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Problem of Functional Extension and Space-Like Surfaces in Minkowski Space

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Abstract. Let $\Xi(x)$ be the distribution of convex sets over a domain $D \subset \mathbb{R}^n$ and let $\phi : \partial D \to \mathbb{R}$ be a function. We consider the existence problem of locally Lipschitz functions f defined in the domain D so that $f|_{\partial D} = \phi$ and $\nabla f(x) \in \Xi(x)$ almost everywhere in D. These questions are related to the existence problem for space-like surfaces of arbitrary codimension with prescribed boundary in Minkowski space.

Keywords: *Lipschitz function, pseudometric, Finsler space, Minkowski space, space-like surface*

AMS subject classification: 54C30

1. Introduction

Let $\mathbb{R}_1^{n+1}$ be an $(n+1)$-dimensional Minkowski space, that is an $(n+1)$-dimensional pseudo-Euclidean space with a metric of signature $(1, n)$. Let $x = (x_1, x_2, \dots, x_n)$ and $\chi = (t, x) \in \mathbb{R}_1^{n+1}$. For an arbitrary pair of vectors $\chi' = (t', x')$ and $\chi'' = (t'', x'')$ in $\mathbb{R}_1^{n+1}$ we will set the inner product to be

$$\langle \chi', \chi'' \rangle = -t't'' + \sum_{i=1}^{n} x_i' x_i''$$

and the scalar square of a vector $\chi \in \mathbb{R}_1^{n+1}$ to be

$$|\chi|^2 = \langle \chi, \chi \rangle.$$

We say that a non-zero vector $\chi \in \mathbb{R}_1^{n+1}$ is *space-like, time-like or light-like* depending on the realization of the conditions $|\chi|^2 > 0$, $|\chi|^2 < 0$ or $|\chi|^2 = 0$. The set of light-like vectors $\chi \in \mathbb{R}_1^{n+1}$ from the origin forms a light cone. The space-like vectors lie outside that cone, but the time-like ones lie inside it.

Let $t = f(x)$ be a C^1-function defined in a domain $D \subset \mathbb{R}^n$, and let F be its graph. The surface F is called *space-like* if any tangent vector to it is space-like. It

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is not difficult to see that F is space-like if and only if $|\nabla f(x)| < 1$ everywhere in D. The area of the space-like graph can be calculated by the integral

$$\int_D \sqrt{1 - |\nabla f|^2}\, dx. \tag{1}$$

The problem of describing the sets of admissible functions in the variational problem for the area functional (1) with Dirichlet boundary condition $f|_{\partial D} = \phi$ is transformed into the following problem of extension for functions under restrictions on the gradient:

Let $\phi : \partial D \to \mathbb{R}$ be a function. We are required to give conditions for the existence of a function $f : \overline{D} \to \mathbb{R}$ such that $f \in C^0(\overline{D}) \cap C^1(D)$, $f|_{\partial D} = \phi$ and $|\nabla f(x)| < 1$ everywhere in D.

From the point of view of these variational problems for sets of admissible functions for the functional (1) it is enough to study locally Lipschitz functions.

Let $\operatorname{Lip} D$ be the set of functions $f : D \to \mathbb{R}$ satisfying the Lipschitz condition on any compact subset of D. According to the Rademacher theorem, any function $f \in \operatorname{Lip} D$ has a total differential almost everywhere in D. So, in the case of functions belonging to the class $\operatorname{Lip} D$ we may extend the conception of space-like graph $t = f(x)$ by supposing

$$\operatorname*{ess\,sup}_{x \in K} |\nabla f(x)| < 1 \quad \text{for any compact } K \subset D, \tag{2}$$

and on the extension problem formulated above we may consider functions having property (2) instead of functions $f \in C^0(\overline{D}) \cap C^1(D)$. In the case of convex domains $D \subset \mathbb{R}^n$ the solution of the problem follows immediately from the classical Kirsbraun theorem about extension of Lipschitz functions (see [3 : Theorem 2.10.43] and new results [6, 7]).

In the present paper we study the following general problem of extension for functions under restrictions on the gradient.

Let $D \subset \mathbb{R}^n$ be a domain. Suppose that for every point $\chi = (t, x) \in \mathbb{R}^{n+1}_1$ with $x \in D$ the set $\Xi(x, t) \subset \mathbb{R}^n$ is defined. We will say that the distribution of sets $\Xi(x, t)$ is *locally uniformly bounded* over the domain D, if for every point $x_0 \in D$ there is a neighborhood $U(x_0)$ of that point and a number $R > 0$ such that for all $x \in U(x_0)$ the sets $\Xi(x, t)$ contain inside an n-dimensional ball $B(0, R) \subset \mathbb{R}^n$.

Let us fix an arbitrary distribution of sets $\Xi(x, t)$ over the domain $D \subset \mathbb{R}^n$. Then $\phi : \partial D \to \mathbb{R}$ is a boundary function. We require to find conditions for the existence of a function $f \in \operatorname{Lip} D$ with the property $f|_{\partial D} = \phi$ and such that

$$\nabla f(x) = \Big(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)\Big) \quad \text{is in } \Xi(x, f(x)) \text{ a.e. in } D. \tag{3}$$

The case in which the sets $\Xi = \Xi(x)$ are uniformly bounded, convex and symmetric was studied in [4]. In that paper, the authors build some Finsler metric ρ by a prescribed continuous distribution of $\Xi(x)$, and they show a criterion for the existence of the extension for functions $\phi : \partial D \to \mathbb{R}$ to a locally Lipschitz function $f : D \to \mathbb{R}$, defined on the domain D and having property (3).

In the present paper we give a criterion for the solvability of problem (3) in the general case of locally bounded convex sets $\Xi = \Xi(x)$, not necessarily open and symmetric. We also replace the condition of continuity of the distribution $\Xi(x)$ by the weaker integral condition.

The problem is the key problem for the description of admissible functions for the area functional of space-like and time-like surfaces in Minkowski space and in warped Lorentz products. A partial solution of the problem in [4] led to very general theorems of existence and uniqueness for solutions of the Dirichlet problem with singularities for the maximal surface equation in Minkowski space [5].

