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Level Sets of Hölder Functions and Hausdorff Measures

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Abstract. In this paper we investigate some connections between Hausdorff measures, Hölder functions and analytic sets in terms of images of zero-derivative sets and level sets. We characterize in terms of Hausdorff measures and descriptive complexity subsets $M \subseteq \mathbb{R}$ which are

(1) the image under some $C^{n,\alpha}$ function f of the set of points where the derivatives of first n orders are zero

(2) the set of points where the level sets of some $C^{n,\alpha}$ function are perfect

(3) the set of points where the level sets of some $C^{n,\alpha}$ function are uncountable.

Keywords: *Level sets, $C^{n,\alpha}$ functions*

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1. Introduction

Several authors have studied level sets of continuous functions and smooth functions and, critical sets. For example, Bruckner and Garg [3] and Darji and Morayne [7] have proved results concerning how big is the set of points where the level sets of a "typical" continuous function (in the category sense) and of a typical C^n $(n \geq 1)$ function, respectively, are large. The present author and Darji [5, 6], in terms of Hausdorff measures and descriptive complexity, have characterized subsets $M \subseteq \mathbb{R}$ which are

1) the image under some C^n function f of the set of points where the derivatives of first n orders are zero

2) the set of points where the level sets of some C^n function are perfect, and

3) the set of points where the level sets of some C^n function are uncountable.

In this paper we consider the case of Hölder functions. In Section 2 we "parametrize" the Hausdorff dimension of certain closed subsets of $[0, 1]$ with Hölder functions. In

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Section 3 we characterize the set of points where the level sets of a $C^{n,\alpha}$ function ($1 \le n < \infty, 0 < \alpha \le 1$) are perfect.

It is a very old result of Mazurkiewicz and Sierpinski [12] that a set $M \subseteq [0,1]$ is analytic if and only if it is equal to the set $\{y : f^{-1}(\{y\}) \text{ is uncountable}\}$ for some continuous function f. In Section 4 we characterize such sets M for $C^{n,\alpha}$ functions. At last, in Section 5 we characterize the set of points where the level sets of a Lipschitz function are perfect.

2. Images of zero-derivative sets

In this section we characterize images of zero-derivative sets of Hölder functions. We first need few definitions and some terminology.

Definition 2.1. Let f be a C^n ($1 \le n < \infty$) function on a closed interval $I \subset \mathbb{R}$, $f^{(0)} = f, f^{(i)}$ ($1 \le i \le n$) the $i-$th derivative of f and

$$Z_{(f,n)} = \left\{x \in I : f^{(i)}(x) = 0 \text{ for all } 1 \le i \le n\right\}$$

the so-called *zero-derivative set*. We use $\|f\|_n$ to denote the *n-norm* of f, i.e. $\|f\|_n = \sum_{i=0}^{n} \|f^{(i)}\|$ where $\|\cdot\|$ denotes the supremum norm.

Definition 2.2. If $0 < \alpha \le 1$, we denote by $C^{0,\alpha}(I)$ the space of Hölder functions on a closed interval $I \subset \mathbb{R}$, i.e. the space of functions f such that

$$[f]_{0,\alpha} = \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} < \infty.$$

More generally, we denote by $C^{n,\alpha}(I)$ the space of $C^n(I)$ functions with Hölder n-th derivatives and denote $[f]_{n,\alpha} = [f^{(n)}]_{0,\alpha}$. Clearly, $C^{0,1}(I)$ is the space of Lipschitz functions on I. In $C^{n,\alpha}(I)$ we consider the norm $\|f\|_{n,\alpha} = \|f\|_n + \sum_{k=0}^{n} [f]_{k,\alpha}$.

Lemma 2.3. *Suppose $I = [a,b] \subset [0,1], 1 \le n < \infty, 0 < \alpha \le 1$, and $f : I \to \mathbb{R}$ is a $C^{n,\alpha}$ function with $f'(a) = \ldots = f^{(n)}(a) = 0$. Then*

$$|f(x) - f(a)| \le [f]_{n,\alpha} |x-a|^{n+\alpha}$$

for every $x \in [a,b]$.

Proof. The proof of this easy lemma is left to the reader ∎

Throughout we use λ to denote the Lebesgue measure on $\mathbb{R}$.

Lemma 2.4. *Suppose $I = [a,b] \in \mathbb{R}, 1 \le n < \infty, 0 < \alpha \le 1$, and $f, g : I \to \mathbb{R}$ are $C^{n,\alpha}$ functions with $f^{(i)}(a) = g^{(i)}(a)$ ($0 \le i \le n$), $|f^{(n)}(x) - g^{(n)}(x)| < \varepsilon$ for all $x \in I$ and $[f-g]_{n,\alpha} < \varepsilon$. Then*

$$\|f - g\|_{n,\alpha} < \varepsilon \left(\sum_{k=0}^{n} \lambda(I)^k + \sum_{k=0}^{n} \lambda(I)^{k+\alpha} \right).$$

In particular, if $I \subset [0,1]$, then $\|f-g\|_{n,\alpha} < 2\varepsilon(n+1)$.

Proof. By [6: Lemma 2.2],

$$\|f-g\|_n < \varepsilon \sum_{k=0}^{n} \lambda(I)^k.$$

We shall prove that also

$$\sum_{k=0}^{n} [f-g]_{k,\alpha} < \varepsilon \sum_{k=0}^{n} \lambda(I)^{k+\alpha}.$$

Indeed, for every $x, y \in I$,

$$\begin{aligned}
&\big|(f-g)^{(n-1)}(x) - (f-g)^{(n-1)}(y)\big| \\
&\quad = \left| \int_x^y (f-g)^{(n)}(t)\, dt \right| \\
&\quad \le \left| \int_x^y \big|(f-g)^{(n)}(t)\big|\, dt \right| \\
&\quad = \left| \int_x^y \big|(f-g)^{(n)}(t) - (f-g)^{(n)}(a)\big|\, dt \right| \\
&\quad < \left| \int_x^y \varepsilon |t-a|^{\alpha} dt \right| \\
&\quad \le \varepsilon |x-y|^{\alpha+1}.
\end{aligned}$$

Arguing in this way we obtain that, for every $x, y \in I$ and every $0 \le k \le n$,

$$\big|(f-g)^{(n-k)}(x) - (f-g)^{(n-k)}(y)\big| < \varepsilon |x-y|^{\alpha+k}.$$

Therefore, $[f-g]_{n-k,\alpha} < \varepsilon \lambda(I)^{k+\alpha}$ and the result follows ∎

Our goal in this section is to characterize the following class $\mathcal{A}_{n,\alpha}$.

Definition 2.5. We define $\mathcal{A}_{n,\alpha}$ $(1 \le n < \infty, 0 < \alpha \le 1)$ to be the collection of all sets $P \subseteq [0,1]$ such that $P = f(Z_{(f,n)})$ for some $C^{n,\alpha}$ function $f : [0,1] \to [0,1]$.

We provide a characterization of $\mathcal{A}_{n,\alpha}$ in terms of Hausdorff measures and a condition β defined below.

Definition 2.6. If $M \subset \mathbb{R}$ and $s > 0$, then $\mathcal{H}^s(M)$ is the *s-dimensional Hausdorff measure* of M.

Definition 2.7. Suppose $I \subset \mathbb{R}$ is a closed interval and $P \subset \mathbb{R}$ is a closed set. Then

$$\beta_{n,\alpha}(P, I) = \sum_{i=1}^{\infty} \lambda(S_i)^{\frac{1}{n+\alpha}}$$

where S_i are components of $I \setminus P$.

We first have the following

Lemma 2.8. *Let $P \in \mathcal{A}_{n,\alpha}$. Then:*

1. $\beta_{n,\alpha}(P, [0,1]) < \infty$.

