$ called right monogenic. However, one can treat these two function classes in a very similar way.

It is essential to mention that the usual powers of the hypercomplex number $z = z_0 + \sum_{q=1}^{k} z_q e_q$ are neither left nor right monogenic. In Clifford analysis the positive powers are substituted by the so-called Fueter polynomials (first mentioned by R. Fueter in [4])

$$V_{n_1,\dots,n_k}(z) = \frac{1}{(n_1+\ldots+n_k)!} \sum_{\pi \in \mathrm{perm}(n_1,\dots,n_k)} Z_{\pi(n_1)} Z_{\pi(n_2)} \cdots Z_{\pi(n_k)}$$

where $\mathrm{perm}(n_1,\dots,n_k)$ denotes the set of permutations of the sequence $(n_1,\dots,n_k)$, $Z_q = z_q - z_0 e_q$ for $q = 1,\dots,k$ and $V_{0,\dots,0}(z) = 1$. The negative powers are substituted (cf. [2, 4]) by

$$q_{0,\dots,0}(z) = \frac{\overline{z}}{\|z\|^{k+1}}$$
$$q_{n_1,\dots,n_k}(z) = \frac{\partial^{n_1+\ldots+n_k}}{\partial z_1^{n_1}\ldots\partial z_k^{n_k}} q_{\mathbf{0}}(z) \quad (n_1+\ldots+n_k \geq 1).$$

In analogy to the planar case, one can represent a real differentiable function f that is left monogenic in an annular domain centred around a point $z^* \in \mathcal{A}_{k+1}$ by a unique Laurent series built with the functions $V_{n_1,\dots,n_k}$ and $q_{n_1,\dots,n_k}$, i.e.

$$\begin{aligned} f(z) = &\sum_{n=0}^{\infty} \sum_{n=n_1+\ldots+n_k} V_{n_1,\dots,n_k}(z-z^*) a_{n_1,\dots,n_k} \\ &+ \sum_{n=0}^{\infty} \sum_{n=n_1+\ldots+n_k} q_{n_1,\dots,n_k}(z-z^*) b_{n_1,\dots,n_k} \end{aligned}$$

where $a_{n_1,\dots,n_k}$ and $b_{n_1,\dots,n_k}$ are Clifford numbers.

The functions $q_{n_1,\dots,n_k}$ are the building stones for the monogenic generalization of the classical Eisenstein series in Clifford analysis considered in [8 - 11]. These monogenic Eisenstein series provide e.g. monogenic generalizations of the tangent, cotangent, secant, cosecant and the elliptic functions and also variants of the Riemann zeta function in Clifford analysis. For the analysis of this function class the structure of the functions $q_{\mathbf{n}}$ is important. In [1, 3] it has been shown that the functions $q_{n_1,\dots,n_k}$ can be expressed in terms of spherical harmonic functions, more precisely, in terms of Lagrange and Gegenbauer polynomials. However, the formulas given there do not provide a closed and explicit representation of the functions $q_{\mathbf{n}}$ in terms of a finite explicit sum of explicitly determined functions.

In Section 2 we introduce the most important notions.

In Subsection 3.1, we use Fourier transform and plane wave decomposition methods to derive a compact integral representation formula for the derivatives of $q_{0,\dots,0}(z)$ in a half-space, and apply it to provide a useful upper bound on their values. Two important questions remained open for quite a long time: It is due to H. Malonek [13] that the

Fueter polynomials can be written explicitly in terms of permutational products of hypercomplex variables. More precisely,

$$V_{n_1,\dots,n_k}(z) = \frac{1}{n_1! \cdots n_k!} Z_1^{n_1} \times \cdots \times Z_k^{n_k}. \tag{1}$$

One first natural question is therefore to ask whether or not there is also an analogue of this kind of representation for the generalized negative power functions $q_{n_1,\dots,n_k}(z)$. In [10] a positive answer has been given to this question. An explicit recurrence formula for the functions $q_{\mathbf{n}}$ in terms of finite permutational products of $2k$ hypercomplex variables has been established there.

We recall the basic results in Subsection 3.2 and discuss some basic applications and consequences. The recurrence relations can be applied to derive directly the estimates of the $q_{n_1,\dots,n_k}$-functions playing a crucial role in the analysis of convergence of the generalized monogenic Eisenstein series discussed in [8 - 11]. Furthermore, we observe by this recurrence formula that every meromorphic function can be written in terms of permutational products of a finite number of special hypercomplex variables.

A second question is to ask for explicit closed finite representation formulas for the $q_{\mathbf{n}}$ functions. In Subsection 3.3 we derive an explicit and non-recurrent closed formula for $q_{n_1,\dots,n_k}(z)$. In Subsection 3.4 we apply this formula in combination with the inversion formula from [3] to obtain closed and non-recurrent representations in terms of permutational products, but involving different and fewer hypercomplex variables than used in the recurrence relations discussed in Subsection 3.2.

In Section 4 we apply the closed representation formula of Subsection 3.3 to number theoretical problems.

Summing up the functions $q_{n_1,\dots,n_k}$ over a positive half-lattice in $\mathcal{A}_{k+1}$ leads to a generalization of the classical Riemann zeta function. This variant of the Riemann zeta function has been introduced and discussed in [8, 10]. The new closed formula for the $q_{n_1,\dots,n_k}$ allows us to deduce a closed representation for these functions in terms of a finite sum of real-valued Dirichlet series standing in close relationship with Epstein zeta functions.

2. Preliminaries

We introduce the most important notions. For detailed information about Clifford algebras and their function theory we refer, for example, to [1, 3, 5].

By $\{e_1, e_2, \dots, e_k\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^k$. The attached real Clifford algebra $\mathbf{Cl}_{0k}$ is the free algebra generated by $\mathbb{R}^k$ modulo the relation

$$\mathbf{z}^2 = -\|\mathbf{z}\|^2 e_0$$

where $\mathbf{z} \in \mathbb{R}^k$ and e_0 is the neutral element with respect to multiplication of the Clifford algebra $\mathbf{Cl}_{0k}$. In the Clifford algebra $\mathbf{Cl}_{0k}$ the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0 \qquad (i, j = 1, \dots, k)$$

hold where δ_{ij} is the Kronecker symbol. A basis for the Clifford algebra $\mathbf{Cl}_{0k}$ is given by the set $\{e_A : A \subseteq \{1, \dots, k\}\}$ with $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \le l_1 < \dots < l_r \le k, e_\emptyset =$

$e_0 = 1$. Every $a \in \mathbf{Cl}_{0k}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. Two examples of real Clifford algebras are the complex number field $\mathbb{C}$ and the Hamiltonian skew field $\mathbb{H}$.

The conjugation anti-automorphism in the Clifford algebra $\mathbf{Cl}_{0k}$ is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -e_j$ for $j = 1, \ldots, k$, $\overline{e}_0 = e_0 = 1$.

By

$$\mathcal{A}_{k+1} := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_k\} = \mathbb{R} \oplus \mathbb{R}^k \subset \mathbf{Cl}_{0k}$$

we denote the space of hypercomplex numbers

$$z = z_0 + z_1 e_1 + z_2 e_2 + \ldots + z_k e_k$$

often called paravectors. In this paper we denote pure vectors by a boldface letter, and scalars, paravectors or Clifford numbers by a normal letter. In this notation the hypercomplex number z is represented in the form $z = z_0 + \mathbf{z}$ with $\mathrm{Sc}(z) = z_0$ and $\mathrm{Vec}(z) = \mathbf{z}$. A scalar product between two Clifford numbers $a, b \in \mathbf{Cl}_{0k}$ is defined by $\langle a, b\rangle = \mathrm{Sc}(a\overline{b})$ and the Clifford norm of an arbitrary $a = \sum_A a_A e_A$ is $\|a\| = (\sum_A |a_A|^2)^{1/2}$.

Any element $z \in \mathcal{A}_{k+1} \backslash \{0\}$ has an inverse element in $\mathcal{A}_{k+1}$ given by $z^{-1} = \frac{\overline{z}}{\|z\|^2}$.