Below we apply our results to the general existence problem for k-dimensional space-like surfaces with prescribed boundary in the Minkowski space $\mathbb{R}^{n+1}_1$ $(2 \le k \le n)$ and in warped Lorentz products $M \times_\delta \hat{R}$ with a warping function of the general form $\delta = \delta(m, t)$.

The basis of our approach to the extension problem with restrictions on the gradient is the reduction of this problem to some problem about Lipschitz extension in Finsler spaces [9] associated with the distribution of convex sets $\Xi(x)$ over the domain D. Moreover, abandoning conditions of symmetry and uniform boundedness of sets $\Xi(x)$ substantially complicate the problem, because the Finsler pseudometrics which arise do not satisfy the traditional axioms of a metric space. So, asymmetry of the sets $\Xi(x)$ implies the omission of the symmetry axiom for the Finsler pseudometric. Giving up of the local uniform boundedness condition for the distribution of the sets $\Xi(x)$ implies giving up the identity axiom. Besides, in the general case, the (pseudo)metric can take values on the extended line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.

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2. Pseudometric spaces

The general extension problem of functions with restrictions on the gradient can be reduced to the problem of Lipschitz extension for functions in some special pseudometric spaces. Below we research this problem.

Let $\mathcal{X}$ be an arbitrary non-empty set and let $p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ be a function, having the following properties:

α) $p(x, x) = 0$ and $p(x, y) \ge 0$ for all $x, y \in \mathcal{X}$.

β) $p(x, y) \le p(x, z) + p(z, y)$ for all $x, y, z \in \mathcal{X}$.

The pair $(\mathcal{X}, p)$ is called a *pseudometric space*, and the function p is called a *pseudometric.* We observe that here we do not suppose symmetry of the pseudometric p, that is, $p(x, y) \ne p(y, x)$ in the general case. On the set $\mathcal{X}$ we may introduce a topology associated with the pseudometric p as the topology determined by the system of neighborhoods

$$U_\varepsilon(x) = \big\{y \in \mathcal{X} : p(x,y) < \varepsilon\big\}.$$

Thus, the concept of limit for the function $f : \mathcal{X} \to \mathbb{R}$ at a point and the concepts of continuity and uniform continuity can be introduced by standard way.

Let $\mathcal{S}$ be a subset of $\mathcal{X}$. A function $\phi : \mathcal{S} \to \mathbb{R}$ is called *p-Lipschitz* if there is a constant $L < +\infty$ such that

$$-L\, p(y,x) \le \phi(x) - \phi(y) \le L\, p(x,y) \qquad (x, y \in \mathcal{S}).$$

The smallest of the constants L we shall call a *Lipschitz constant* and denote it by $\mathrm{Lip}\,(\phi, \mathcal{S})$. Further, we restrict the study to functions ϕ for which $\mathrm{Lip}\,(\phi, \mathcal{S}) \le 1$.

We define some additional notions. For an arbitrary triple of points $x, y, z \in \mathcal{X}$ we set

$$\Lambda(x,y,z) = \frac{p(x,y)}{p(x,z) + p(z,y)}.$$

Since p is a pseudometric, then $\Lambda(x,y,z) \le 1$. The condition $\Lambda(x,y,z) = 1$ implies that the points x, y, z are situated on a 'geodesic line' with respect to the pseudometric p. We will call by a *pseudodistance* from set $\mathcal{P}$ to set $\mathcal{S}$ the quantity

$$p(\mathcal{P}, \mathcal{S}) = \inf\big\{p(x,y) : x \in \mathcal{P} \text{ and } y \in \mathcal{S}\big\}.$$

We will call by a *distance* between sets $\mathcal{P}, \mathcal{S} \subset \mathcal{X}$ the quantity

$$\mathrm{dist}\,(\mathcal{P}, \mathcal{S}) = \max\big\{p(\mathcal{P}, \mathcal{S}), p(\mathcal{S}, \mathcal{P})\big\}.$$

A set $U \in \mathcal{X}$ is said to be *p-compact*, if a subsequence convergent to some point $x_0 \in U$ may be chosen from every sequence $\{x_m\}$ of points of the given set U.

Lemma 1. *Let $\mathcal{S} \subset \mathcal{X}$ be an arbitrary set and let $\phi : \mathcal{S} \to \mathbb{R}$ be a function with*

$$-p(y,x) \le \phi(x) - \phi(y) \le p(x,y) \qquad (x, y \in \mathcal{S}).$$

Then there is a function $f : \mathcal{X} \to \mathbb{R}$ with $f|_{\mathcal{S}} = \phi$ such that

$$-p(y,x) \le f(x) - f(y) \le p(x,y) \qquad (x, y \in \mathcal{X}).$$

Proof. We set

$$f(x) = \inf_{y \in \mathcal{S}} \big\{\phi(y) + p(x,y)\big\}.$$

Then for an arbitrary $\varepsilon > 0$ and any $x', x'' \in \mathcal{X}$ there are points $y', y'' \in \mathcal{S}$ such that

$$f(x') > \phi(y') + p(x', y') - \varepsilon$$
$$f(x'') > \phi(y'') + p(x'', y'') - \varepsilon.$$

Hence

$$\begin{aligned} f(x') - f(x'') &> \phi(y') + p(x', y') - \varepsilon - \phi(y') - p(x'', y') \\ &= p(x', y') - p(x'', y') - \varepsilon \\ &\ge -p(x'', x') - \varepsilon. \end{aligned}$$

Similarly, we obtain

$$f(x') - f(x'') < p(x', x'') + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the lemma is proved ∎

Remark 1. In the case when $\mathcal{X}$ is a metric space, this expression is formulated in Federer's monograph [3: Section 2.10.44].

Our main problem in the present section is the extension problem of functions from the boundary into a domain with Lipschitz constant strictly separated from 1 on every compact subset of the domain. The lemma formulated below is a key one in the construction algorithm for this p-Lipschitz extension.