2. *P is a closed set with $\mathcal{H}^{\frac{1}{n+\alpha}}(P) = 0$.*

Proof. Let us first show that Condition 1 holds. For this, let $S_1, S_2, \ldots$ be the components of $[0,1] \setminus P$. Without loss of generality we may assume that $\{0,1\} \subset P$. Fix $N \in \mathbb{N}$, let $S_i = (c_i, d_i)$ and $a_i', b_i' \in Z_{(f,n)}$ be such that $f(a_i') = c_i$ and $f(b_i') = d_i$ for $1 \le i \le N$. Applying [6: Lemma 2.6] to the sequence formed by ordering the set $\{a_i', b_i' : 1 \le i \le N\}$ from the left to the right, we may choose non-overlapping intervals $I_i = [a_i, b_i]$ $(i = 1, \ldots, N)$ such that their end-points are in $\{a_i', b_i' : 1 \le i \le N\} \subseteq Z_{(f,n)}$ and $\lambda(S_i) = |d_i - c_i| \le |f(b_i) - f(a_i)|$. Then, using the fact that $f^{(1)}(a_i) = f^{(2)}(a_i) = \ldots = f^{(n)}(a_i) = 0$ and Lemma 2.3 we obtain

$$\lambda(S_i) \le |f(a_i) - f(b_i)| \le [f]_{n,\alpha} |b_i - a_i|^{n+\alpha}.$$

Since $\{I_i\}_{i=1}^N$ is a sequence of non-overlapping intervals contained in $[0,1]$, we get

$$\sum_{i=1}^N \lambda(S_i)^{\frac{1}{n+\alpha}} \le ([f]_{n,\alpha})^{\frac{1}{n+\alpha}} \sum_{i=1}^N |b_i - a_i| \le ([f]_{n,\alpha})^{\frac{1}{n+\alpha}}.$$

Hence Condition 1 follows.

Since Condition 1 holds and since by [8: Theorem 3.4.3] $\mathcal{H}^{\frac{1}{n}}(P) = 0$ and hence $\lambda(P) = 0$, by [1: Lemma 2] Condition 2 follows ∎

For the convenience of the reader, throughout the paper we recall some definitions and necessary terminology from [6]. Afterwards, the rest of this section is devoted to proving the converse of the above result.

We now recall the definition of chain and introduce new notions. Throughout, π_1 and π_2 denote coordinate projections.

Definition 2.9. A *box* is a set of the form $B = I \times J$ where $I, J \subset \mathbb{R}$ are compact intervals. For $1 \le n < \infty$ and $0 \le \alpha \le 1$, $sl_{n,\alpha}(B) = \frac{\lambda(J)}{\lambda(I)^{n+\alpha}}$ is its (n,α)-*slope*.

Definition 2.10 [6: Definition 2.9]. A *basic building block function* is a C^∞ function $\phi : [0,1] \to [0,1]$ with

1. $\phi(0) = 0$ and $\phi(1) = 1$

2. $\phi^{(1)}(x) > 0$ for all $0 < x < 1$

3. $\phi^{(i)}(0) = \phi^{(i)}(1) = 0$ for all $i \ge 1$.

If $B = I \times J$ is a box, then $\phi_B = \psi_1 \circ \phi \circ \psi_2$ where ψ_1 and ψ_2 are the linear increasing homeomorphisms from $[0,1]$ onto J and from I onto $[0,1]$, respectively. Note that ϕ_B is simply a congruent copy of ϕ in B. Moreover, for $i \ge 1$ there exists a map $x \mapsto p_x$ from $\pi_1(B)$ onto $[0,1]$ such that $\phi_B^{(i)}(x) = \phi^{(i)}(p_x)\, sl_i(B)$. From this, $\|\phi_B^{(i)}\| = \|\phi^{(i)}\|\, sl_i(B)$.

For the remainder of this section we shall use ϕ to denote some fixed basic building block function.

Definition 2.11 [6: Defintion 2.11]. Suppose $B = I \times J \subseteq [0,1] \times [0,1]$ is a box. A collection $\mathcal{G} = \{G_1, G_2, \dots, G_t\}$ is a *chain* in B if there are partitions $\{I_1, I_2, \dots, I_t\}$ of I and $\{J_1, J_2, \dots, J_t\}$ of J (both sequences ordered from the left to the right) such that, for all i, $G_i = I_i \times J_i$. The intervals I and J are the *domain* and the *range* of $\mathcal{G}$ and we denote them by $\operatorname{dom}(\mathcal{G})$ and $\operatorname{ran}(\mathcal{G})$, respectively. To say that $\mathcal{G}$ is a chain means that $\mathcal{G}$ is a chain in some box.

Definition 2.12 [6: Definition 2.12]. A function f is ϕ-like in the chain $\mathcal{G}$ if $\operatorname{dom}(f) = \operatorname{dom}(\mathcal{G})$, $\operatorname{ran}(f) = \operatorname{ran}(\mathcal{G})$ and, for each box $B = I \times J$ in $\mathcal{G}$, $f|I = \phi_B$.

Definition 2.13 [6: Definition 2.14]. Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are chains. We say that $\mathcal{G}_2$ *refines* $\mathcal{G}_1$, denoted by $\mathcal{G}_2 \ll \mathcal{G}_1$, if every element of $\mathcal{G}_2$ is contained in some element of $\mathcal{G}_1$, $\operatorname{dom}(\mathcal{G}_1) = \operatorname{dom}(\mathcal{G}_2)$ and $\operatorname{ran}(\mathcal{G}_1) = \operatorname{ran}(\mathcal{G}_2)$.

Definition 2.14. Suppose B_1 and B_2 are boxes with $B_2 \subseteq B_1$, $1 \le n < \infty$ and $0 \le \alpha \le 1$. We define

$$\Delta_{n,\alpha}(B_1, B_2) = \begin{cases} 0 & \text{if } B_1 = B_2 \\ sl_{n,\alpha}(B_1) + sl_{n,\alpha}(B_2) & \text{else.} \end{cases}$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are chains with $\mathcal{G}_2 \ll \mathcal{G}_1$, then we define

$$\Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2) = \max\Big\{\Delta_{n,\alpha}(B_1, B_2) : \ B_i \in \mathcal{G}_i \ (i = 1, 2) \text{ and } B_2 \subseteq B_1\Big\}.$$

For sake of symmetry, we let $\Delta_{n,\alpha}(B_2, B_1) = \Delta_{n,\alpha}(B_1, B_2)$ and $\Delta_{n,\alpha}(\mathcal{G}_2, \mathcal{G}_1) = \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2)$. When $\alpha = 0$, as in [6: Definition 2.15] we denote $\Delta_n(\mathcal{G}_2, \mathcal{G}_1) = \Delta_{n,\alpha}(\mathcal{G}_2, \mathcal{G}_1)$. Clearly, if all boxes are contained in $[0,1] \times [0,1]$, then $\Delta_n(\mathcal{G}_1, \mathcal{G}_2) \le \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2)$.

From now on we shall consider only chains contained $[0,1] \times [0,1]$.