Further, we recall that the permutational product of arbitrary Clifford numbers $a_1, \ldots, a_n$ is defined by

$$a_1 \times a_2 \times \ldots \times a_n = \frac{1}{n!} \sum_{\mathrm{perm}(i_1,\ldots,i_n)} a_{i_1} \cdot a_{i_2} \cdot \;\cdots\; \cdot a_{i_n}.$$

For details we refer to [12]. One further uses the abbreviation

$$\underbrace{a_1 \times \ldots \times a_1}_{k_1 \ times} \times \ldots \times \underbrace{a_n \times \ldots \times a_n}_{k_n \ times} = [a_1]^{k_1} \times [a_2]^{k_2} \times \ldots \times [a_n]^{k_n}.$$

In order to distinguish powers in terms of the permutational product from powers in the usual sense, one sets round brackets when meaning ordinary powers. One has to write, for example, $[a_1]^2 \times a_2 = a_1 \times a_1 \times a_2$, but $(a_1)^2 \times a_2 = (a_1 \cdot a_1) \times a_2$.

In order to present many calculations in a more suggestive way, the following notations will be used, where $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ and $\mathbf{j} = (j_1, \ldots, j_k) \in \mathbb{N}_0^k$ are k-dimensional multi-indices:

$$\mathbf{z}^{\mathbf{n}} := z_1^{n_1} \ldots z_k^{n_k}, \quad \mathbf{n}! = n_1! \ldots n_k!, \quad |\mathbf{n}| = n_1 + \ldots + n_k$$
$$\binom{\mathbf{n}}{\mathbf{j}} = \binom{n_1}{j_1} \cdots \binom{n_k}{j_k} \quad (\mathbf{j} \leq \mathbf{n};\ \mathbf{j} \leq \mathbf{n} \ \Leftrightarrow\ j_1 \leq n_1, \ldots, j_k \leq n_k).$$

By $\tau(j)$ we denote the multi-index $\mathbf{n} = (n_1, \ldots, n_k)$ for which $n_i = \delta_{ij}$, δ_{ij} being the Kronecker symbol. We also write $(a)_p$ for the product $a(a+1)\ldots(a+p-1)$.

If z is any paravector, we follow e.g. [6] in writing

$$e^z = \sum_{j=0}^{+\infty} \frac{z^j}{j!} = e^{z_0}\left(\cos(\|\mathbf{z}\|) + \frac{\mathbf{z}}{\|\mathbf{z}\|}\sin(\|\mathbf{z}\|)\right). \tag{2}$$

3. Formulas for the functions $q_{\mathbf{n}}$

In this section we discuss several representation formulas for the functions $q_{\mathbf{n}}(z)$.

3.1 Fourier and plane wave representations. For our purposes we need the following

Lemma 1. *The Fourier transform of $q_{\mathbf{0}}(z)$ with respect to the k coordinates $z_1, \ldots, z_k$ equals*

$$\int_{\mathbf{z}\in\mathbb{R}^k} q_{\mathbf{0}}(z_0+\mathbf{z})e^{-i\langle \mathbf{z},\mathbf{v}\rangle}dz_1\cdots dz_k = \frac{A_{k+1}}{2}e^{-\|\mathbf{v}\|z_0}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right) \tag{3}$$

for $z_0>0$.

Proof (see also [8, 12, 14]). Rename the integration variable w and rewrite the integral as

$$\left(\int_{\substack{\mathbf{w}\in\mathbb{R}^k\\ w_0=z_0}} q_{\mathbf{0}}(w)\,\underbrace{e^{-i\langle \mathbf{w},\mathbf{v}\rangle+i\mathbf{v}w_0}}_{monogenic\ in\ w}\,dw_1\cdots dw_k\right)e^{-i\mathbf{v}z_0}.$$

Then split

$$\begin{aligned}
e^{i\mathbf{v}w_0} &= \frac{1}{2}\Big((e^{i\mathbf{v}w_0}+e^{-i\mathbf{v}w_0})+(e^{i\mathbf{v}w_0}-e^{-i\mathbf{v}w_0})\Big)\\
&= \frac{1}{2}\Big((e^{\|\mathbf{v}\|w_0}+e^{-\|\mathbf{v}\|w_0})+\frac{i\mathbf{v}}{\|\mathbf{v}\|}(e^{\|\mathbf{v}\|w_0}-e^{-\|\mathbf{v}\|w_0})\Big)\\
&= \frac{1}{2}e^{\|\mathbf{v}\|w_0}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)+\frac{1}{2}e^{-\|\mathbf{v}\|w_0}\left(1-\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)
\end{aligned}$$

so that

$$e^{-i\langle \mathbf{w},\mathbf{v}\rangle+i\mathbf{v}w_0} = \frac{1}{2}e^{-i\langle \mathbf{w},\mathbf{v}\rangle+\|\mathbf{v}\|w_0}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)+\frac{1}{2}e^{-i\langle \mathbf{w},\mathbf{v}\rangle-\|\mathbf{v}\|w_0}\left(1-\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right).$$

Each of these two terms is monogenic in w. For the first term, we consider the integral

$$\int_{\substack{\mathbf{w}\in\mathbb{R}^k\\ w_0=z_0}} q_{\mathbf{0}}(w)\frac{1}{2}e^{-i\langle \mathbf{w},\mathbf{v}\rangle+\|\mathbf{v}\|w_0}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)dw_1\cdots dw_k$$

as a boundary integral for the domain $w_0<z_0$; since $z_0>0$, the pole $w=0$ occurs inside the domain, and in view of the residue of $q_{\mathbf{0}}(w)$ at $w=0$ being 1, we obtain by the residue theorem for this integral $\frac{A_{k+1}}{2}\big(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\big)$.

For the second term, we consider the integral

$$\int_{\substack{\mathbf{w}\in\mathbb{R}^k\\ w_0=z_0}} q_{\mathbf{0}}(w)\frac{1}{2}e^{-i\langle \mathbf{w},\mathbf{v}\rangle-\|\mathbf{v}\|w_0}\left(1-\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)dw_1\cdots dw_k$$

as a boundary integral for the domain $w_0 > z_0$. Since $z_0 > 0$, the pole $w = 0$ does not occur inside the domain. In view of the residue theorem, we obtain zero for this integral. The totality of the Fourier transform is then given by

$$\begin{aligned}
&\frac{A_{k+1}}{2}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)e^{-i\mathbf{v}z_0}\\
&\quad=\frac{A_{k+1}}{2}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)\left(\frac{1}{2}e^{-\|\mathbf{v}\|z_0}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)+\frac{1}{2}e^{\|\mathbf{v}\|z_0}\left(1-\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)\right)\\
&\quad=\frac{A_{k+1}}{2}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)e^{-\|\mathbf{v}\|z_0}.
\end{aligned}$$

and the lemma is proved ∎

Lemma 2. *For $q_{\mathbf{0}}$, the integral representation*

$$q_{\mathbf{0}}(z)=\frac{A_{k+1}}{(2\pi)^k}\int_{\mathbf{v}\in\mathbb{R}^k}e^{i\langle\mathbf{z},\mathbf{v}\rangle-\|\mathbf{v}\|z_0}\frac{1}{2}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)dv_1\cdots dv_k \tag{4}$$

holds for $z_0 > 0$.

Proof. This follows at once from (3) by applying the inverse Fourier transform over $\mathbb{R}^k$ ∎

Lemma 3. *For $q_{\mathbf{0}}$, the real plane wave integral representation*

$$q_{\mathbf{0}}(z)=\frac{A_{k+1}}{2(2\pi)^k}\int_{\mathbf{v}\in\mathbb{R}^k}e^{-\frac{\mathbf{v}\langle\mathbf{z},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|z_0}dv_1\cdots dv_k \tag{5}$$

holds for $z_0 > 0$ where the exponential of a paravector is the one defined by (2).