Fix a set $\mathcal{S} \subset \mathcal{X}$. Let ϕ be a p-Lipschitz function with $\mathrm{Lip}\,(\phi, \mathcal{S}) \le 1$. For arbitrary $\delta > 0$ and μ with $0 < \mu < 1$ we set

$$\begin{aligned} A_\delta^\mu &= A_\delta^\mu(\phi, \mathcal{S}) \\ &= \Big\{(x,y) \in \mathcal{S} \times \mathcal{S} :\, p(x,y) \ge \delta \text{ and } \phi(x) - \phi(y) \ge (1-\mu)\, p(x,y)\Big\}. \end{aligned}$$

Note that, for $\mathrm{Lip}\,(\phi, \mathcal{S}) < 1$, $A_\delta^\mu(\phi, \mathcal{S}) = \emptyset$ for μ sufficiently close to 0.

Lemma 2. *Let $\mathcal{P}, \mathcal{Q}, \mathcal{S} \subset \mathcal{X}$ be mutually disjoint sets and $p(\mathcal{P}, \mathcal{S}) > 0$. Let $\phi : \mathcal{S} \to \mathbb{R}$ be a p-Lipschitz function having the property*

$$\forall\, \delta > 0\; \exists\, \mu \in (0,1) : \quad \sup \Big\{\Lambda(x,y,\zeta) : (x,y) \in A_\delta^\mu(\phi, \mathcal{S}) \text{ and } \zeta \in \mathcal{P}\Big\} < 1. \tag{4}$$

Let $g : \mathcal{Q} \to \mathbb{R}$ be a function such that for some $L < 1$

$$-L\, p(\eta, \xi) \le g(\xi) - g(\eta) \le L\, p(\xi, \eta) \qquad (\xi, \eta \in \mathcal{Q}) \tag{5}$$

$$-L\, p(x, \xi) \le g(\xi) - \phi(x) \le L\, p(\xi, x) \qquad (\xi \in \mathcal{Q}, x \in \mathcal{S}). \tag{6}$$

Then there is a function $f : \mathcal{P} \to \mathbb{R}$ and a constant $L_0 < 1$ such that

$$-L_0\, p(\eta, \xi) \le f(\xi) - f(\eta) \le L_0\, p(\xi, \eta) \qquad (\xi, \eta \in \mathcal{P}) \tag{7}$$

$$-L_0\, p(\zeta, \xi) \le f(\xi) - g(\zeta) \le L_0\, p(\xi, \zeta) \qquad (\xi \in \mathcal{P}, \zeta \in \mathcal{Q}) \tag{8}$$

$$-L_0\, p(x, \xi) \le f(\xi) - \phi(x) \le L_0\, p(\xi, x) \qquad (\xi \in \mathcal{P}, x \in \mathcal{S}). \tag{9}$$

Proof. We put $\Delta = p(\mathcal{P}, \mathcal{S})$. By condition (4), for $\delta = \frac{1}{2}\Delta$ there is $\mu \in (0,1)$, for which

$$\sup \Big\{\Lambda(x,y,\zeta) :\, (x,y) \in A_\delta^\mu \text{ and } \zeta \in \mathcal{P}\Big\} = L_1 < 1. \tag{10}$$

We set

$$L_0 = \max \big\{L_1, L, 1-\mu, \tfrac{1}{2}\big\}$$

and consider the function

$$f(\xi) = \inf_{x \in \mathcal{S} \cup \mathcal{Q}} \big\{\psi(x) + L_0\, p(\xi, x)\big\}$$

where

$$\psi(x) = \begin{cases} \phi(x) & \text{for } x \in \mathcal{S} \\ g(x) & \text{for } x \in \mathcal{Q}. \end{cases}$$

We fix arbitrary $\xi, \eta \in \mathcal{P}$. For given $\varepsilon > 0$ points $x_\varepsilon, y_\varepsilon \in \mathcal{Q} \cup \mathcal{S}$ are found such that

$$f(\xi) \geq \psi(x_\varepsilon) + L_0\, p(\xi, x_\varepsilon) - \varepsilon$$
$$f(\eta) \geq \psi(y_\varepsilon) + L_0\, p(\eta, y_\varepsilon) - \varepsilon.$$

Hence,

$$f(\xi) - f(\eta) \leq L_0\big(p(\xi, y_\varepsilon) - p(\eta, y_\varepsilon)\big) + \varepsilon \leq L_0 p(\xi, \eta) + \varepsilon$$
$$f(\xi) - f(\eta) \geq L_0\big(p(\xi, x_\varepsilon) - p(\eta, x_\varepsilon)\big) - \varepsilon \geq -L_0 p(\eta, \xi) - \varepsilon$$

and by virtue of the arbitrary choice of $\varepsilon > 0$, we obtain (7).

We shall prove inequalities (8) - (9) simultaneously. Let $\xi \in \mathcal{P}$ and $x \in \mathcal{S} \cup \mathcal{Q}$. Then

$$f(\xi) - \psi(x) \leq L_0 p(\xi, x)$$

which implies the validity of inequalities in the right parts of (8) - (9). Let us show that

$$f(\xi) - \psi(x) \geq -L_0 p(x, \xi).$$

We have

$$\forall\ \varepsilon > 0\ \exists\ x_\varepsilon \in \mathcal{S} \cup \mathcal{Q}: \quad f(\xi) \geq \psi(x_\varepsilon) + L_0 p(\xi, x_\varepsilon) - \varepsilon.$$

We carry out the following arguments separately depending on the placement of the points x, x_ε on the sets $\mathcal{Q}$ and $\mathcal{S}$.

α) Let $x, x_\varepsilon \in \mathcal{Q}$. Then by (5)

$$\begin{aligned} f(\xi) - \psi(x) &\geq \psi(x_\varepsilon) - \psi(x) + L_0 p(\xi, x_\varepsilon) - \varepsilon \\ &= g(x_\varepsilon) - g(x) + L_0 p(\xi, x_\varepsilon) - \varepsilon \\ &\geq -L_0 p(x, x_\varepsilon) + L_0 p(\xi, x_\varepsilon) - \varepsilon \\ &\geq -L_0 p(x, \xi) - \varepsilon. \end{aligned}$$