Proposition 2.15. *Suppose* $1 \le n < \infty$ *and* $0 < \alpha \le 1$. *Then there is a constant* $K_{n,\alpha}$ *such that, whenever* $\mathcal{G}_2 \ll \mathcal{G}_1$ *and* f_i *is* ϕ*-like in* $\mathcal{G}_i$, *then*

$$\|f_1 - f_2\|_{n,\alpha} \le K_{n,\alpha}\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2).$$

Proof. By [6: Proposition 2.16] there exists a constant K_n such that

$$\|f_1 - f_2\|_n \le K_n \Delta_n(\mathcal{G}_1, \mathcal{G}_2) \le K_n\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2).$$

We shall prove that there exists a constant $T_{n,\alpha}$ such that

$$\sum_{k=0}^{n} [f_1 - f_2]_{k,\alpha} \le T_{n,\alpha}\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2).$$

Let $B_i \in \mathcal{G}_i$ $(i = 1, 2)$ with $B_2 \subseteq B_1$, let $I = \pi_1(B_2)$ and $x, y \in I$. If $B_1 = B_2$, then $f_1|I = f_2|I$ and

$$\frac{\big|(f_1 - f_2)^{(n)}(x) - (f_1 - f_2)^{(n)}(y)\big|}{|x - y|^{\alpha}} = 0 = \Delta_{n,\alpha}(B_1, B_2).$$

Let us now consider the case $B_1 \neq B_2$. Then

$$\big|(f_1 - f_2)^{(n)}(x) - (f_1 - f_2)^{(n)}(y)\big| = \big|\big(\phi_{B_1}^{(n)}(x) - \phi_{B_2}^{(n)}(x)\big) - \big(\phi_{B_1}^{(n)}(y) - \phi_{B_2}^{(n)}(y)\big)\big|$$

because $f_i = \phi_{B_i}$ on $\pi_1(B_i)$ $(i = 1, 2)$. So

$$\begin{aligned}
&\frac{\big|(f_1 - f_2)^{(n)}(x) - (f_1 - f_2)^{(n)}(y)\big|}{|x-y|^\alpha} \\
&\quad \le \frac{\big|\phi_{B_1}^{(n)}(x) - \phi_{B_1}^{(n)}(y)\big|}{|x-y|^\alpha} + \frac{\big|\phi_{B_2}^{(n)}(x) - \phi_{B_2}^{(n)}(y)\big|}{|x-y|^\alpha} \\
&\quad \le [\phi]_{n,\alpha}\big(sl_{n,\alpha}(B_1) + sl_{n,\alpha}(B_2)\big) \\
&\quad \le [\phi]_{n,\alpha}\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2).
\end{aligned}$$

What we have just shown is that, whenever x and y belong to the first projection of the same box in $\mathcal{G}_2$, then the inequality

$$\frac{\big|(f_1 - f_2)^{(n)}(x) - (f_1 - f_2)^{(n)}(y)\big|}{|x-y|^\alpha} \le [\phi]_{n,\alpha}\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2)$$

holds. Now let us consider the case when $x \in \pi_1(B_2^x)$ and $y \in \pi_1(B_2^y)$ with B_2^x and B_2^y in $\mathcal{G}_2$ and $B_2^x \neq B_2^y$. Let B_1^x and B_1^y be two boxes in $\mathcal{G}_1$ such that $B_2^x \subseteq B_1^x$ and $B_2^y \subseteq B_1^y$. Then two cases are to consider.

Case 1: $B_1^x = B_1^y$. In this case, setting $B = B_1^x = B_1^y$, we have

$$\begin{aligned}
&\frac{\big|(f_1 - f_2)^{(n)}(x) - (f_1 - f_2)^{(n)}(y)\big|}{|x-y|^\alpha} \\
&\quad = \frac{\big|\big(\phi_{B_1}^{(n)}(x) - \phi_{B_2^x}^{(n)}(x)\big) - \big(\phi_{B_1}^{(n)}(y) - \phi_{B_2^y}^{(n)}(y)\big)\big|}{|x-y|^\alpha} \\
&\quad = \frac{\big|\big(\phi_{B_1}^{(n)}(x) - \phi_{B_1}^{(n)}(y)\big) - \big(\phi_{B_2^x}^{(n)}(x) - \phi_{B_2^y}^{(n)}(y)\big)\big|}{|x-y|^\alpha} \\
&\quad \le \frac{\big|\phi_{B_1}^{(n)}(x) - \phi_{B_1}^{(n)}(y)\big|}{|x-y|^\alpha} + \frac{|\phi_{B_2^x}^{(n)}(x)|}{|x-y|^\alpha} + \frac{|\phi_{B_2^y}^{(n)}(y)|}{|x-y|^\alpha} \\
&\quad \le [\phi]_{n,\alpha}\, sl_{n,\alpha}(B_1) + \frac{|\phi_{B_2^x}^{(n)}(x)|}{|x-y|^\alpha} + \frac{|\phi_{B_2^y}^{(n)}(y)|}{|x-y|^\alpha}.
\end{aligned}$$

Without loss of generality we can assume $x < y$. Then, let be

p_2^x the right end-point of $\pi_1(B_2^x)$

p_2^y the left end-point of $\pi_1(B_2^y)$.

Since, by construction,

$$\phi_{B_2^x}^{(n)}(p_2^x) = \phi_{B_2^y}^{(n)}(p_2^y) = 0$$

and

$$\max\left\{|x-p_2^x|, |y-p_2^y|\right\} \le |x-y|$$

we get

$$\begin{aligned}
&\frac{|\phi^{(n)}_{B_2^x}(x)|}{|x-y|^\alpha} + \frac{|\phi^{(n)}_{B_2^y}(y)|}{|x-y|^\alpha} \\
&\quad = \frac{\left|\phi^{(n)}_{B_2^x}(x) - \phi^{(n)}_{B_2^x}(p_2^x)\right|}{|x-y|^\alpha} + \frac{\left|\phi^{(n)}_{B_2^y}(y) - \phi^{(n)}_{B_2^y}(p_2^y)\right|}{|x-y|^\alpha} \\
&\quad \le \frac{\left|\phi^{(n)}_{B_2^x}(x) - \phi^{(n)}_{B_2^x}(p_2^x)\right|}{|x-p_2^x|^\alpha} + \frac{\left|\phi^{(n)}_{B_2^y}(y) - \phi^{(n)}_{B_2^y}(p_2^y)\right|}{|y-p_2^y|^\alpha} \\
&\quad \le [\phi]_{n,\alpha}\left(sl_{n,\alpha}(B_2^x) + sl_{n,\alpha}(B_2^y)\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{\left|(f_1-f_2)^{(n)}(x) - (f_1-f_2)^{(n)}(y)\right|}{|x-y|^\alpha} \\
&\quad \le [\phi]_{n,\alpha}\, sl_{n,\alpha}(B_1) + [\phi]_{n,\alpha}\left(sl_{n,\alpha}(B_2^x) + sl_{n,\alpha}(B_2^y)\right) \\
&\quad \le [\phi]_{n,\alpha}\left(sl_{n,\alpha}(B_1) + sl_{n,\alpha}(B_2^x)\right) + [\phi]_{n,\alpha}\left(sl_{n,\alpha}(B_1) + sl_{n,\alpha}(B_2^y)\right) \\
&\quad \le 2[\phi]_{n,\alpha}\, \Delta_{n,\alpha}(\mathcal{G}_1, \mathcal{G}_2).
\end{aligned}$$

Case 2: $B_1^x \ne B_1^y$. Then

$$\begin{aligned}
&\frac{\left|(f_1-f_2)^{(n)}(x) - (f_1-f_2)^{(n)}(y)\right|}{|x-y|^\alpha} \\
&\quad = \frac{\left|\left(\phi^{(n)}_{B_1^x}(x) - \phi^{(n)}_{B_2^x}(x)\right) - \left(\phi^{(n)}_{B_1^y}(y) - \phi^{(n)}_{B_2^y}(y)\right)\right|}{|x-y|^\alpha} \\
&\quad \le \frac{|\phi^{(n)}_{B_1^x}(x)|}{|x-y|^\alpha} + \frac{|\phi^{(n)}_{B_2^x}(x)|}{|x-y|^\alpha} + \frac{|\phi^{(n)}_{B_1^y}(y)|}{|x-y|^\alpha} + \frac{|\phi^{(n)}_{B_2^y}(y)|}{|x-y|^\alpha}.
\end{aligned}$$

Without loss of generality we can assume $x < y$. Then let be

p_2^x the right end-point of $\pi_1(B_2^x)$
p_1^x the right end-point of $\pi_1(B_1^x)$
p_2^y the left end-point of $\pi_1(B_2^y)$
p_1^y the left end-point of $\pi_1(B_1^y)$.