Proof. Since i does not occur on the left-hand side of (4), the formula must also hold with i replaced by $-i$. Adding this to the original (4) yields an integrand of the form

$$\frac{1}{2}\left(1+\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)e^{i\langle\mathbf{z},\mathbf{v}\rangle-\|\mathbf{v}\|z_0}+\frac{1}{2}\left(1-\frac{i\mathbf{v}}{\|\mathbf{v}\|}\right)e^{-i\langle\mathbf{z},\mathbf{v}\rangle-\|\mathbf{v}\|z_0}=e^{-\frac{\mathbf{v}\langle\mathbf{z},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|z_0}.$$

and the statement is proved ∎

Theorem 1. *Let $w^{(0)},\dots,w^{(k)}$ be unit paravectors of $\mathbb{R}^{k+1}$, and let n be a multi-index. Then for $z_0 > 0$ we have*

$$\begin{aligned}
&\langle w^{(0)},D_z\rangle^{n_0}\cdots\langle w^{(k)},D_z\rangle^{n_k}q_{\mathbf{0}}(z)\\
&\quad=\frac{A_{k+1}}{2(2\pi)^k}\int_{\mathbf{v}\in\mathbb{R}^k}\Big(\frac{\mathbf{v}\langle\mathbf{w}^{(0)},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|w_0^{(0)}\Big)^{n_0}\cdots\Big(\frac{\mathbf{v}\langle\mathbf{w}^{(k)},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|w_0^{(k)}\Big)^{n_k}\\
&\quad\quad\times e^{-\frac{\mathbf{v}\langle\mathbf{z},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|z_0}dv_1\cdots dv_k.
\end{aligned}$$

In particular, choosing $n_0 = 0$ and $w_j = e_j$, we obtain for $q_{\mathbf{n}}(z)$ the formula

$$q_{\mathbf{n}}(z)=\frac{A_{k+1}}{2(2\pi)^k}\int_{\mathbf{v}\in\mathbb{R}^k}\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)^{|\mathbf{n}|}\mathbf{v}^{\mathbf{n}}e^{-\frac{\mathbf{v}\langle\mathbf{z},\mathbf{v}\rangle}{\|\mathbf{v}\|}-\|\mathbf{v}\|z_0}dv_1\cdots dv_k$$

for $z_0 > 0$.

Proof. This follows directly by applying the derivations $\langle w^{(j)},D_z\rangle$ repeatedly in (5) ∎

Lemma 4. *Let $w^{(0)}, \ldots, w^{(k)}$ be unit paravectors of $\mathbb{R}^{k+1}$, and let n be a multi-index. Then*

$$\left\| \langle w^{(0)}, D_z\rangle^{n_0} \cdots \langle w^{(k)}, D_z\rangle^{n_k} q_{\mathbf{0}}(z) \right\|_{z=z_0} \leq \frac{k(k+1)\cdots(k+|n|-1)}{|z_0|^{k+|n|}}.$$

Proof. Because $q_{\mathbf{0}}(-z) = -q_{\mathbf{0}}(z)$ we can consider without loss of generality the domain $z_0 > 0$. Specializing the result of Theorem 1 to the case where $z = z_0$ so that $\mathbf{z} = \mathbf{0}$,

$$\begin{aligned} &\langle w^{(0)}, D_z\rangle^{n_0} \cdots \langle w^{(k)}, D_z\rangle^{n_k} q_{\mathbf{0}}(z)\big|_{z=z_0} \\ &\quad = \frac{A_{k+1}}{2(2\pi)^k} \int_{\mathbf{v}\in\mathbb{R}^k} \Big(\frac{\mathbf{v}\langle \mathbf{w}^{(0)}, \mathbf{v}\rangle}{\|\mathbf{v}\|} - \|\mathbf{v}\| w_0^{(0)} \Big)^{n_0} \cdots \Big(\frac{\mathbf{v}\langle \mathbf{w}^{(k)}, \mathbf{v}\rangle}{\|\mathbf{v}\|} - \|\mathbf{v}\| w_0^{(k)} \Big)^{n_k} \\ &\quad\quad \times e^{-\|\mathbf{v}\| z_0} dv_1 \ldots dv_k. \end{aligned}$$

For $j = 0, \ldots, k$,

$$\begin{aligned} \left\| \frac{\mathbf{v}\langle \mathbf{w}^{(j)}, \mathbf{v}\rangle}{\|\mathbf{v}\|} - \|\mathbf{v}\| w_0^{(j)} \right\|^2 &= \langle \mathbf{w}^{(j)}, \mathbf{v}\rangle^2 + \|\mathbf{v}\|^2 (w_0^{(j)})^2 \\ &\leq \|\mathbf{w}^{(j)}\|^2 \|\mathbf{v}\|^2 + \|\mathbf{v}\|^2 (w_0^{(j)})^2 \\ &= \|\mathbf{v}\|^2 \end{aligned}$$

so that

$$\left\| \langle w^{(0)}, D_z\rangle^{n_0} \cdots \langle w^{(k)}, D_z\rangle^{n_k} q_{\mathbf{0}}(z) \right\|_{z=z_0} \leq \frac{A_{k+1}}{2(2\pi)^k} \int_{\mathbf{w}\in\mathbb{R}^k} \|\mathbf{v}\|^{|n|} e^{-\|\mathbf{v}\| z_0} dv_1 \cdots dv_k.$$

Now consider (5) in the case where $z = z_0$:

$$\frac{1}{z_0^k} = q_{\mathbf{0}}(z_0) = \frac{A_{k+1}}{2(2\pi)^k} \int_{\mathbf{v}\in\mathbb{R}^k} e^{-\|\mathbf{v}\| z_0} dv_1 \cdots dv_k \qquad (z_0 > 0)$$

and apply the $|n|$-th derivative with respect to z_0 to get

$$(-1)^{|n|} \frac{k(k+1)\cdots(k+|n|-1)}{z_0^{k+|n|}} = \frac{A_{k+1}}{2(2\pi)^k} \int_{\mathbf{v}\in\mathbb{R}^k} (-1)^{|n|} \|\mathbf{v}\|^{|n|} e^{-\|\mathbf{v}\| z_0} dv_1 \cdots dv_k$$

for $z_0 > 0$ from which the stated estimate follows ∎

Theorem 2. *For all $(n_0, n_1, \ldots, n_k) \in \mathbb{N}^{k+1}$,*

$$\left\| \frac{\partial^{n_0+n_1+\ldots+n_k}}{\partial z_0^{n_0} \partial z_1^{n_1} \cdots \partial z_k^{n_k}} q_{\mathbf{0}}(z) \right\| \leq \frac{k(k+1)\cdots(k+|n|-1)}{\|z\|^{k+|n|}}.$$

Proof. The previous lemma proves this fact for all z in the positive e_0 direction by taking $w^{(0)} = e_0, \ldots, w^{(k)} = e_k$. For any other direction defined by a unit vector $\hat{z}$, consider a rotation that maps $\hat{z}$ to e_0 and take for $w^{(j)}$ the image of e_j to prove the estimate in that direction ∎

3.2 A recurrence formula in terms of permutational products and applications. In [10] the following recurrence formula has been proved providing a representation formula of the $q_{\mathbf{n}}$ functions in terms of permutational products. We quote:

Theorem 3. *Let $\alpha \in \{1,\dots,k\}$ and $\mathbf{n} \in \mathbb{N}_0^k$. Then*

$$\begin{aligned} q_{\mathbf{n}+\tau(\alpha)}(z) = \sum_{\mathbf{0}\le\mathbf{j}\le\mathbf{n}} \binom{\mathbf{n}}{\mathbf{j}} |\mathbf{j}|!\, q_{\mathbf{n}-\mathbf{j}}(z) \\ \cdot \Big\{ \frac{k-1}{2} [\overline{z}^{-1}e_k]^{j_k} \times \cdots \times [\overline{z}^{-1}e_1]^{j_1} \cdot (\overline{z}^{-1}e_\alpha) \\ + (-1)^{|\mathbf{j}|+1} \frac{k+1}{2} (e_\alpha z^{-1}) \cdot [e_1 z^{-1}]^{j_1} \times \cdots \times [e_k z^{-1}]^{j_k} \Big\}. \end{aligned} \tag{6}$$

For the detailed proof we refer to [10].

Consequences and Applications.

1. By Theorem 3 and (1) we further observe that every function being monogenic in an annular domain can be written in terms of permutational products of a finite number of the special hypercomplex variables $Z_q = z_q - z_0 e_q, \zeta_q = \overline{z}^{-1} e_q$ and $\eta_q = e_q z^{-1}$. This is the analogue to the fact that every meromorphic function in the complex plane can be represented in terms of products of z and z^{-1}.

2. For multi-indices $\mathbf{n} \in \mathbb{N}_0^k$ formula (6) can be used (cf. [10]) to deduce immediately the estimate of Theorem 2 by applying only elementary calculations:

Theorem 4. *For all multi-indices $\mathbf{n} \in \mathbb{N}_0^k$ the estimate*

$$\left\| \frac{\partial^{|\mathbf{n}|}}{\partial \mathbf{z}^{\mathbf{n}}} q_{\mathbf{0}}(z) \right\| \le \frac{k(k+1)\cdots(k+|\mathbf{n}|-1)}{\|z\|^{k+|\mathbf{n}|}} \tag{7}$$

holds for all $z \in \mathcal{A}_{k+1}$.