β) Let $x \in \mathcal{S}$ and $x_\varepsilon \in \mathcal{Q}$, or $x \in \mathcal{Q}$ and $x_\varepsilon \in \mathcal{S}$. Then, as above, by virtue of (6) we have

$$f(\xi) - \psi(x) \geq -L_0 p(x, \xi) - \varepsilon.$$

γ) Let $x, x_\varepsilon \in \mathcal{S}$ and $p(x, x_\varepsilon) \leq \delta = \frac{1}{2}\Delta$. Then

$$\begin{aligned} f(\xi) - \psi(x) &\geq \phi(x_\varepsilon) - \phi(x) + L_0 p(\xi, x_\varepsilon) - \varepsilon \\ &\geq -p(x, x_\varepsilon) + L_0 p(\xi, x_\varepsilon) - \varepsilon \\ &\geq -\tfrac{\Delta}{2} + L_0 \Delta - \varepsilon \\ &\geq -\varepsilon \\ &\geq -L_0 p(x, \xi) - \varepsilon. \end{aligned}$$

δ) Let $(x, x_\varepsilon) \in A^\mu_\delta$. It follows from (10) that

$$\frac{p(x, x_\varepsilon)}{p(x, \zeta) + p(\zeta, x_\varepsilon)} \leq L_1 \leq L_0 \qquad (\zeta \in \mathcal{P}).$$

Moreover,

$$\begin{aligned}f(\xi)-\psi(x) &\geq \phi(x_\varepsilon)-\phi(x)+L_0p(\xi,x_\varepsilon)-\varepsilon\\ &\geq -p(x,x_\varepsilon)+L_0p(\xi,x_\varepsilon)-\varepsilon\\ &\geq -L_0\big(p(x,\xi)+p(\xi,x_\varepsilon)\big)+L_0p(\xi,x_\varepsilon)-\varepsilon\\ &= -L_0p(x,\xi)-\varepsilon.\end{aligned}$$

ε) Now, if the point (x,x_ε) does not belong to the set A^μ_δ but $p(x,x_\varepsilon)>\delta$, then by definition of the set A^μ_δ

$$\phi(x_\varepsilon)-\phi(x)\geq -(1-\mu)\,p(x,x_\varepsilon).$$

From here we obtain

$$\begin{aligned}f(\xi)-\psi(x) &\geq \phi(x_\varepsilon)-\phi(x)+L_0p(\xi,x_\varepsilon)-\varepsilon\\ &\geq -(1-\mu)\,p(x,x_\varepsilon)+L_0p(\xi,x_\varepsilon)-\varepsilon\\ &\geq -L_0p(x,\xi)-\varepsilon.\end{aligned}$$

Combining the cases $\alpha)-\varepsilon)$ and passing to the limit as $\varepsilon\to 0$, we get

$$f(\xi)-\psi(x)\geq -L_0p(x,\xi)\qquad (\xi\in\mathcal{P},x\in\mathcal{S}\cup\mathcal{Q})$$

that implies the validity of the left inequalities in relations (8) - (9) and the lemma is proved ∎

Suppose that the pseudometric space $(\mathcal{X},p)$ is a so-called *arcwise connected* one and that the pseudometric p coincides with the so-called *pseudointrinsic distance.*

Let us explain the terminology. By a arcwise connected spaces $(\mathcal{X},p)$ we designate spaces with the property that for all $x,y\in\mathcal{X}$ there is a continuous mapping $\gamma:[0,1]\to(\mathcal{X},p)$ such that $\gamma(0)=x$ and $\gamma(1)=y$. We also say that p coincides with the pseudointrinsic distance in $\mathcal{X}$ if $p(x,y)=\inf_\gamma|\gamma|_p$, where the infimum is taken over all curves γ joining the points x and y. Moreover,

$$|\gamma|_p=\sup\sum_{i=1}^n p\big(\gamma(t_i),\gamma(t_{i+1})\big)$$

where the supremum is calculated over all partitions of the segment $[0,1]$ by points $0=t_1\leq t_2\leq\ldots\leq t_{n+1}=1$. We note that the length $|\gamma|_p$ of a curve depends on how one traces the curve γ.

The following statement provides the main result of this section.

Lemma 3. *Let $(\mathcal{X},p)$ be a pseudometric space with the described properties. Let $\mathcal{S}\subset\mathcal{X}$ be an arbitrary subset and let $\phi:\mathcal{S}\to\mathbb{R}$ be a p-Lipschitz function. In order that the function ϕ be a trace of some function $f:\mathcal{X}\to\mathbb{R}$ satisfying the conditions*

$$\begin{aligned}&\forall\, U\subset\mathcal{X}\ \textit{with}\ \operatorname{dist}(U,\mathcal{S})>0\ \exists\ \textit{constant}\ L_U<1:\\ &\limsup_{y\to x}\frac{f(x)-f(y)}{p(x,y)}\leq L_U\ \forall\, x\in U &(11)\\ &\liminf_{y\to x}\frac{f(x)-f(y)}{p(y,x)}\geq -L_U &(12)\end{aligned}$$

it is sufficient that

$$\begin{aligned}&\forall\ p\text{-}compact\ U\subset\mathcal{X}\ \ with\ \ \operatorname{dist}(U,\mathcal{S})>0\ \ and\ \forall\ \delta>0\\ &\exists\ \mu\in(0,1)\ \ such\ that\ \sup\big\{\Lambda(x,y,z):\ (x,y)\in A^{\mu}_{\delta},z\in U\big\}<1.\end{aligned}\tag{13}$$

In the case when either the function ϕ or the set $\mathcal{S}$ is bounded, condition (13) *is also necessary.*

Proof. First we prove sufficiency. For an arbitrary $k\in\mathbb{N}$ we set

$$\begin{aligned}\Omega_k&=\big\{x\in\mathcal{X}:\ \operatorname{dist}(x,\mathcal{S})>\tfrac{1}{k}\big\}\\ S_k&=\big\{x\in\mathcal{X}:\ \operatorname{dist}(x,\mathcal{S})=\tfrac{1}{k}\big\}\\ \overline{\Omega}_k&=S_k\cup\Omega_k.\end{aligned}$$