Since, by construction,

$$\phi^{(n)}_{B_2^x}(p_2^x) = \phi^{(n)}_{B_1^x}(p_1^x) = \phi^{(n)}_{B_2^y}(p_2^y) = \phi^{(n)}_{B_1^y}(p_1^y) = 0$$

and

$$\max\left\{|x-p_2^x|, |x-p_1^x|, |y-p_2^y|, |y-p_1^y|\right\} \le |x-y|$$

we get

$$
\begin{aligned}
&\frac{|\phi^{(n)}_{B^x_1}(x)|}{|x-y|^\alpha}+\frac{|\phi^{(n)}_{B^x_2}(x)|}{|x-y|^\alpha}+\frac{|\phi^{(n)}_{B^y_1}(y)|}{|x-y|^\alpha}+\frac{|\phi^{(n)}_{B^y_2}(y)|}{|x-y|^\alpha}\\
&\quad=\frac{\left|\phi^{(n)}_{B^x_1}(x)-\phi^{(n)}_{B^x_1}(p^x_1)\right|}{|x-y|^\alpha}+\frac{\left|\phi^{(n)}_{B^x_2}(x)-\phi^{(n)}_{B^x_2}(p^x_2)\right|}{|x-y|^\alpha}\\
&\qquad+\frac{\left|\phi^{(n)}_{B^y_1}(y)-\phi^{(n)}_{B^y_1}(p^y_1)\right|}{|x-y|^\alpha}+\frac{\left|\phi^{(n)}_{B^y_2}(y)-\phi^{(n)}_{B^y_2}(p^y_2)\right|}{|x-y|^\alpha}\\
&\quad\le\frac{\left|\phi^{(n)}_{B^x_1}(x)-\phi^{(n)}_{B^x_1}(p^x_1)\right|}{|x-p^x_1|^\alpha}+\frac{\left|\phi^{(n)}_{B^x_2}(x)-\phi^{(n)}_{B^x_2}(p^x_2)\right|}{|x-p^x_2|^\alpha}\\
&\qquad+\frac{\left|\phi^{(n)}_{B^y_1}(y)-\phi^{(n)}_{B^y_1}(p^y_1)\right|}{|y-p^y_1|^\alpha}+\frac{\left|\phi^{(n)}_{B^y_2}(y)-\phi^{(n)}_{B^y_2}(p^y_2)\right|}{|y-p^y_2|^\alpha}\\
&\quad\le[\phi]_{n,\alpha}\left(sl_{n,\alpha}(B^x_1)+sl_{n,\alpha}(B^x_2)\right)+[\phi]_{n,\alpha}\left(sl_{n,\alpha}(B^y_1)+sl_{n,\alpha}(B^y_2)\right)\\
&\quad\le 2[\phi]_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2).
\end{aligned}
$$

Hence we can conclude that, for each $x, y \in \operatorname{dom}(\mathcal{G}_1)$,

$$\frac{\left|(f_1-f_2)^{(n)}(x)-(f_1-f_2)^{(n)}(y)\right|}{|x-y|^\alpha}\le 2[\phi]_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2)$$

and so

$$[f_1-f_2]_{n,\alpha}\le 2[\phi]_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2).$$

Since $f_1^{(k)}(a) = f_2^{(k)}(a) = 0$ for all $1 \le k \le n$ and $f_1(a) = f_2(a)$ where $a = \inf\operatorname{dom}(\mathcal{G}_1)$, by Lemma 2.4 it follows that

$$\sum_{k=0}^{n}[f_1-f_2]_{k,\alpha}\le 2^2(n+1)[\phi]_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2)=T_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2)$$

where $T_{n,\alpha} = 2^2(n+1)[\phi]_{n,\alpha}$. Hence,

$$\|f_1-f_2\|_{n,\alpha}\le K_{n,\alpha}\,\Delta_{n,\alpha}(\mathcal{G}_1,\mathcal{G}_2)$$

where $K_{n,\alpha} = K_n + T_{n,\alpha}$ ∎

Definition 2.16. Suppose $1 \le n < \infty, 0 < \alpha \le 1$ and $\{\mathcal{G}_k\}$ a sequence of chains with $\mathcal{G}_{k+1} \ll \mathcal{G}_k$ for all k. We say that $\{\mathcal{G}_k\}$ is a (n, α)-*Cauchy sequence* if for all $\varepsilon > 0$ there is a $M \in \mathbb{N}$ such that if $M < m_1, m_2 \in \mathbb{N}$, then $\Delta_{n,\alpha}(\mathcal{G}_{m_1}, \mathcal{G}_{m_2}) < \varepsilon$. A sequence $\{f_k\}$ of $C^{n,\alpha}$ functions to be a (n, α)-*Cauchy sequence means that it is a Cauchy sequence in the norm* $\|\cdot\|_{n,\alpha}$.

Definition 2.17. Suppose $1 \le n < \infty$ and $0 < \alpha \le 1$. For sake of notational convenience, the triple $(\{\mathcal{G}_k\}, \{f_k\}, \phi)$ is called (n, α)-*proper* if ϕ is a basic building block function, $\{\mathcal{G}_k\}$ is a (n, α)-Cauchy sequence and f_k is ϕ-like in $\mathcal{G}_k$ for all k.

Proposition 2.18. *Suppose* $1 \le n < \infty, 0 < \alpha \le 1$ *and* $(\{\mathcal{G}_k\}, \{f_k\}, \phi)$ *is* (n, α)*-proper. Then* $\{f_k\}$ *is a* (n, α)*-Cauchy sequence and hence converges to some* $C^{n,\alpha}$ *function* f*.*

Proof. This follows from the definition of a (n, α)-Cauchy sequence and Proposition 2.15 ∎

Definition 2.19 [6: Definition 2.20]. Let $\mathcal{G}$ be a chain and $I \times J$ some box in $\mathcal{G}$. Then $\mathcal{E}_Y(\mathcal{G})$ is the set of all endpoints y of J and $\mathcal{E}_X(\mathcal{G}))$ is the set of all endpoints x of I.

Definition 2.20 [6: Definition 2.21]. Let $\{\mathcal{G}_k\}$ be a sequence of chains with $\mathcal{G}_{k+1} \ll \mathcal{G}_k$ for all k. Then $\mathcal{C}_Y(\{\mathcal{G}_k\})$ and $\mathcal{C}_X(\{\mathcal{G}_k\}))$ are the sets of all y and x, respectively, such that there are an increasing sequence of integers $\{k_i\}$ and two sequences of boxes $\{B_i\}$ and $\{B_i'\}$ such that, for all i,

1. $y \in \pi_2(B_i) \cap \pi_2(B_i')$ and $x \in \pi_1(B_i) \cap \pi_1(B_i')$, respectively

2. $B_i \in \mathcal{G}_{k_i}$ and $B_i' \in \mathcal{G}_{k_i+1}$

3. B_i' is a proper subset of B_i

4. $B_{i+1} \subseteq B_i' \subseteq B_i$.