Proof. At first one considers the special case where $\mathbf{n} = (n,0,\dots,0)$. By a direct calculation one verifies that (7) is true for $n=0$ and $n=1$. Now, we assume $n \ge 1$ and apply Theorem 3 for the special case $\mathbf{n} = (n,0,\dots,0)$ which leads to

$$\begin{aligned} &\left\| \frac{\partial^{n+1}}{\partial z_1^{n+1}} q_{\mathbf{0}}(z) \right\| \\ &= \left\| \sum_{j=0}^{n} \binom{n}{j} j!\, q_{(n-j)\tau(1)}(z) \Big[\frac{k-1}{2} (\overline{z}^{-1}e_1)^{j+1} + (-1)^{j+1} \frac{k+1}{2} (e_1 z^{-1})^{j+1} \Big] \right\| \\ &\le \frac{k}{\|z\|^{k+n+1}}\, n! \left[\sum_{j=0}^{n} \binom{(k-1)+n-j}{n-j} \right] \\ &= k(k+1)\cdots(k+n) \frac{1}{\|z\|^{k+n+1}}. \end{aligned}$$

With the estimates

$$\big\| (e_1 z^{-1}) \cdot [e_1 z^{-1}]^{j_1} \times \cdots \times [e_k z^{-1}]^{j_k} \big\| \le \|e_1 z^{-1}\|^{|\mathbf{j}|+1}$$
$$\big\| [\overline{z}^{-1} e_k]^{j_k} \times \cdots \times [\overline{z}^{-1} e_1]^{j_1} \cdot (\overline{z}^{-1} e_1) \big\| \le \|\overline{z}^{-1} e_1\|^{|\mathbf{j}|+1}$$

and the formula $\sum_{\mathbf{j}\in\mathbb{N}_0^k, |\mathbf{j}|=j} \binom{n_1}{j_1} \cdots \binom{n_k}{j_k} = \binom{|\mathbf{n}|}{j}$ together with Theorem 3 we infer further with a simple induction argument that $\|q_{\mathbf{n}}(z)\| \le \|q_{|\mathbf{n}|,0,\cdots,0}(z)\|$ for all $\mathbf{n} \in \mathbb{N}_0^k$ ∎

Remarks. Formula (7) provides actually a more precise estimate than that given in [1]. The estimate in (7) is furthermore stronger than the estimate proved in [5] by R. Fueter for the quaternionic case, i.e.

$$\|q_{\mathbf{n}}(z)\| \leq (|\mathbf{n}|+2)!\,\|z\|^{-(|\mathbf{n}|+3)} \tag{8}$$

proved in [5] by R. Fueter for the quaternionic case. R. Fueter's method for his proof is based on the formula

$$q_{\mathbf{n}}(\zeta) = \zeta^{-1}\frac{\partial^{|\mathbf{n}|}}{\partial \mathbf{z}^{\mathbf{n}}}\left[\Delta_z\{(z\zeta^{-1})^{n+2}\}\right] \qquad (z \in \mathbb{H}, \zeta \in \mathbb{H}\backslash\{0\}) \tag{9}$$

where Δ_z denotes the Laplace operator with respect to the variable z.

Estimate (7) plays a crucial role in the analysis of convergence of the monogenic Eisenstein series. For details see [8 - 11].

3.3 Closed formula as a finite sum. Now we want to deduce a closed formula for the functions $q_{\mathbf{n}}$. To this end we prove

Lemma 5. *Let f be a C^∞ function of a single real variable x. Then*

$$\left(\frac{d}{dx}\right)^n f(x) = \sum_{0\leq 2p\leq n} \frac{(2x)^{n-2p}n!}{(n-2p)!\,p!}\left(\frac{d}{d(x^2)}\right)^{n-p} f(x). \tag{10}$$

Proof. It is immediate to verify that

$$\left(\frac{d}{dx}\right) f(x) = 2x\left(\frac{d}{d(x^2)}\right) f(x)$$

$$\left(\frac{d}{dx}\right)^2 f(x) = 2\left(\frac{d}{d(x^2)}\right) f(x) + 4x^2\left(\frac{d}{d(x^2)}\right)^2 f(x)$$

and, in general,

$$\left(\frac{d}{dx}\right)^n f(x) = \sum_{r=0}^{+\infty} c_{n,r}(x)\left(\frac{d}{d(x^2)}\right)^r f(x) \tag{11}$$

where only a finite number of terms are non-zero, and the $c_{n,q}$ are functions of x that must still be determined. Applying equation (11) to $f(x) = e^{ax^2}$, we obtain

$$\left(\frac{d}{dx}\right)^n e^{ax^2} = \left(\sum_{r=0}^{+\infty} c_{n,r}(x)a^r\right) e^{ax^2}. \tag{12}$$

From this equation, the $c_{n,r}$ could be obtained in terms of Hermite polynomial coefficients. We complete the proof directly, though. We multiply both sides of equation (12) by $\frac{b^n}{n!}$ and sum over $n \geq 0$; on the left-hand side this is a Taylor expansion of $e^{a(x+b)^2}$ at $b = 0$, so we get

$$e^{a(x+b)^2} = \left(\sum_{n,r=0}^{+\infty} c_{n,r}(x)\frac{a^r b^n}{n!}\right) e^{ax^2}. \tag{13}$$

Dividing both sides by e^{ax^2} gives

$$e^{ab(2x+b)} = \sum_{n,r=0}^{+\infty} c_{n,r}(x) \frac{a^r b^n}{n!} \tag{14}$$

and, introducing the series form of the exponential on the left-hand side,

$$\sum_{q=0}^{+\infty} \frac{a^q b^q (2x+b)^q}{q!} = \sum_{n,r=0}^{+\infty} c_{n,r}(x) \frac{a^r b^n}{n!}. \tag{15}$$

Expanding $(2x+b)^q$ herein using the binomial formula yields

$$\sum_{0 \le p \le q < \infty} \frac{a^q b^{p+q} (2x)^{q-p}}{(q-p)!\, p!} = \sum_{n,r=0}^{+\infty} c_{n,r}(x) \frac{a^r b^n}{n!}. \tag{16}$$

Identifying terms with the same powers of a and b on both sides, we find $p+q=n$ and $q=r$, i.e. $q-p=n-2p$, so that

$$\sum_{r=0}^{+\infty} c_{n,r}(x) a^r = \sum_{0 \le 2p \le n} \frac{(2x)^{n-2p} n!}{(n-2p)!\, p!} a^{n-p}$$

from which equation (10) follows at once through equation (11) ∎

Theorem 5. *Let $f(\|z\|)$ be a C^∞ radial function. Then*

$$\partial_{\mathbf{n}}(f(\|z\|)) = \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} \frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}} \mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\, \mathbf{p}!} \left(\frac{d}{d(\|z\|^2)} \right)^{|\mathbf{n}|-|\mathbf{p}|} f(\|z\|).$$

Proof. Applying Lemma 5 to each of the coordinates $z_1, \ldots, z_k$ for the $n_1, \ldots, n_k$ -th derivative, respectively, we find

$$\partial_{\mathbf{n}}(f(\|z\|)) = \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} \frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}} \mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\, \mathbf{p}!} \left(\frac{\partial}{\partial(z_1^2)} \right)^{n_1-p_1} \cdots \left(\frac{\partial}{\partial(z_k^2)} \right)^{n_k-p_k} f(\|z\|).$$

Since $f(\|z\|)$ is radial, $f(\|z\|) = g(\|z\|^2) = g(z_0^2 + \ldots + z_k^2)$, so that all derivatives with respect to a z_q^2 coincide with the derivative with respect to $\|z\|^2$, and therefore

$$\left(\frac{\partial}{\partial(z_1^2)} \right)^{n_1-p_1} \cdots \left(\frac{\partial}{\partial(z_k^2)} \right)^{n_k-p_k} f(\|z\|) = \left(\frac{d}{d(\|z\|^2)} \right)^{|\mathbf{n}|-|\mathbf{p}|} f(\|z\|)$$

from which the theorem follows ∎

Example. When applied to a radial function,

$$
\begin{aligned}
\partial_{z_1^2 z_2^3 z_3^5} &= 1024 z_1^2 z_2^3 z_3^5 \Big(\frac{\partial}{\partial(\|z\|^2)}\Big)^{10} \\
&\quad + 512\big(10 z_1^2 z_2^2 + 3 z_1^2 z_3^2 + z_2^2 z_3^2\big) z_2 z_3^3 \Big(\frac{\partial}{\partial(\|z\|^2)}\Big)^{9} \\
&\quad + 256\big(15 z_1^2 z_2^2 + 30 z_1^2 z_3^2 + 10 z_2^2 z_3^2 + 3 z_3^4\big) z_2 z_3 \Big(\frac{\partial}{\partial(\|z\|^2)}\Big)^{8} \\
&\quad + 1920\big(3 z_1^2 + z_2^2 + 2 z_3^2\big) z_2 z_3 \Big(\frac{\partial}{\partial(\|z\|^2)}\Big)^{7} \\
&\quad + 2880 z_2 z_3 \Big(\frac{p}{\partial(\|z\|^2)}\Big)^{6}.
\end{aligned}
$$