Note that $S_i\cap S_j=\emptyset$ and $\operatorname{dist}(S_i,S_j)>0$ for an arbitrary $i\neq j$. It is not hard to see that the sets $\overline{\Omega}_k$ are closures of the sets Ω_k in the topology determined by the pseudometric p. Using property (13) and Lemma 2 for $\mathcal{P}=\overline{\Omega}_1$ and $\mathcal{Q}=\emptyset$, we find a constant $L_1<1$ and a function $u_1:\overline{\Omega}_1\to\mathbb{R}$ such that

$$\begin{aligned}-L_1p(\eta,\xi)\le u_1(\xi)-u_1(\eta)\le L_1p(\xi,\eta)&\qquad(\xi,\eta\in\overline{\Omega}_1)\\ -L_1\,p(x,\xi)\le u_1(\xi)-\phi(x)\le L_1p(\xi,x)&\qquad(\xi\in\overline{\Omega}_1,x\in\mathcal{S}).\end{aligned}$$

Now, using (13) and Lemma 2 for $\mathcal{P}=\overline{\Omega}_2\setminus\overline{\Omega}_1$ and $\mathcal{Q}=\overline{\Omega}_1$, we find a constant $L_2<1$ and a function $u_2:\overline{\Omega}_2\setminus\Omega_1\to\mathbb{R}$ such that

$$\begin{aligned}-L_2\,p(\eta,\xi)\le u_2(\xi)-u_2(\eta)\le L_2p(\xi,\eta)&\qquad(\xi,\eta\in\overline{\Omega}_2\setminus\Omega_1)\\ -L_2p(\zeta_1,\xi)\le u_2(\xi)-u_1(\zeta_1)\le L_2p(\xi,\zeta_1)&\qquad(\xi\in\overline{\Omega}_2\setminus\Omega_1,\zeta_1\in\overline{\Omega}_1)\\ -L_2p(x,\xi)\le u_2(\xi)-\phi(x)\le L_2p(\xi,x)&\qquad(\xi\in\overline{\Omega}_2\setminus\overline{\Omega}_1,x\in\mathcal{S}).\end{aligned}$$

Continuing this process for an arbitrary $k>1$ we find a constant $L_k<1$ and a function $u_k:\overline{\Omega}_k\setminus\overline{\Omega}_{k-1}\to\mathbb{R}$ for which

$$\begin{aligned}-L_kp(\eta,\xi)\le u_k(\xi)-u_k(\eta)\le L_kp(\xi,\eta)&\qquad\big(\xi,\eta\in\overline{\Omega}_k\setminus\overline{\Omega}_{k-1}\big)\\ -L_kp(\zeta_i,\xi)\le u_k(\xi)-u_i(\zeta_i)\le L_kp(\xi,\zeta_i)&\qquad\begin{pmatrix}\xi\in\overline{\Omega}_k\setminus\Omega_{k-1}\\ \zeta_i\in\overline{\Omega}_i\ (i=1,\dots,k-1)\end{pmatrix}\\ -L_k\ p(x,\xi)\le u_k(\xi)-\phi(x)\le L_kp(\xi,x)&\qquad\big(\xi\in\overline{\Omega}_k\setminus\overline{\Omega}_{k-1},x\in\mathcal{S}\big).\end{aligned}$$

Thus, the function $\tilde{f}$ is equal u_k on $\overline{\Omega}_k$ and is defined on the set $\cup_{k=1}^{\infty}\Omega_k$. It is clear that $\tilde{f}$ has properties (11) - (12) by construction.

We put $\mathcal{S}'=\{x\in\mathcal{X}:\ \operatorname{dist}(x,\mathcal{S})=0\}$. It is obvious that $\cup_{k=1}^{\infty}\Omega_k\cup\mathcal{S}'=\mathcal{X}$. Let $x\in\mathcal{S}'$. There exists a sequence of points $\{y_k\}_{k=1}^{\infty}\subset\mathcal{S}$ for which $p(x,y_k)\to0$ for $k\to\infty$. We set $f_0(x)=\lim_{k\to\infty}\phi(y_k)$. Let us show that the limit exists and does not depend on the choice of the sequence. As

$$\begin{aligned}&\phi(y_k)-\phi(y_{k+l})\le p(y_k,y_{k+l})\le p(y_k,x)+p(x,y_{k+l})\to0\\ &\phi(y_k)-\phi(y_{k+l})\ge-p(y_{k+l},y_k)\ge-\big(p(y_{k+l},x)+p(x,y_k)\big)\to0,\end{aligned}$$

the sequence $\phi(y_k)$ is fundamental in $\mathbb{R}$ and has a limit. The independence of this limit from the sequence $\{y_k\}$ can be established similarly.

It remains to show that the function

$$f(x) = \begin{cases} \tilde{f}(x) & \text{for } x \in \cup_{k=1}^{\infty}\Omega_k \\ f_0(x) & \text{for } x \in \mathcal{S}' \end{cases}$$

coincides with ϕ on $\mathcal{S}$. Let $x \in \mathcal{S}$ and $y \in \mathcal{X}$. If $y \in \cup_{k=1}^{\infty}\Omega_k$, then

$$f(y) - \phi(x) = \tilde{f}(y) - \phi(x) < p(y, x)$$
$$f(y) - \phi(x) = \tilde{f}(y) - \phi(x) > -p(x, y).$$

If $y \in \mathcal{S}' \setminus \mathcal{S}$, then there is a sequence $y_k \in \mathcal{S}$ for which $f(y) = f_0(y) = \lim_{k\to\infty} \phi(y_k)$. Hence,

$$\begin{aligned} f(y) - \phi(x) &= \lim_{k\to\infty} \phi(y_k) - \phi(x) \\ &\le \lim_{k\to\infty} p(y_k, x) \\ &\le \lim_{k\to\infty} \big(p(y_k, y) + p(y, x)\big) \\ &= p(y, x). \end{aligned}$$

Similarly,

$$f(y) - \phi(x) = f_0(y) - \phi(x) \ge -p(x, y).$$

From here we conclude that $f|_{\mathcal{S}} = \phi$.