Definition 2.21 [6: Definition 2.22]. Let $\{\mathcal{G}_k\}$ be a sequence of chains with $\mathcal{G}_{k+1} \ll \mathcal{G}_k$ for all k. Then

$$\mathcal{F}_Y(\{\mathcal{G}_k\}) = \mathcal{C}_Y(\{\mathcal{G}_k\}) \cup \Big(\cup_{i=1}^{\infty} \mathcal{E}_Y(\mathcal{G}_i) \Big)$$
$$\mathcal{F}_X(\{\mathcal{G}_k\}) = \mathcal{C}_X(\{\mathcal{G}_k\}) \cup \Big(\cup_{i=1}^{\infty} \mathcal{E}_X(\mathcal{G}_i) \Big).$$

Proposition 2.22. *Let* $(\{\mathcal{G}_k\}, \{f_k\}, \phi)$ *be* (n, α)*-proper,* $1 \le n < \infty$ *and* $0 < \alpha \le 1$*. Then* $\mathcal{F}_Y(\{\mathcal{G}_k\}) = f(Z_{(f,n)})$ *and* $\mathcal{F}_X(\{\mathcal{G}_k\}) = Z_{(f,n)}$ *where* f *is the limit of* $\{f_k\}$*.*

Proof. This follows from the fact that the convergence in $C^{n,\alpha}$ is stronger than that in C^n and from [6: Propositions 2.23 and 2.24] ∎

Lemma 2.23. *Let* $J \subset \mathbb{R}$ *be a closed interval and* $P \subset \mathbb{R}$ *a closed set such that* $\beta_{n,\alpha}(P, J) < \infty$*. Then for every* $\varepsilon > 0$ *there exists* $h > 0$ *such that if* $J_1, \dots, J_t$ *is a finite collection of non-overlapping intervals contained in* J *and covering* $P \cap J$ *with* $J_k \cap P \ne \emptyset$ *and* $\lambda(J_k) < h$ *for all* $1 \le k \le t$*, then* $\sum_{k=1}^{t} \beta_{n,\alpha}(P, J_k) < \varepsilon$*.*

Proof. The proof of this lemma is analogous to that of [6: Lemma 2.25] ∎

Lemma 2.24. *Suppose* $B = I \times J$ *is a box,* $P \subset \mathbb{R}$ *a closed set with* $\mathcal{H}^{\frac{1}{n+\alpha}}(P) = 0$*, and the end-points of* J *are in* P*. Moreover, suppose* $M > L > 0$ *are such that* $L^{\frac{1}{n+\alpha}} \beta_{n,\alpha}(P, J) < \lambda(I)$*. Then there exists a chain* $\mathcal{G}$ *in* B *such that:*

1. $\mathcal{E}_Y(\mathcal{G}) \subseteq P$*.*

2. $sl_{n+\alpha}(B') \le \frac{1}{L}$ *for all* $B' \in \mathcal{G}$*.*

3. *If* $B' \in \mathcal{G}'$*, then* $\lambda(\pi_2(B')) < \frac{1}{M}$*,* $M^{\frac{1}{n+\alpha}} \beta_{n,\alpha}(P, \pi_2(B')) < \lambda(\pi_1(B'))$ *and* $\sum_{B' \in \mathcal{G}'} \lambda(\pi_1(B')) < \frac{1}{M}$*, where* $\mathcal{G}'$ *is the set of all boxes* B' *in* $\mathcal{G}$ *such that the interior of* $\pi_2(B')$ *contains a point of* P*.*

Proof. The proof of this lemma is a simple modification of the proof of [6: Lemma 2.26] and can be carried out by applying Lemma 2.23 ∎

Lemma 2.25. *Let $1 \leq n < \infty$ and $0 < \alpha \leq 1$, and suppose $P \subseteq [0,1]$ is a closed set with $\mathcal{H}^{\frac{1}{n+\alpha}}(P) = 0$ and $\beta_{n,\alpha}(P,[0,1]) < \infty$. Then there exists a sequence of chains $\{\mathcal{G}_k\}$ so that*

(i) *$\{\mathcal{G}_k\}$ is a (n,α)-Cauchy sequence*

(ii) $P = \mathcal{F}_Y(\{\mathcal{G}_k\})$

(iii) $\lambda(\mathcal{F}_X(\{\mathcal{G}_k\})) = 0$.

Proof. We construct the sequence $\{\mathcal{G}_k\}$ using induction and Lemma 2.24. Without loss of generality we can assume $\{0,1\} \subseteq P$. We first construct $\mathcal{G}_0$. Let $L > 0$ be such that $L^{\frac{1}{n+\alpha}}\beta_{n,\alpha}(P,[0,1]) < 1$ and $M \geq 2$. Applying Lemma 2.24 to $[0,1]\times[0,1]$, L and M, we obtain a chain $\mathcal{G}_0$ which satisfies the conclusions of Lemma 2.24. Now suppose $k \geq 1$ and $\mathcal{G}_1,\dots,\mathcal{G}_k$ have been already constructed so that, for $1 \leq l \leq k$ denoting

$$\mathcal{T}_l = \Big\{B \in \mathcal{G}_l : \text{ the interior of } \pi_2(B) \text{ contains a point of } P\Big\},$$

the following conditions are satisfied:

1. $\mathcal{G}_k \ll \mathcal{G}_{k-1}$.

2. $\mathcal{E}_Y(\mathcal{G}_k) \subseteq P$.

3. If $B \in \mathcal{G}_k$ and $B \subseteq B' \in \mathcal{G}_{k-1} \setminus \mathcal{T}_{k-1}$, then $B = B'$.

4. If $B \in \mathcal{G}_k$ and $B \subseteq B' \in \mathcal{G}_{k-1} \cap \mathcal{T}_{k-1}$, then $sl_{n,\alpha}(B) < \frac{1}{2^k}$.

5. If $B \in \mathcal{T}_k$, then $(2^{k+1})^{\frac{1}{n+\alpha}}\beta_{n,\alpha}(P,\pi_2(B)) < \lambda(\pi_1(B))$ and $\lambda(\pi_2(B)) < \frac{1}{2^k}$.

6. $\sum_{B\in\mathcal{T}_k}\lambda(\pi_1(B)) < \frac{1}{2^k}$.

Let us now construct $\mathcal{G}_{k+1}$. Let $B \in \mathcal{G}_k$. If $B \notin \mathcal{T}_k$, then we let $\mathcal{G}^B_{k+1} = \{B\}$. If $B \in \mathcal{T}_k$, then we apply Lemma 2.24 to B, $L = 2^{k+1}$ and $M = \max\{2^{k+2}, \frac{2^{k+2}}{\lambda(\pi_1(B))}\}$. Let $\mathcal{G}^B_{k+1}$ be the resulting chain and $\mathcal{G}_{k+1} = \cup_{B\in\mathcal{G}_k}\mathcal{G}^B_{k+1}$. By construction, $\mathcal{G}_{k+1}$ satisfies the induction hypotheses.

Let us now show that $\{\mathcal{G}_k\}$ is a (n,α)-Cauchy sequence. For this let $B' \in \mathcal{G}_{k-1}$ and $B \in \mathcal{G}_k$ with $B \subseteq B'$. If $B' \notin \mathcal{T}_{k-1}$, then by induction hypothesis 3 $B = B'$ and hence $\Delta_{n,\alpha}(B,B') = 0$. If $B' \in \mathcal{T}_{k-1}$, then by induction hypothesis 4 $sl_{n,\alpha}(B) < \frac{1}{2^k}$. Let $B'' \in \mathcal{G}_{k-2}$ be such that $B' \subseteq B''$. Since $B' \in \mathcal{T}_{k-1}$, $B'' \in \mathcal{T}_{k-2}$ and by hypothesis 4 at stage $k-1$ we have $sl_{n,\alpha}(B') < \frac{1}{2^{k-1}}$. Therefore we have just shown that $\Delta_{n,\alpha}(B,B') < \frac{1}{2^k} + \frac{1}{2^{k-1}}$. Hence, $\Delta_{n,\alpha}(\mathcal{G}_{k-1},\mathcal{G}_k) < \frac{1}{2^k} + \frac{1}{2^{k-1}}$. Therefore, $\mathcal{G}_k$ is a (n,α)-Cauchy sequence. The rest of the proof is the same as in the proof of [6: Lemma 2.27] ∎

Theorem 2.26. *Let $P \subseteq [0,1]$, $1 \leq n < \infty$ and $0 < \alpha \leq 1$. Then the following assertions are equivalent:*

1. $P \in \mathcal{A}_{n,\alpha}$.