Theorem 6. *For $k > 1$, we have*

$$
\begin{aligned}
& q_{\mathbf{n}}(z) \\
&= -\frac{1}{k-1}\overline{D}_z \left(\sum_{\mathbf{0}\le 2\mathbf{p}\le \mathbf{n}} \frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}}\mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!}(-1)^{|\mathbf{n}|-|\mathbf{p}|} \quad \frac{k-1}{2} \quad {}_{|\mathbf{n}|-|\mathbf{p}|} \; \frac{1}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} \right) \qquad (17) \\
&= \frac{1}{k-1} \sum_{\mathbf{0}\le 2\mathbf{p}\le \mathbf{n}} \frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}}\mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!}(-1)^{|\mathbf{n}|-|\mathbf{p}|} \quad \frac{k-1}{2} \quad {}_{|\mathbf{n}|-|\mathbf{p}|} \; \frac{1}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} \qquad (18) \\
&\quad \cdot \quad \frac{k-1+2|\mathbf{n}|-2|\mathbf{p}|}{z} + \sum_{q=1}^{k} \frac{n_q - 2p_q}{z_q} e_q \Bigg)
\end{aligned}
$$

Proof. Since

$$
q_{\mathbf{0}}(z) = \frac{\overline{z}}{|z|^{k+1}} = -\frac{1}{k-1}\overline{D}_z \quad \frac{1}{\|z\|^{k-1}} \quad ,
$$

the first equality follows from application of Theorem 5 to $\frac{1}{\|z\|^{k-1}} = (\|z\|^2)^{-\frac{k-1}{2}}$. In the second equality, the first term originates in the application of $\overline{D}_z$ to the factor $\frac{1}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}}$, the second in its application to the factor $(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}}$ ∎

3.4 A further closed representation formula involving permutational products. According to Theorem 3 the functions $q_{\mathbf{n}}(z)$ can be written in terms of permutational products involving the variables $\zeta_q = \overline{z}^{-1} e_q$ and $\eta_q = e_q z^{-1}$ multiplied with $q_{\mathbf{0}}$ from the left side. The question that arises is whether it is possible to reduce the number of hypercomplex variables used in the permutational products. In this section we derive a further, essentially different formula for $q_{\mathbf{n}}(z)$ involving also permutational products but using fewer hypercomplex variables.

The starting point is the inversion formula of [3] by which we know that there must be uniquely defined Clifford numbers $a_{\mathbf{n},\mathbf{m}}$ such that

$$
q_{\mathbf{0}}(z) q_{\mathbf{n}}(z^{-1}) = \sum_{\mathbf{0}\le \mathbf{m}, |\mathbf{m}|=|\mathbf{n}|} V_{\mathbf{m}}(z) a_{\mathbf{n},\mathbf{m}}. \qquad (19)
$$

The functions $V_{\mathbf{m}}$ can be written themselves in terms of permutational products of the variables $Z_q = z_q - z_0 e_q$ as $V_{\mathbf{m}}(z) = \frac{[Z_1]^{m_1} \times \cdots \times [Z_k]^{m_k}}{\mathbf{m}!}$, or $\frac{\mathbf{Z}^{\mathbf{m}}}{\mathbf{m}!}$ for short. Then one can write

$$q_{\mathbf{n}}(z) = (-1)^{|\mathbf{n}|} \frac{q_{\mathbf{0}}(z)}{\|z\|^{2|\mathbf{n}|}} \sum_{\mathbf{0} \le \mathbf{m}, |\mathbf{m}| = |\mathbf{n}|} \frac{1}{\mathbf{m}!} \overline{\mathbf{Z}}^{\mathbf{m}} a_{\mathbf{n},\mathbf{m}} \tag{20}$$

where $\overline{\mathbf{Z}}^{\mathbf{m}} = [\overline{Z}_1]^{m_1} \times \cdots \times [\overline{Z}_k]^{m_k}$ with $\overline{Z}_q = z_q + z_0 e_q$ for $q = 1, \ldots, k$. With the closed representation formula one can now determine explicitly the coefficients $a_{\mathbf{n},\mathbf{m}}$, which leads to an explicit closed formula for $q_{\mathbf{n}}(z)$ in terms of permutational products with the variables $\overline{Z}_q$ multiplied with $\frac{q_{\mathbf{0}}(z)}{\|z\|^{2|\mathbf{n}|}}$ from the left. In the sequel we set:

$$a(k, \mathbf{n}, \mathbf{p}) = (-1)^{|\mathbf{n}|-|\mathbf{p}|} \frac{1}{k-1} \frac{2^{|\mathbf{n}|-2|\mathbf{p}|} \mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\, \mathbf{p}!} \left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|} \cdot (k-1+2|\mathbf{n}|-2|\mathbf{p}|) \tag{21}$$

$$b_q(k, \mathbf{n}, \mathbf{p}) = (-1)^{|\mathbf{n}|-|\mathbf{p}|} \frac{1}{k-1} \frac{2^{|\mathbf{n}|-2|\mathbf{p}|} \mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\, \mathbf{p}!} \left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|} \cdot (n_q - 2p_q) \quad (q = 1, \ldots, k). \tag{22}$$

Theorem 7. *For all multi-indices $\mathbf{n} \in \mathbb{N}_0^k$ and $z \in \mathcal{A}_{k+1} \backslash \{0\}$ the representation formula*

$$q_{\mathbf{n}}(z) = \frac{q_{\mathbf{0}}(z)}{\|z\|^{2|\mathbf{n}|}} \Bigg\{ \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} \Bigg(a(k, \mathbf{n}, \mathbf{p}) \sum_{|\mathbf{r}| = |\mathbf{p}|} \overline{\mathbf{Z}}^{\mathbf{n}-2\mathbf{p}+2\mathbf{r}} \frac{|\mathbf{r}|!}{\mathbf{r}!} \Bigg) + \sum_{j,q=1}^{k} \Bigg(b_q(k, \mathbf{n}, \mathbf{p}) \sum_{|\mathbf{r}| = |\mathbf{p}|} \overline{\mathbf{Z}}^{\mathbf{n}-2\mathbf{p}+2\mathbf{r}+\tau(j)-\tau(q)} \frac{|\mathbf{r}|!}{\mathbf{r}!} e_j e_q \Bigg) \Bigg\} \tag{23}$$

holds.

Proof. With abbreviations (21) and (22) we can write the closed representation formula as

$$q_{\mathbf{n}}(z) = \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} a(k, \mathbf{n}, \mathbf{p}) \frac{\mathbf{z}^{\mathbf{n}-2\mathbf{p}}}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} \frac{1}{z} + \sum_{q=1}^{k} b_q(k, \mathbf{n}, \mathbf{p}) \frac{\mathbf{z}^{\mathbf{n}-2\mathbf{p}-\tau(q)}}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} e_q.$$