Finally, we turn to the proof of necessity. Assume that the function $\phi : \mathcal{S} \to \mathbb{R}$ is a trace of some function $f : \mathcal{X} \to \mathbb{R}$ satisfying condition (11) on the set $\mathcal{S} \subset \mathcal{X}$. Then for all p-compact U with $\operatorname{dist}(U, S) > 0$ there is $\varepsilon > 0$ and constant $L < 1$ such that for any $\xi \in U$ under every $\eta \in \{x \in \mathcal{X} : p(\xi, x) = \varepsilon\} = C_\varepsilon$ we have

$$-L\, p(\eta, \xi) \le f(\xi) - f(\eta) \le L\, p(\xi, \eta).$$

a) First we consider the case when the set $\mathcal{S}$ is bounded, that is

$$\sup_{x,y\in\mathcal{S}} p(x, y) = M < +\infty.$$

Let γ be a path leading from x to ξ and such that $p(\xi, x) > |\gamma|_p - \delta \quad (\delta > 0)$ and $\eta \in \gamma \cap C_\varepsilon$. Then

$$\begin{aligned} f(\xi) - \phi(x) &= f(\xi) - f(\eta) + f(\eta) - \phi(x) \\ &\le L\, p(\xi, \eta) + p(\eta, x) \\ &= \big(p(\xi, \eta) + p(\eta, x)\big) \frac{L\, p(\xi, \eta) + p(\eta, x)}{p(\xi, \eta) + p(\eta, x)} \\ &\le \big(p(\xi, x) + \delta\big) \frac{L\, p(\xi, \eta) + p(\eta, x)}{p(\xi, \eta) + p(\eta, x)} \\ &= \big(p(\xi, x) + \delta\big) \frac{L + \frac{1}{\varepsilon} p(\eta, x)}{1 + \frac{1}{\varepsilon} p(\eta, x)} \end{aligned}$$

and by arbitrarity in the choice of $\delta > 0$ we obtain

$$f(\xi) - \phi(x) \le \frac{L + \frac{M}{\varepsilon}}{1 + \frac{M}{\varepsilon}}\, p(\xi, x). \tag{14}$$

Suppose that property (13) does not hold. There is a sequence of points $x_m, y_m \in \mathcal{S}$ and z_m such that $\inf_m \operatorname{dist}(z_m, \mathcal{S}) = \varepsilon > 0$. And there is a $\delta_1 > 0$ such that, for all $m \in \mathbb{N}$, $(x_m, y_m) \in A_{\delta_1}^{\frac{1}{m}}$ and

$$\frac{p(x_m, y_m)}{p(x_m, z_m) + p(z_m, y_m)} \to 1 \qquad (m \to \infty).$$

From the condition $(x_m, y_m) \in A_{\delta_1}^{\frac{1}{m}}$ we have

$$\phi(x_m) - \phi(y_m) \ge (1 - \tfrac{1}{m})\, p(x_m, y_m). \tag{15}$$

On the other hand, it follows from (14) that

$$\begin{aligned} \phi(x_m) - \phi(y_m) &= \phi(x_m) - f(z_m) + f(z_m) - \phi(y_m) \\ &\le \frac{p(x_m, z_m) + p(z_m, y_m)}{p(x_m, y_m)} \, \tfrac{L + \frac{M}{\varepsilon}}{1 + \frac{M}{\varepsilon}}\, p(x_m, y_m). \end{aligned}$$

This contradicts (15) for large m.

b) Suppose that the function ϕ is bounded. Under realization of this condition the function f is bounded. Let

$$M_0 = \sup_{x \in \mathcal{S}} |\phi(x)| = \sup_{x \in \mathcal{X}} |f(x)|.$$

If $x \in \mathcal{S}$, then let γ be a path joining the points ξ and x such that $p(\xi, x) > |\gamma|_p - \delta$ $(\delta > 0)$. Assume that $p(\xi, x) \ge 4M_0$. We have

$$f(\xi) - \phi(x) \le 2M_0 \le \tfrac{1}{2}\, p(\xi, x).$$

If $p(\xi, x) < 4M_0$, then

$$\begin{aligned} f(\xi) - \phi(x) &\le \big(p(\xi, \eta) + p(\eta, x)\big)\, \frac{L\, p(\xi, \eta) + p(\eta, x)}{p(\xi, \eta) + p(\eta, x)} \\ &\le \big(p(\xi, x) + \delta\big)\, \frac{p(\xi, x) + \delta - (1 - L)\varepsilon}{p(\xi, x)}. \end{aligned}$$

Setting $\delta \to 0$ we obtain

$$\begin{aligned} f(\xi) - \phi(x) &\le p(\xi, x)\, \frac{p(\xi, x) - (1 - L)\varepsilon}{p(\xi, x)} \\ &\le p(\xi, x)\, \frac{4M_0 - (1 - L)\varepsilon}{4M_0}. \end{aligned}$$

Hence, we can find a constant $L_1 < 1$ such that for all $x \in \mathcal{S}$

$$f(\xi) - \phi(x) \le L_1 p(\xi, x).$$

Repeating arguments of section a), we come to (13). The lemma is completely proved ∎

We need the following construction. Let $\overline{p} : \mathcal{X} \times \mathcal{X} \to \overline{\mathbb{R}}$ be a function having properties $\alpha) - \beta)$ of a pseudometric described above. We introduce the space $\mathcal{X}$ as union $\cup_{a\in\mathcal{A}}\mathcal{X}_a$ of subsets $\mathcal{X}_a \subset \mathcal{X}$ such that for all $a \in \mathcal{A}$ and for all $x, y \in \mathcal{X}_a$ the values $\overline{p}(x, y)$ and $\overline{p}(y, x)$ are finite. On each of the sets $\mathcal{X}_a$ the function $\overline{p}$ induces a pseudometric. We shall say that the space $(\mathcal{X}, \overline{p})$ is *arcwise connected* if any pseudometric space $(\mathcal{X}_a, \overline{p})$ is arcwise connected. In the case when for every $a \in \mathcal{A}$ the pseudometric $\overline{p}$ is an intrinsic distance in the space $(\mathcal{X}_a, \overline{p})$ we say that the function $\overline{p}$ is an *intrinsic distance* in $\mathcal{X}$.

The next lemma follows immediately from Lemma 3.