2. *P is a closed set with $\mathcal{H}^{\frac{1}{n+\alpha}}(P) = 0$ and $\beta_{n,\alpha}(P,[0,1]) < \infty$.*

Moreover, if $P \subseteq [0,1]$ satisfies Condition 2, then there is a $C^{n,\alpha}$ homeomorphism f from $[0,1]$ onto $[0,1]$ such that $P = f(Z_{(f,n)})$ and $\lambda(Z_{(f,n)}) = 0$.

Proof. Assertion (1) $\Rightarrow$ (2) is simply Lemma 2.8.

Assertion (2) $\Rightarrow$ (1): By Lemma 2.25 we may choose a sequence of chains $\{\mathcal{G}_k\}$ such that $\{\mathcal{G}_k\}$ is a (n,α)-Cauchy sequence, $P = \mathcal{F}_Y(\{\mathcal{G}_k\})$ and $\lambda(\mathcal{F}_X(\{\mathcal{G}_k\})) = 0$. By [6: Proposition 2.13] there is a unique function f_k which is ϕ-like in $\mathcal{G}_k$. Then $(\{\mathcal{G}_k\}, \{f_k\}, \phi)$ is (n,α)-proper. Let f be the limit of $\{f_k\}$. Then $f \in C^{n,\alpha}$ and, by Proposition 2.22, $\mathcal{F}_Y(\{\mathcal{G}_k\}) = f(Z_{(f,n)})$. Hence $P = f(Z_{(f,n)})$. By Proposition 2.22, $\lambda(Z_{(f,n)}) = \lambda(\mathcal{F}_X(\{\mathcal{G}_k\})) = 0$. Since f is a non-decreasing function and $\lambda(Z_{(f,n)}) = 0$, f is a homeomorphism ∎

Theorem 2.27. *Let $1 \le n < \infty$. The collection $\mathcal{A}_{n,\alpha}$ forms an ideal of compact sets.*

Proof. The proof is a simple modification of the proof of [6: Theorem 2.30] ∎

Example 2.28. Denote by $\dim_{\mathcal{H}}$ the Hausdorff dimension and let C_γ be the "Cantor sets" obtained by removing the middle γ-th percentage every time. Then $\dim_{\mathcal{H}}(C_\gamma) = -\frac{\log 2}{\log \frac{1-\gamma}{2}}$. Clearly, if $\gamma > 1 - \frac{1}{2^{n+\alpha-1}}$, then $\mathcal{H}^{\frac{1}{n+\alpha}}(C_\gamma) = 0$. Moreover, for such γ, $\beta_{n,\alpha}(C_\gamma, [0,1]) < \infty$. Hence $C_\gamma \in \mathcal{A}_{n,\alpha}$ for $\gamma > 1 - \frac{1}{2^{n+\alpha-1}}$.

Remark 2.29. In [6], by $\mathcal{A}_n$ $(1 \le n < \infty)$ there is denoted the collection of all sets $P \subseteq [0,1]$ such that $P = f(Z_{(f,n)})$ for some C^n function $f : [0,1] \to [0,1]$, and by $\mathcal{A}_\infty$ there is denoted the collection of all sets $P \subseteq [0,1]$ such that $P = f(Z_{(f,\infty)})$ for some C^∞ function $f : [0,1] \to [0,1]$ where $Z_{(f,\infty)} = \{x \in \mathbb{R} : f^{(i)}(x) = 0 \text{ for all } 1 \le i\}$. From [6: Theorems 2.28 and 2.33] it follows that $\cap_n \mathcal{A}_n = \mathcal{A}_\infty$. On the other hand, it is also clear that $\cap_n \mathcal{A}_{n,\alpha} = \mathcal{A}_\infty$.

Theorem 2.30. *Let $P \subseteq [0,1]$ and $1 \le n < \infty$. Then $\mathcal{A}_{n,1} = \mathcal{A}_{n+1}$ and the following assertions are equivalent:*

1. *There is a $C^{n,1}$ function $f : [0,1] \to [0,1]$ such that $P = f(Z_{(f,n)})$.*
2. *P is a closed set with $\mathcal{H}^{\frac{1}{n+1}}(P) = 0$ and $\beta_{n,1}(P, [0,1]) < \infty$.*
3. *There is a C^{n+1} function $f : [0,1] \to [0,1]$ such that $P = f(Z_{(f,n+1)})$.*

Proof. Assertion (1) $\Leftrightarrow$ (2) is a consequence of Theorem 2.26. Assertion (2) $\Leftrightarrow$ (3) follows from the fact that, by definition, $\beta_{n,1}(P, [0,1]) = \beta_{n+1}(P, [0,1])$ and from [6: Theorem 2.28] ∎

3. Perfect level sets

In this section we characterize the set of points where level sets of a given $C^{n,\alpha}$ function are perfect. We first recall some basic definitions and results.

Definition 3.1 [6: Definition 3.1]. Let $f : I \to \mathbb{R}$ be a continuous function and G the union of all open (relative to I) intervals S such that f is monotone on S. We call $p \in I \setminus G$ a *turning point* of f and denote by T_f the union of the set of all turning points of f and all end-points of I.

Definition 3.2. Let $f : I \to \mathbb{R}$ be a continuous function. For $\gamma \geq 1$ we define the *γ-variation* of f by

$$V_\gamma(f) = \sup \left\{ \sum_{i=1}^{k} |f(x_{i+1}) - f(x_i)|^{\frac{1}{\gamma}} \right\}$$

where the supremum is taken over points $x_i \in T_f$ such that $x_1 < x_2 < \ldots < x_{k+1}$.

Theorem 3.3 [10, 11]. *Suppose $1 \leq n < \infty$ and $0 < \alpha \leq 1$, and let $f : [0,1] \to \mathbb{R}$ be a continuous function of bounded variation with $V_{n+\alpha}(f) < \infty$. Then there is a homeomorphism $h : [0,1] \to [0,1]$ such that $f \circ h$ is a $C^{n,\alpha}$ function.*

Definition 3.4 [6: Definition 3.13]. Suppose P and Q are closed sets. We say that P is *strongly contained* in Q, denoted by $P \preceq Q$, if $P \subseteq Q$, $Q \setminus P$ is countable and every point of P is a bilateral limit point of Q.

Lemma 3.5. *Suppose $1 \leq n < \infty$ and $P \in \mathcal{A}_{n,\alpha}$. Then there is $Q \in \mathcal{A}_{n,\alpha}$ such that $P \preceq Q$.*

Proof. The proof is the same as that of [6: Lemma 3.14] ∎

We now proceed towards the goal of this section.

Theorem 3.6. *Let $M \to [0,1]$, $1 \leq n < \infty$ and $0 < \alpha \leq 1$. Then the following assertions are equivalent:*

1. *M is the union of a G_δ set and a countable set and there is $P \in \mathcal{A}_{n,\alpha}$ such that $M \subset P$.*

2. *There is a $C^{n,\alpha}$ function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

Proof. Assertion (2) $\Rightarrow$ (1): By [6: Lemma 3.7], M is the union of a G_δ set and a countable set. From the fact that $f^{-1}(\{y\})$ is perfect for all $y \in M$ it follows that $M \subseteq f(Z_{(f,n)}) \in \mathcal{A}_{n,\alpha}$.