For a vector $\mathbf{z} \in \mathbb{R}^k \backslash \{\mathbf{0}\}$ we get

$$q_{\mathbf{0}}(\mathbf{z}) q_{\mathbf{n}}(\mathbf{z}^{-1}) = (-1)^{|\mathbf{n}|} \Bigg(\sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} a(k, \mathbf{n}, \mathbf{p}) \mathbf{z}^{\mathbf{n}-2\mathbf{p}} \|\mathbf{z}\|^{2|\mathbf{p}|} + \sum_{q=1}^{k} b_q(k, \mathbf{n}, \mathbf{p}) \mathbf{z} \cdot \mathbf{z}^{\mathbf{n}-2\mathbf{p}-\tau(q)} \|\mathbf{z}\|^{2|\mathbf{p}|} e_q \Bigg)$$

which, after having applied the multinomial formula on the $\|\mathbf{z}\|^{2|\mathbf{p}|}$, leads to

$$q_{\mathbf{0}}(\mathbf{z})q_{\mathbf{n}}(\mathbf{z}^{-1}) = (-1)^{|\mathbf{n}|}\Bigg\{ \sum_{\mathbf{0}\leq 2\mathbf{p}\leq \mathbf{n}} a(k,\mathbf{n},\mathbf{p}) \sum_{|\mathbf{r}|=|\mathbf{p}|} \mathbf{z}^{\mathbf{n}-2\mathbf{p}+2\mathbf{r}} \frac{|\mathbf{r}|!}{\mathbf{r}!}$$
$$+ \sum_{q=1}^{k} b_q(k,\mathbf{n},\mathbf{p}) \sum_{j=1}^{k} \sum_{|\mathbf{r}|=|\mathbf{p}|} \mathbf{z}^{\mathbf{n}-2\mathbf{p}+2\mathbf{r}+\tau(j)-\tau(q)} \frac{|\mathbf{r}|!}{\mathbf{r}!} e_j e_q \Bigg\}.$$

Note that the function $q_{\mathbf{0}}(\mathbf{z})q_{\mathbf{n}}(\mathbf{z}^{-1})$ is a polynomial in $\mathbb{R}^k$, so it has a unique left monogenic Cauchy-Kowalewski extension to $\mathcal{A}_{k+1}$. Since the Cauchy-Kowalewski extension of $\mathbf{z}^{\mathbf{n}}$ is precisely $\mathbf{n}!\, V_{\mathbf{n}}(z)$, we get for all $z \in \mathcal{A}_{k+1}$

$$q_{\mathbf{0}}(z)q_{\mathbf{n}}(z^{-1}) = (-1)^{|\mathbf{n}|}\Bigg\{ \sum_{\mathbf{0}\leq 2\mathbf{p}\leq \mathbf{n}} a(k,\mathbf{n},\mathbf{p}) \sum_{|\mathbf{r}|=|\mathbf{p}|} V_{\mathbf{n}-2\mathbf{p}+2\mathbf{r}}(z)(\mathbf{n}-2\mathbf{p}+2\mathbf{r})! \frac{|\mathbf{r}|!}{\mathbf{r}!}$$
$$+ \sum_{q=1}^{k} b_q(k,\mathbf{n},\mathbf{p}) \sum_{j=1}^{k} \sum_{|\mathbf{r}|=|\mathbf{p}|} V_{\mathbf{n}-2\mathbf{p}+2\mathbf{r}+\tau(j)-\tau(q)}(z)$$
$$\cdot \; \mathbf{n}-2\mathbf{p}+2\mathbf{r}+\tau(j)-\tau(q)\; ! \frac{|\mathbf{r}|!}{\mathbf{r}!} e_j e_q \Bigg\}.$$

Multiplying this relation from the left by $q_{\mathbf{0}}(z)^{-1}$ and then replacing z by z^{-1} leads to the stated result ∎

Theorem 8. *The coefficients $a_{\mathbf{n},\mathbf{m}}$ of* (19)*, where $|\mathbf{n}| = |\mathbf{m}|$, are given explicitly in the entangled permutational product form*

$$a_{\mathbf{n},\mathbf{m}} = (-1)^{|\mathbf{m}|} \sum_{j=0}^{|\mathbf{m}|} \frac{1}{j!\,(|\mathbf{m}|-j)!} \left(\frac{k-1}{2}\right)_j \left(\frac{k+1}{2}\right)_{|\mathbf{m}|-j}$$
$$\cdot \sum_{\sigma_1,\sigma_2} \left(e_{\sigma_1(1)}e_{\sigma_2(1)}\cdots e_{\sigma_1(j)}e_{\sigma_2(j)}\right)\left(e_{\sigma_2(j+1)}e_{\sigma_1(j+1)}\cdots e_{\sigma_2(|\mathbf{n}|)}e_{\sigma_1(|\mathbf{m}|)}\right)$$

where σ_1 ranges over all mappings from $\{1,\ldots,|\mathbf{m}|\}$ to $\mathbb{N}_0$ that reach m_1 times the value $1,\ldots,m_k$ times the value k, and σ_2 is similar but for $\mathbf{n}$ instead of $\mathbf{m}$.

Proof. In (19) only the constants $a_{\mathbf{n},\mathbf{m}}$ with $|\mathbf{n}| = |\mathbf{m}|$ occur, so we define $a_{\mathbf{n},\mathbf{m}} = 0$ when $|\mathbf{n}| \neq |\mathbf{m}|$ and rewrite (19) as

$$q_{\mathbf{0}}(z)q_{\mathbf{n}}(z^{-1}) = \sum_{\mathbf{0}\leq \mathbf{m}} V_{\mathbf{m}}(z)a_{\mathbf{n},\mathbf{m}}.$$

Substituting $z = \mathbf{u}$, i.e. a pure vector, non-zero and of sufficiently small norm, leads in view of $V_{\mathbf{m}}(\mathbf{u}) = \mathbf{u}^{\mathbf{m}}$ to

$$q_{\mathbf{0}}(\mathbf{u})q_{\mathbf{n}}(\mathbf{u}^{-1}) = \sum_{\mathbf{0}\leq \mathbf{m}} \frac{1}{\mathbf{m}!} \mathbf{u}^{\mathbf{m}} a_{\mathbf{n},\mathbf{m}}.$$

Let $\mathbf{v}$ also be a pure vector of sufficiently small norm; multiply the last result by $\frac{\mathbf{v}^{\mathbf{n}}}{\mathbf{n}!}$ and sum over all $\mathbf{n}$: in view of the Taylor expansion formula, we get

$$q_{\mathbf{0}}(\mathbf{u})q_{\mathbf{0}}(\mathbf{u}^{-1}+\mathbf{v}) = q_{\mathbf{0}}(\mathbf{u})\sum_{\mathbf{0}\le\mathbf{n}} q_{\mathbf{n}}(\mathbf{u}^{-1})\frac{\mathbf{v}^{\mathbf{n}}}{\mathbf{n}!} = \sum_{\mathbf{0}\le\mathbf{m},\mathbf{n}} \frac{\mathbf{u}^{\mathbf{m}}\mathbf{v}^{\mathbf{n}}a_{\mathbf{n},\mathbf{m}}}{\mathbf{n}!}.$$

The left-hand side simplifies to $\frac{1+\mathbf{uv}}{\|1+\mathbf{uv}\|^{k+1}}$, so

$$a_{\mathbf{n},\mathbf{m}} = \left(\frac{\partial^{|\mathbf{m}|+|\mathbf{n}|}}{\partial\mathbf{u}^{\mathbf{m}}\partial\mathbf{v}^{\mathbf{n}}}\left(\frac{1+\mathbf{uv}}{\|1+\mathbf{uv}\|^{k+1}}\right)\right)_{\mathbf{u}=\mathbf{v}=\mathbf{0}}.$$

To compute this derivative, we note that $\mathbf{uv}$ and $\mathbf{vu}$ commute. So (for $\mathbf{u},\mathbf{v}$ near $\mathbf{0}$)

$$\begin{aligned}\frac{1+\mathbf{uv}}{\|1+\mathbf{uv}\|^{k+1}} &= (1+\mathbf{uv})^{-\frac{k-1}{2}}(1+\mathbf{vu})^{-\frac{k+1}{2}}\\ &= \sum_{p=0}^{+\infty}(-1)^p\sum_{j=0}^{p}\left(\frac{k-1}{2}\right)_j\left(\frac{k+1}{2}\right)_{p-j}\frac{(\mathbf{uv})^j(\mathbf{vu})^{p-j}}{j!\,(p-j)!}.\end{aligned}$$

Considering the $p=|\mathbf{m}|$ term then leads to the stated result ∎

Theorem 9. *For $q_{\mathbf{n}}(z)$, the alternative representation formula*

$$\begin{aligned}q_{\mathbf{n}}(z) = -\frac{1}{k-1}\overline{D}_z\Bigg(&\frac{1}{\|z\|^{k-1+2|\mathbf{n}|}}\\ &\cdot\sum_{\substack{\mathbf{0}\le 2\mathbf{p}\le\mathbf{n}\\ \mathbf{0}\le\mathbf{r},|\mathbf{r}|=|\mathbf{p}|}}\frac{2^{|\mathbf{n}|-2|\mathbf{p}|}(-1)^{|\mathbf{n}|-|\mathbf{p}|}\mathbf{n}!\,|\mathbf{r}|!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!\,\mathbf{r}!}\left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|}\\ &\cdot(\mathbf{n}+2\mathbf{p}-2\mathbf{r})!\,\mathrm{Sc}(V_{\mathbf{n}+2\mathbf{p}-2\mathbf{r}}(z))\Bigg)\end{aligned}$$

holds.