Lemma 4. *Let $(\mathcal{X}, \overline{p})$ be arcwise connected and $\overline{p}$ be an intrinsic distance on $\mathcal{X}$. Let $\mathcal{S} \subset \mathcal{X}$ be an arbitrary set and let $\phi : \mathcal{S} \to \mathbb{R}$ be a function satisfying the condition*

$$-\overline{p}(y, x) \leq \phi(x) - \phi(y) \leq \overline{p}(x, y) \qquad (x, y \in \mathcal{S}).$$

In order that the function ϕ be the trace of a function $f : \mathcal{X} \to \mathbb{R}$ satisfying conditions (11) – (12) *it is sufficient for ϕ to have property* (13) *on every subset $\mathcal{S} \cap \mathcal{X}_a$ with $a \in \mathcal{A}$. In the case when on every $\mathcal{X}_a$ either the function ϕ or the set $\mathcal{S} \cap \mathcal{X}_a$ is bounded, condition* (13) *is also necessary.*

In the case when the pseudometric p is a metric, i.e. it satisfies the axioms of identity and symmetry, the existence criterion of p-Lipschitz extensions of a function can be formulated in a clearer manner.

For an arbitrary pair of points $x_1, x_2 \in \mathcal{X}$ we put

$$\Gamma(x_1, x_2) = \big\{x \in \mathcal{X} : p(x_1, x_2) = p(x_1, x) + p(x, x_2)\big\}.$$

Note that the set $\Gamma(x_1, x_2)$ is non-empty because at least $x_1, x_2 \in \Gamma(x_1, x_2)$.

Theorem 1. *Let $K \subset \mathcal{X}$ be a p-compact set. A function $\phi : K \to \mathbb{R}$ is the trace on K of a function $f : \mathcal{X} \to \mathbb{R}$ satisfying the condition*

$$\limsup_{y\to x} \frac{|f(x) - f(y)|}{p(x, y)} < 1$$

if and only if ϕ has the properties

$$|\phi(x_1) - \phi(x_2)| \leq p(x_1, x_2) \quad (x_1, x_2 \in K) \tag{16}$$

$$|\phi(x_1) - \phi(x_2)| < p(x_1, x_2) \quad \textit{if } p(x_1, x_2) > 0, \Gamma(x_1, x_2) \cap (\mathcal{X} \setminus K) \neq \emptyset. \tag{17}$$

Proof. Condition (16) means that ϕ is a Lipschitz function on K. Therefore, to prove the theorem we establish that conditions (17) and (13) are equivalent. In fact, suppose that (17) holds. We shall show that for any set U with $\overline{U} \subset \mathcal{X} \setminus K$ and for any $\delta > 0$ there is number m_0 for which

$$\sup\Big\{L(x, y, z) : (x, y) \in A_\delta^{\frac{1}{m_0}} \text{ and } z \in U\Big\} < 1.$$

We suppose the opposite, i.e. there are $\delta > 0$, points $x_m, y_m \in K$ and $z_m \in U$ such that $p(x_m, y_m) \geq \delta$ and

$$\frac{\rho(x_m, y_m)}{\rho(x_m, z_m) + \rho(z_m, y_m)} \to 1 \qquad (m \to +\infty). \tag{18}$$

From the assumptions for the sets K and U, there are points $x_0, y_0 \in K$ and $z_0 \in U$ for which

$$\left.\begin{array}{r} p(x_m, x_0) + p(x_0, x_m) \to 0 \\ p(y_m, y_0) + p(y_0, y_m) \to 0 \\ p(z_m, z_0) + p(z_0, z_m) \to 0 \end{array}\right\} \qquad (m \to +\infty).$$

From this and (18) we get $p(x_0, y_0) = p(x_0, z_0) + p(z_0, y_0)$ which means $z_0 \in \Gamma(x_0, y_0)$. On the other hand, since $(x_m, y_m) \in A_\delta^{\frac{1}{m}}$, then

$$\phi(x_m) - \phi(y_m) \geq \big(1 - \tfrac{1}{m}\big) p(x_m, y_m).$$

Passing to the limit as $m \to \infty$ we obtain $\phi(x_0) - \phi(y_0) \geq p(x_0, y_0) \geq \delta$. Therefore, as ϕ is Lipschitz, we have the equality $\phi(x_0) - \phi(y_0) = p(x_0, y_0)$ for $z_0 \in \Gamma(x_0, y_0) \cap \mathcal{X})$ what contradicts to (17). Hence condition (13) holds.

Inversely, let us suppose that condition (13) holds. Then we shall show that condition (17) holds too. Again we suppose the opposite. Then there are points $x_0, y_0 \in K$ and $z_0 \in \Gamma(x_0, y_0) \cap \mathcal{X}$, for which $\phi(x_0) - \phi(y_0) = p(x_0, y_0) > 0$. We put $U = \{z_0\}$. From (13) for $\delta = \rho(x_0, y_0)$ we have $L(x_0, y_0, z_0) < 1$. Therefore $z_0 \notin \Gamma(x_0, y_0)$ and we obtain a contradiction. The theorem is proved ∎

3. Finsler metric

The extension problem of functions with restrictions on the gradient can be reduced to the problem about Lipschitz extensions in Finsler spaces. Using results from the previous section, we obtain very general theorems answering the formulated problem.

Let $\Omega \subset \mathbb{R}^n$ be a domain and let Φ be a function, determined in $\Omega \times \mathbb{R}^n$, which takes values in $\overline{\mathbb{R}}$ and such that the following conditions are fulfilled:

a) $\Phi(x, \xi) \geq 0$.

b) $\Phi(x, \lambda\xi) = \lambda\Phi(x, \xi)$ for all $\lambda \geq 0$, $\xi \in \mathbb{R}^n$ and $x \in \Omega$.

c) $\Xi(x) = \big\{\xi \in \mathbb{R}^n : \Phi(x, \xi) < 1\big\}$ are convex for every $x \in \Omega$.

Determine the dual function H by

$$H(x, \eta) = \sup_{\Phi(x,\xi)=1} \langle \eta, \xi \rangle$$

(see [8: Section 15]) and then set

$$h(x) = \sup_{|\eta|=1} \sup_{\Phi(x,\xi)=1} \langle \eta, \xi \rangle.$$

It is clear that the function H has properties a) - c). We define the set

$$C(x) = \big\{\eta \in \mathbb{R}^n : H(x,\eta) < 1\big\}.$$

We also note the formula

$$\Phi(x,\xi) = \sup_{H(x,\eta)\neq 0} \frac{\langle \xi, \eta\rangle}{H(x,\eta)}.$$

In the general case the function H takes on $\Omega \times \mathbb{R}^n$ values in $\overline{\mathbb{R}}$. Infinite values of H arise in the cases when the convex set $\Xi(x)$ is unbounded. On the other hand, it is not difficult to see that the set $\Xi(x)$ is bounded if and only if $h(x) < +\infty$.