Assertion (1) $\Rightarrow$ (2): Let M and P be as described in Assertion 1. By Lemma 3.5 there exists a set $P' \in A_{n,\alpha}$ such that $P \preceq P'$. By Theorem 2.26, there is a $C^{n,\alpha}$ increasing homeomorphism h such that $h(Z_{(h,n)}) = P'$ and $\lambda(Z_{(h,n)}) = 0$. Let $Q' = h^{-1}(P')$, $Q = h^{-1}(P)$ and $N = h^{-1}(M)$. The sets Q', Q and N clearly satisfy the hypotheses of [6: Proposition 3.16]. Now applying this proposition we can obtain a continuous function of bounded variation g such that $g^{-1}(\{y\})$ is perfect for all $y \in N$, finite otherwise and $g(T_g) \subseteq Q'$. Now we consider the function $h \circ g$. First, $(h \circ g)^{-1}(\{y\})$ clearly is perfect for all $y \in M$ and finite otherwise. Next we want to observe that $h \circ g$ satisfies the hypotheses of Theorem 3.3. As h is Lipschitz and g is a continous function of bounded variation, $h \circ g$ is also a continuous function of bounded variation. Now we want to show that $V_{n+\alpha}(h \circ g) < \infty$. Let $x_1 < x_2 < \ldots < x_k < x_{k+1}$ be elements of $T_{h \circ g}$. Since h is a homeomorphism, $x_i \in T_g$ for all i as well. Let $1 \leq i \leq k$. As $g(x_i) \in Q' = Z_{(h,n)}$,

$$h^{(1)}(g(x_i)) = h^{(2)}(g(x_i)) = \ldots = h^{(n)}(g(x_i)) = 0.$$

Then, by Lemma 2.3,

$$\big|(h \circ g)(x_{i+1}) - (h \circ g)(x_i)\big| \le [h \circ g]_{n,\alpha} \, |g(x_{i+1}) - g(x_i)|^{n+\alpha}.$$

Hence,

$$\begin{aligned}
&\sum_{i=1}^{k} \big|(h \circ g)(x_{i+1}) - (h \circ g)(x_i)\big|^{\frac{1}{n+\alpha}} \\
&\qquad \le \sum_{i=1}^{k} \left([h \circ g]_{n,\alpha} \big|g(x_{i+1}) - g(x_i)\big|^{n+\alpha}\right)^{\frac{1}{n+\alpha}} \\
&\qquad = ([h \circ g]_{n,\alpha})^{\frac{1}{n+\alpha}} \sum_{i=1}^{k} \big|g(x_{i+1}) - g(x_i)\big| \\
&\qquad \le ([h \circ g]_{n,\alpha})^{\frac{1}{n+\alpha}} \, V(g).
\end{aligned}$$

As g is a function of bounded variation, $V_{n+\alpha}(h \circ g) < \infty$. Now, applying Theorem 3.3 to $h \circ g$, we obtain a homeomorphism h_1 of $[0,1]$ such that $h \circ g \circ h_1$ is a $C^{n,\alpha}$ function and $f = h \circ g \circ h_1$ is the desired function ∎

Theorem 3.7. *Let $M \subset [0,1]$ and $1 \le n < \infty$. Then the following assertions are equivalent:*

1. There is a $C^{n,1}$ function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.

2. *M is the union of a G_δ set and a countable set and there is $P \in \mathcal{A}_{n,1}$ such that $M \subset P$.*

3. *There is a C^{n+1} function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

Proof. Assertion (1) $\Leftrightarrow$ (2) is a consequence of Theorem 3.6 and Assertion (2) $\Leftrightarrow$ (3) follows from the fact that $\mathcal{A}_{n,1} = \mathcal{A}_{n+1}$ and from [6: Theorem 3.17] ∎

4. Uncountable level sets

In this section we characterize the set of points where level sets of a given $C^{n,\alpha}$ function are uncountable.

Theorem 4.1. *Let $M \subset [0,1], 1 \le n < \infty$ and $0 < \alpha \le 1$. Then the following assertions are equivalent:*

(1) *M is an analytic set and there is $P \in \mathcal{A}_{n,\alpha}$ such that $M \subset P$.*

(2) *There is a $C^{n,\alpha}$ function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

Proof. Assertion (2) $\Rightarrow$ (1): By [9: p. 498/Theorem 2], M is analytic. Since $f^{-1}(\{y\})$ contains a perfect set for each $y \in M$, $y \in f(Z_{(f,n)})$. Hence $M \subset f(Z_{(f,n)})$. The proof of assertion (1) $\Rightarrow$ (2) is analogous to that Theorem 3.6 and can be carried out by applying [6: Proposition 4.2] ∎

Theorem 4.2. *Let $M \subset [0,1]$ and $1 \leq n < \infty$. Then the following assertions are equivalent:*

1. *There is a $C^{n,1}$ function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

2. *M is an analytic set and there is $P \in \mathcal{A}_{n,1}$ such that $M \subset P$.*

3. *There is a C^{n+1} function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is uncountable for every $y \in M$ and countable otherwise.*

Proof. Assertion (1) $\Leftrightarrow$ (2) is a consequence of Theorem 4.1 and Assertion (2) $\Leftrightarrow$ (3) follows from the fact that $\mathcal{A}_{n,1} = \mathcal{A}_{n+1}$ and from [6: Theorem 4.3] ∎

5. The Lipschitz case

In this section we characterize the set of points where level sets of a given Lipschitz function are perfect. We first have the following definitions and propositions.

Definition 5.1. For a Lipschitz function f on a closed interval $I \subset \mathbb{R}$ we set

$$\begin{aligned} P_f &= \{y \in \mathbb{R} : \ f^{-1}(\{y\}) \text{ is perfect } \} \\ D_f &= \{x \in \mathbb{R} : \ f \text{ is differentiable at } x\} \\ \tilde{Z}_{(f,1)} &= \{x \in D_f : \ f^{(1)}(x) = 0\}. \end{aligned}$$

Clearly, if $f \in C^1(I)$, then $\tilde{Z}_{(f,1)} = Z_{(f,1)}$.

Theorem 5.2 [4: Lemma 1]. *If f is a continuous function of bounded variation on $[0,1]$, there exists a homeomorphism h of $[0,1]$ onto itself such that $f \circ h$ is a Lipschitz function.*

Definition 5.3. Let $B = I \times J$ be a box. We use I_L, I_M, I_R to denote the left third, middle third and right third intervals of I, respectively, and define $B_L = I_L \times J, B_M = I_M \times J, B_R = I_R \times J$ – the so-called *vertical splitting* of B. A continuous function f is

- *diagonal* to B if the restriction of f to B is a linear function which passes through the diagonal corners of B
- *jagged inside* B if it is diagonal to each of B_L, B_M, B_R.

Definition 5.4. Let $f : [0,1] \to [0,1]$ be a continuous function and J a closed interval in $[0,1]$. Then we define $\alpha_f(J)$ to be the extended positive integer equal to the number of components of $f^{-1}(J)$.

Proposition 5.5. *Suppose G is a G_δ set with $\lambda(G) = 0$. Then there is a continuous function of bounded variation $f : [0,1] \to [0,1]$ such that*

1. *$f^{-1}(\{y\})$ is perfect for all $y \in G$*

2. *$f^{-1}(\{y\})$ is finite for all $y \notin G$.*

Proof. Since G is a G_δ set with $\lambda(G) = 0$, there exists a decreasing sequence of open sets $\{A_k\}$ with $\lambda(A_k) < \frac{1}{3^{2k}}$ and $G = \cap_{k=1}^{\infty} A_k$. We will construct our desired function f as the uniform limit of an appropriately chosen sequence $\{f_k\}$.