Proof. Define $h_{\mathbf{n}}(z) = \frac{\partial^{|\mathbf{n}|}}{\partial z^{\mathbf{n}}}\|z\|^{-(k-1)}$. We proved (17) by applying Theorem 5 to $\frac{1}{\|z\|^{k-1}}$, yielding the result

$$h_{\mathbf{n}}(z) = \sum_{\mathbf{0}\le 2\mathbf{p}\le\mathbf{n}}\frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}}\mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!}(-1)^{|\mathbf{n}|-|\mathbf{p}|}\left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|}\frac{1}{\|z\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}}.$$

The function $h_{\mathbf{n}}(z)$ is a derivative of the harmonic function $\frac{1}{\|z\|^{k-1}}$, hence $\frac{h_{\mathbf{n}}(\overline{z}^{-1})}{\|z\|^{k-1}}$ is also harmonic. Similarly, it follows from $(\frac{\partial}{\partial z_0}\frac{1}{\|z\|^{k-1}})_{z=\mathbf{z}}=0$ that $(\frac{\partial}{\partial z_0}h_{\mathbf{n}}(z))_{z=\mathbf{z}}=0$. In the pure vector case, $z=\mathbf{z}$, we have

$$\frac{h_{\mathbf{n}}(\overline{\mathbf{z}}^{-1})}{\|\mathbf{z}\|^{k-1}} = \sum_{\mathbf{0}\le 2\mathbf{p}\le\mathbf{n}}\frac{(2\mathbf{z})^{\mathbf{n}-2\mathbf{p}}\mathbf{n}!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!}(-1)^{|\mathbf{n}|-|\mathbf{p}|}\left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|}\|\mathbf{z}\|^{2|\mathbf{p}|}$$

or, using the multinomial formula to expand $\|\mathbf{z}\|^{2|\mathbf{p}|}$,

$$\frac{h_{\mathbf{n}}(\overline{\mathbf{z}}^{-1})}{\|\mathbf{z}\|^{k-1}} = \sum_{\substack{0\leq 2\mathbf{p}\leq \mathbf{n}\\ 0\leq \mathbf{r},|\mathbf{r}|=|\mathbf{p}|}} \frac{2^{|\mathbf{n}|-2|\mathbf{p}|}(-1)^{|\mathbf{n}|-|\mathbf{p}|}\mathbf{n}!\,|\mathbf{r}|!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!\,\mathbf{r}!}\left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|}\mathbf{z}^{\mathbf{n}+2\mathbf{p}-2\mathbf{r}}.$$

Since $\mathrm{Sc}(V_{\mathbf{n}+2\mathbf{p}-2\mathbf{r}}(z))$ is harmonic, coincides with $\frac{\mathbf{z}^{\mathbf{n}+2\mathbf{p}-2\mathbf{r}}}{\mathbf{n}+2\mathbf{p}-2\mathbf{r})!}$ when $z_0=0$, and has zero derivative with respect to z_0 there,

$$\frac{h_{\mathbf{n}}(\overline{z}^{-1})}{\|z\|^{k-1}} = \sum_{\substack{0\leq 2\mathbf{p}\leq \mathbf{n}\\ 0\leq \mathbf{r},|\mathbf{r}|=|\mathbf{p}|}} \frac{2^{|\mathbf{n}|-2|\mathbf{p}|}(-1)^{|\mathbf{n}|-|\mathbf{p}|}\mathbf{n}!\,|\mathbf{r}|!}{(\mathbf{n}-2\mathbf{p})!\,\mathbf{p}!\,\mathbf{r}!}\left(\frac{k-1}{2}\right)_{|\mathbf{n}|-|\mathbf{p}|}$$
$$\cdot\,(\mathbf{n}+2\mathbf{p}-2\mathbf{r})!\,\mathrm{Sc}(V_{\mathbf{n}+2\mathbf{p}-2\mathbf{r}}(z)).$$

Because $h_{\mathbf{n}}(z)$ is homogeneous of degree $-(k-1+|\mathbf{n}|)$ in z, the left-hand side simplifies to $h_{\mathbf{n}}(z)\|z\|^{k-1+2|\mathbf{n}|}$. Dividing both sides by $\|z\|^{k-1+2|\mathbf{n}|}$ and relying on $q_{\mathbf{n}}(z)=-\overline{D}_z\frac{h_{\mathbf{n}}(z)}{k-1}$ then proves the stated result ∎

Remark. The expression in Theorem 9 is still related closely to permutational products since

$$\mathbf{m}!\,\mathrm{Sc}(V_{\mathbf{m}}(z)) = \frac{\mathbf{m}!}{2}\big(V_{\mathbf{m}}(z)+\overline{V_{\mathbf{m}}(z)}\big) = \frac{1}{2}\big(\mathbf{Z}^{\mathbf{m}}+\overline{\mathbf{Z}}^{\mathbf{m}}\big).$$

Note that in this formula we use as many variables as in the recurrence relation described in Section 3.2.

4. Applications to a variant of the Riemann zeta function in Clifford Analysis

In the sequel we assume p to be an integer satisfying $1\leq p\leq k+1$. Let $\omega_1,\ldots,\omega_p$ denote $\mathbb{R}$-linear independent paravectors in $\mathcal{A}_{k+1}$. The $\mathbb{Z}$-module

$$\Omega_p = \mathbb{Z}\omega_1+\ldots+\mathbb{Z}\omega_p$$

is then a p-dimensional lattice in $\mathcal{A}_{k+1}$. Like in [8 - 10], we split this lattice into a positive and a negative part. The positive part of Ω_p is said to be the set

$$\begin{aligned}\Omega_p^+ &= \mathbb{N}\omega_1+\mathbb{Z}\omega_2+\mathbb{Z}\omega_3+\ldots+\mathbb{Z}\omega_p\\ &\cup\mathbb{N}\omega_2+\mathbb{Z}\omega_3+\ldots+\mathbb{Z}\omega_p\\ &\vdots\\ &\cup\mathbb{N}\omega_p.\end{aligned}$$

The negative part of the lattice Ω_p is defined by

$$\Omega_p^- = (\Omega_p\backslash\{0\})\backslash\Omega_p^+.$$

We observe that $z\in\Omega_p^+$ if and only if $-z\in\Omega_p^-$ and that $\Omega_p^+\cup\Omega_p^-\cup\{0\}=\Omega_p$. In particular, for $k=1$ and $\omega_1=1$ one has

$$\Omega_1^+=\mathbb{N},\qquad \Omega_1^-=-\mathbb{N},\qquad \Omega_1=\mathbb{N}\cup-\mathbb{N}\cup\{0\}=\mathbb{Z}.$$

With this notation we introduce

Definition 1 (Generalized Riemann zeta function of Clifford analysis in $\mathcal{A}_{k+1}$; cf. [10]). Let $p \in \mathbb{N}$ with $1 \leq p \leq k+1$. Let further $\mathbf{l} \in \mathbb{N}_0^k$ be a multi-index and suppose for $p = k$ that $|\mathbf{l}| \geq 1$ and for $p = k+1$ that $|\mathbf{l}| \geq 2$. Ω_p^+ denotes the positive semi-lattice. Then the generalized Riemann zeta function of Clifford analysis in $\mathcal{A}_{k+1}$ is defined by

$$\zeta_M^{\Omega_p}(\mathbf{l}) = \sum_{\omega \in \Omega_p^+} q_{\mathbf{l}}(\omega). \tag{24}$$

The series converge absolutely under the given conditions. Note that in the case $|\mathbf{l}| \equiv 1(2)$ we obtain

$$2\zeta_M^{\Omega_p}(\mathbf{l}) = \sum_{\omega \in \Omega_p \setminus \{0\}} q_{\mathbf{l}}(\omega).$$

As in [10] it has been shown, there are actually indices $\mathbf{n}$ for which the associated series does not vanish. To see this it is crucial to observe that these series appear as Laurent coefficient of the generalized p-fold periodic monogenic cotangent in the case $p \leq k$ or of the generalized monogenic Weierstraß' $\wp$-function which is $(k+1)$-fold periodic. For detailed information about these multiperiodic monogenic functions we refer to [8, 9, 11]. In [10] it has already been shown that this variant of the Riemann zeta function is closely related to the Epstein zeta function. For convenience we recall (cf. [7, 15]) that the Epstein zeta function associated with a given $p \times p$ positive definite symmetric matrix S is precisely

$$\zeta_S(s) = \sum_{\mathbf{g} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}} (\mathbf{g}^{\mathrm{T}} S \mathbf{g})^{-s} \qquad \left(s \in \mathbb{C}, \Re(s) > \tfrac{p}{2}\right). \tag{25}$$

The Epstein zeta function is nothing else than a special case of the classical multiple complex valued Dirichlet series associated with a real valued polynomial $P(\mathbf{g})$ which is

$$\delta(P(\,\cdot\,), S, s) := \sum_{\mathbf{g} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}} P(\mathbf{g})(\mathbf{g}^{\mathrm{T}} S \mathbf{g})^{-s}. \tag{26}$$

This series is convergent if $\Re(s) - \deg(P) > \frac{p}{2}$.

We also consider the positive part of the Dirichlet series, which is defined by

$$\mathcal{D}(P(\,\cdot\,), S, s) = \sum_{\mathbf{g} \in \mathbb{Z}^{p+}} P(\mathbf{g})(\mathbf{g}^{\mathrm{T}} S \mathbf{g})^{-s} \tag{27}$$

and provides a canonical generalization of the classical Dirichlet series

$$\mathcal{D}(P(\,\cdot\,), s) = \sum_{n \in \mathbb{N}} P(n) n^{-s}$$

when setting $p = 1$ and $S = (1)$. We observe that $\delta(P(\,\cdot\,), S, s) = 2\mathcal{D}(P(\,\cdot\,), S, s)$ if P is an even function and $\delta(P(\,\cdot\,), S, s) = 0$ if P is odd.

As in [10] has been shown,

$$\|\zeta_M^{\Omega_p}(\mathbf{n})\| \leq \prod_{\mu=0}^{|\mathbf{n}|-1} (k+\mu)\, \zeta_{W^{\mathrm{T}}W}\left(\tfrac{1}{2}(k+|\mathbf{n}|)\right)$$

where W is the matrix built with the columns of the coordinates of the generators of the lattice, i.e. $W = (\omega_1, \ldots, \omega_p)$. For the case $\mathbf{n} = \tau(i)$ it has further been shown, by applying Theorem 3, that

$$\zeta_M^{\Omega_p}(\tau(i)) = \frac{k-1}{4} e_i \zeta_{W^\intercal W}(\tfrac{1}{2}(k+1)) + \frac{k+1}{4} \sum_{\omega \in \Omega_p \setminus \{0\}} \frac{\omega e_i \overline{\omega}}{\|\omega\|^{k+3}}.$$

Now we proceed to apply the closed representation formula for the functions $q_{\mathbf{n}}$ deduced in Section 3.3 in order to derive an explicit and closed representation of the generalized Riemann zeta function of Clifford analysis in terms of a finite sum of the classical Dirichlet series associated with Epstein zeta functions. To this end we observe that one can write

$$\frac{\omega^{\mathbf{n}-2\mathbf{p}}}{\|\omega\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} \cdot \sum_{q=1}^{k} \frac{n_q - 2p_q}{\omega_q} e_q = \sum_{q=1}^{k} (n_q - 2p_q) \frac{\omega^{\mathbf{n}-2\mathbf{p}-\tau(q)}}{\|\omega\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} e_q \tag{28}$$

and also

$$\frac{\omega^{\mathbf{n}-2\mathbf{p}}}{\|\omega\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} \frac{1}{\omega} = \frac{\omega^{\mathbf{n}-2\mathbf{n}+\tau(0)}}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} - \sum_{q=1}^{k} \frac{\omega^{\mathbf{n}-2\mathbf{n}+\tau(q)}}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} e_q. \tag{29}$$

Putting this into the representation formula of Subsection 3.3 and using furthermore abbreviations (21) - (22) we obtain

$$\begin{aligned} q_{\mathbf{n}}(\omega) = \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} \Bigg\{ & a(k, \mathbf{n}, \mathbf{p}) \cdot \left[\frac{\omega^{\mathbf{n}-2\mathbf{p}+\tau(0)}}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} - \sum_{q=1}^{k} \frac{\omega^{\mathbf{n}-2\mathbf{p}+\tau(q)}}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} e_q \right] \\ & + \sum_{q=1}^{k} b_q(k, \mathbf{n}, \mathbf{p}) \frac{\omega^{\mathbf{n}-2\mathbf{p}-\tau(q)}}{\|\omega\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} e_q \Bigg\}. \end{aligned} \tag{30}$$

We further introduce the real-valued polynomials

$$\begin{aligned} A_j(\omega) &= \omega^{\mathbf{n}-2\mathbf{p}+\tau(j)} \\ B_j(\omega) &= \omega^{\mathbf{n}-2\mathbf{p}-\tau(j)}. \end{aligned}$$

This leads to the representation

$$\begin{aligned} \zeta_M^{\Omega_p}(\mathbf{n}) = \sum_{\mathbf{0} \le 2\mathbf{p} \le \mathbf{n}} \Bigg\{ & \sum_{\omega \in \Omega_p^+} \frac{a(k, \mathbf{n}, \mathbf{p}) A_0(\omega)}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} \\ & + \sum_{q=1}^{k} \left[\sum_{\omega \in \Omega_p^+} \frac{b_q(k, \mathbf{n}, \mathbf{p}) B_q(\omega)}{\|\omega\|^{k-1+2|\mathbf{n}|-2|\mathbf{p}|}} - \sum_{\omega \in \Omega_p^+} \frac{a(k, \mathbf{n}, \mathbf{p}) A_q(\omega)}{\|\omega\|^{k+1+2|\mathbf{n}|-2|\mathbf{p}|}} \right] e_q \Bigg\}. \end{aligned}$$

Denoting again by W the $p \times (k+1)$ matrix which maps the generators of the lattice $L_p = \mathbb{Z}e_0 + \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_{p-1}$ onto the generators of Ω_p, i.e. $W = (\omega_1\ \omega_2\ \ldots\ \omega_p)$, we can write

$$\begin{aligned}\zeta_M^{\Omega_p}(\mathbf{n}) = \sum_{\mathbf{0}\le 2\mathbf{p}\le\mathbf{n}} \Bigg\{ & \mathcal{D}\Big(a(k,\mathbf{n},\mathbf{p})A_0(W\,\cdot\,), W^{\mathrm{T}}W, k+1+2|\mathbf{n}|-2|\mathbf{p}|\Big) \\ & + \sum_{q=1}^{k} \Big[\mathcal{D}\Big(b_q(k,\mathbf{n},\mathbf{p})B_q(W\,\cdot\,), W^{\mathrm{T}}W, k-1+2|\mathbf{n}|-2|\mathbf{p}|\Big) \\ & - \mathcal{D}\Big(a(k,\mathbf{n},\mathbf{p})A_q(W\,\cdot\,), W^{\mathrm{T}}W, k+1+2|\mathbf{n}|-2|\mathbf{p}|\Big)e_q\Big]\Bigg\}\end{aligned}$$

where $\mathcal{D}$ is the positive part of the real-valued Dirichlet series introduced in (27). In the case $|\mathbf{n}| \equiv 1 \pmod 2$ we get

$$\begin{aligned}\zeta_M^{\Omega_p}(\mathbf{n}) = \sum_{\mathbf{0}\le 2\mathbf{p}\le\mathbf{n}} \Bigg\{ & \delta\Big(a(k,\mathbf{n},\mathbf{p})A_0(W\,\cdot\,), W^{\mathrm{T}}W, k+1+2|\mathbf{n}|-2|\mathbf{p}|\Big) \\ & + \sum_{q=1}^{k} \Big[\delta\Big(b_q(k,\mathbf{n},\mathbf{p})B_q(W\,\cdot\,), W^{\mathrm{T}}W, k-1+2|\mathbf{n}|-2|\mathbf{p}|\Big) \\ & - \delta\Big(a(k,\mathbf{n},\mathbf{p})A_q(W\,\cdot\,), W^{\mathrm{T}}W, k+1+2|\mathbf{n}|-2|\mathbf{p}|\Big)e_q\Big]\Bigg\}\end{aligned}$$

where δ is the Dirichlet series defined in (26).

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