Let $f_0 : [0,1] \to [0,1]$ be the identity map. Considering A_1 we may obtain a countable collection $\{J_t\}_{t=1}^{\infty}$ of non-overlapping, closed intervals contained in A_1 such that $G \subseteq \cup_{t=1}^{\infty} J_t$. Let f_1 be the modification of f_0 which is linearly jagged inside $f_0^{-1}(J_t) \times J_t$ for all t, and let $\mathcal{G}_1 = \{f_0^{-1}(J_t) \times J_t : t \geq 1\}$. Then, at the end of stage 1, the following properties are satisfied:

- **(i)** f_1 is a continuous function linearly jagged inside each $B \in \mathcal{G}_1$ with $f_1(0) = 0$ and $f_1(1) = 1$.
- **(ii)** The graph of f_1 coincides with the graph of f_0 outside $\cup\mathcal{G}_1$.
- **(iii)** $|f_1^{-1}(\{y\})| \leq 3^1$ for all $y \in [0,1]$.
- **(iv)** f_1 is a continuous function of bounded variation and $V(f_1) \leq V(f_0) + 3\sum_{n=1}^{\infty} \alpha_{f_0}(J_n)\lambda(J_n) = V(f_0) + 3\sum_{n=1}^{\infty} \lambda(J_n) \leq V(f_0) + 3\lambda(A_1)$.
- **(v)** $\pi_2(\cup\mathcal{G}_1) \subseteq A_1$.
- **(vi)** $\lambda(\pi_1(B)) \leq \lambda(\pi_2(B)) \leq \lambda(A_1)$ for every $B \in \mathcal{G}_1$.
- **(vii)** $\|f_1 - f_0\|_0 \leq \lambda(A_1)$.

Now let us assume that we are at stage $k > 1$, f_k and $\mathcal{G}_k$ have been constructed already so that the following properties are satisfied:

- **(i)** f_k is a continuous function linearly jagged inside each $B \in \mathcal{G}_k$ with $f_k(0) = 0$ and $f_k(1) = 1$.
- **(ii)** The graph of f_k coincides with the graph of f_{k-1} outside $\cup\mathcal{G}_k$.
- **(iii)** $|f_k^{-1}(\{y\})| \leq 3^k$ for every $y \in [0,1]$.
- **(iv)** f_k is a continuous function of bounded variation and $V(f_k) \leq V(f_{k-1}) + 3^k\lambda(A_k)$.
- **(v)** $\pi_2(\cup\mathcal{G}_k) \subseteq A_k$.
- **(vi)** $\lambda(\pi_1(B)) \leq \lambda(\pi_2(B)) \leq \lambda(A_k)$ for every $B \in \mathcal{G}_k$.
- **(vii)** $\|f_k - f_{k-1}\|_0 \leq \lambda(A_k)$.
- **(viii)** If $y \in G$ and $B \in \mathcal{G}_{k-1}$ are such that $y \in \pi_2(B)$, then there exist disjoint boxes B_1 and B_2 in $\mathcal{G}_k$ contained in B such that $y \in \pi_2(B_1) \cap \pi_2(B_2)$.
- **(ix)** If $(x, f_k(x))$ is such that $f_k(x) \in G$, then $(x, f_k(x)) \in \cup\mathcal{G}_k$.

Let us now construct f_{k+1}. For this let $B' \in \mathcal{G}_k$ and fix B to be one of B'_L, B'_M or B'_R. Note that f_k is diagonal to B. Let us define f_{k+1} inside B first and construct a collection $\mathcal{G}_{k+1}^B$ of boxes inside B. As before, we obtain a countable collection of non-overlapping closed intervals $\{J_t\}$ contained in $\pi_2(B) \cap A_{k+1}$ such that $\pi_2(B) \cap G \subseteq \cup_{t=1}^{\infty} J_t$. Let $f_{k+1}|_{\pi_1(B)}$ be the modification of $f_k|_{\pi_1(B)}$ which is linearly jagged inside each of $(f_k|_{\pi_1(B)})^{-1}(J_t) \times J_t$ and set

$$\mathcal{G}_{k+1}^B = \{(f_k|_{\pi_1(B)})^{-1}(J_t) \times J_t : t \geq 1\}.$$

We do the above process for each such B and let f_{k+1} be the resulting function. We also set $\mathcal{G}_{k+1} = \cup\mathcal{G}_{k+1}^B$. In order to prove assertion (iv) we notice that

$$V(f_{k+1}) \leq V(f_k) + 3\sum_{n=1}^{\infty} \alpha_{f_k}(J_n)\lambda(J_n) \leq V(f_k) + 3^{k+1}\lambda(A_{k+1}).$$

It is easy to verify that f_{k+1} satisfies all other induction hypotheses of stage $k+1$. By assertion (vii), the sequence $\{f_k\}$ converges uniformly to some continuous function f. By assertion (iv),

$$V(f_k) \le V(f_0) + \sum_{j=1}^{k} 3^j \lambda(A_j) \le V(f_0) + \sum_{j=1}^{k} 3^j \frac{1}{3^{2j}} < V(f_0) + \frac{1}{2}.$$

Hence f is of bounded variation. By assertion (viii), $f^{-1}(\{y\})$ is perfect for all $y \in G$. By assertions (v), (ii) and (iii), $f^{-1}(\{y\})$ is finite for all $y \notin G$ ∎

Proposition 5.6. *Let $M \subseteq [0,1]$ be the union of a G_δ set and a countable set with $\lambda(M) = 0$. Then there exists a Lipschitz function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

Proof. Let $M = G \cup N$, where G is a G_δ set and N is countable, with $G \cap N = \emptyset$. By Proposition 5.6 there exists a continuous function of bounded variation $h : [0,1] \to [0,1]$ such that $h^{-1}(\{y\})$ is perfect for all $y \in G$ and finite otherwise. Since $G \cap N = \emptyset$, $N_1 = h^{-1}(N)$ is countable. By [6: Proposition 3.8], there is a continuous non-decreasing function $g : [0,1] \to [0,1]$ such that $g^{-1}(\{y\})$ is a closed non-degenerate interval for all $y \in N_1$ and $g^{-1}(\{y\})$ is a singleton for all $y \notin N_1$. Clearly, $(h \circ g)^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise and $h \circ g$ is a continuous function of bounded variation. Now, applying Theorem 5.2 we obtain a homeomorphism h_1 such that $h \circ g \circ h_1$ is a Lipschitz function and $f = h \circ g \circ h_1$ is the desired function ∎

Proposition 5.7. *Suppose I is a closed interval contained in $[0,1]$ and f is a Lipschitz function defined on I. Then the set of points where level sets are perfect is the union of a G_δ set and a countable set, and it has Lebesgue measure zero.*

Proof. By [7: Lemma 3.7], P_f is the union of a G_δ and a countable set. Without loss of generality, since countable sets obviously have Lebesgue measure zero, we may assume that P_f is a G_δ set. Let

$$P_f^1 = \big\{ y \in P_f : f(x_y) = y \text{ for some } x_y \in D_f \big\}.$$

Since f is Lipschitz it follows that $\lambda(P_f \setminus {P_f}^1) = 0$. It is clear that $x_y \in \tilde{Z}_{(f,1)}$ for every $y \in {P_f}^1$. By a standard result (e.g. [2: Lemma 7.10]) the set $f(\tilde{Z}_{(f,1)})$ has Lebesgue measure zero. So, since ${P_f}^1$ is a subset of $f(\tilde{Z}_{(f,1)})$, it also has Lebesgue measure zero. Therefore $\lambda(P_f) = 0$ ∎

The following theorem is the goal of this section.

Theorem 5.8. *Let $M \subseteq [0,1]$. Then the following assertions are equivalent:*

1. *There is a Lipschitz function $f : [0,1] \to [0,1]$ such that $f^{-1}(\{y\})$ is perfect for every $y \in M$ and finite otherwise.*

2. *M is the union of a G_δ set and a countable set and $\lambda(M) = 0$.*

Proof. Assertion (1) $\Rightarrow$ (2) is Proposition 5.7 while Assertion (2) $\Rightarrow$ (1) is Proposition 5.6 ∎

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