It is useful to consider the following example.

Example 1. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis in $\mathbb{R}^n_2$ and let $\Phi(x,\xi) = |\langle e_1, \xi\rangle|$. Then

$$\Xi(x) = \{\xi : |\langle e_1, \xi\rangle| < 1\} = \{\xi \in \mathbb{R}^n : |\xi_1| < 1\}.$$

Here the dual function H has the form

$$H(x,\eta) = \begin{cases} |\eta_1| & \text{if } \eta_i = 0; i = 2,3,\ldots,n \\ +\infty & \text{if } \eta_i \neq 0 \text{ for some } i \geq 2 \end{cases}$$

and takes infinite values. The set $C(x)$ is the open interval $(-1,1)$ situated on the axis $O\eta_1$. Here $h(x) \equiv +\infty$.

For arbitrary points $x, y \in \Omega$ we set

$$\overline{\rho}(x,y) = \inf_{\gamma} \int_0^1 H(\gamma(t), \dot{\gamma}(t))\, dt$$

where the infimum is taken over all locally Lipschitz curves $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$. It is clear that in the general case the quantities $\overline{\rho}(x,y)$ and $\overline{\rho}(y,x)$ do not coincide.

Lemma 5. *The function $\overline{\rho}$ has properties α) and β) of a pseudometric.*

Proof. The realization of condition α) is obvious. We show the validity of condition β). Let $x, y, z \in \Omega$ and suppose $\overline{\rho}(x,z), \overline{\rho}(z,y) < \infty$. For every $\varepsilon > 0$ there are curves $\gamma_i : [0,1] \to \Omega \quad (i = 1,2)$ such that

$$\gamma_1(0) = x, \quad \gamma_1(1) = z, \quad \int_0^1 H\big(\gamma_1(t), \dot{\gamma}_1(t)\big)dt < \overline{\rho}(x,z) + \tfrac{\varepsilon}{2}$$
$$\gamma_2(0) = z, \quad \gamma_2(1) = y, \quad \int_0^1 H\big(\gamma_2(t), \dot{\gamma}_2(t)\big)dt < \overline{\rho}(z,y) + \tfrac{\varepsilon}{2}.$$

We put

$$\gamma_3(t) = \begin{cases} \gamma_1(2t) & \text{for } t \in [0, \frac{1}{2}) \\ \gamma_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Then

$$\begin{aligned}\overline{\rho}(x,y) &\le \int_0^1 H\big(\gamma_3(t),\dot\gamma_3(t)\big)dt\\ &= \int_0^1 H\big(\gamma_1(t),\dot\gamma_1(t)\big)dt + \int_0^1 H\big(\gamma_2(t),\dot\gamma_2(t)\big)dt\\ &\le \overline{\rho}(x,z)+\overline{\rho}(z,y)+\varepsilon.\end{aligned}$$

By virtue of the arbitrary choice of $\varepsilon > 0$, the triangle axiom is realized. In the case of conversion in $+\infty$ for even one of the quantities $\overline{\rho}(x,z)$ or $\overline{\rho}(z,y)$ inequality α) is obvious ∎

Later we shall call a pseudometric which has properties α) and β) by *Finsler pseudometric.*

Let us consider the case when the distribution $\Xi(x)$ of convex sets is locally uniformly bounded. Let Ω be a domain in $\mathbb{R}$ and let ρ be a Finsler pseudometric. We shall assume that the function h is locally bounded in Ω, fix a subdomain $\Omega' \subset\subset \Omega$ and set $h' = \sup_{x\in\Omega'} h(x)$. For an arbitrary pair of points $x_1, x_2 \in \Omega'$ such that for the connecting segment $\overline{x_1x_2}$ we have $\overline{x_1x_2} \subset \Omega'$, we get

$$\rho(x_1,x_2) \le \int_0^1 H(x_1+te,e)\,dt \le h'\,|x_2-x_1|$$

where $e = \frac{x_2-x_1}{|x_2-x_1|}$. Therefore, any ρ-Lipschitz in Ω function f is locally Lipschitz in the Euclidean metric. By the Rademacher theorem, the function f has a total differential almost everywhere in Ω. In particular, the vector $(f_{x_1}, f_{x_2}, \ldots, f_{x_n}) = \nabla f(x)$ is defined almost everywhere in Ω.

Let Ω_ρ be the completion of the domain Ω by the pseudometric ρ and let $\partial\Omega_\rho = \Omega_\rho \setminus \Omega$. Assume that the completion Ω_ρ is non-empty.

The following theorem is the main result of the present paper.

Theorem 2. *In order that the function $\phi : \partial\Omega_\rho \to \mathbb{R}$ be the trace on $\partial\Omega_\rho$ for a function $f : \Omega \to \mathbb{R}$ satisfying the condition*

$$\operatorname*{ess\,sup}_U \Phi(x,\nabla f(x)) < 1 \qquad (U \subset \Omega \textit{ compact}) \tag{19}$$

it is sufficient that ϕ is ρ-Lipschitz and has the property

$$\begin{aligned}&\forall\ \delta>0\ \exists\ \mu\in(0,1):\\ &\sup\Big\{L(x,y,z):\ (x,y)\in A^\mu_\delta(\phi,\partial\Omega_\rho)\ \textit{and}\ z\in U\Big\}<1\end{aligned} \tag{20}$$

on every subset $U \subset\subset \Omega$. In the case when the boundary $\partial\Omega_\rho$ or the boundary function ϕ is bounded, condition (20) *is also necessary.*

Proof. By Lemma 3, it is necessary to establish the equivalence of restrictions (19) and (20). Suppose that (19) is held. We fix a set U with $\overline{U} \subset \Omega$ and a subdomain

$\Omega_1 \supset U$ with $\Omega_1 \subset \Omega$. Let $x_1, x_2 \in \Omega_1$ be arbitrary and choose a locally Lipschitz path $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. We have

$$f(x_2) - f(x_1) = \int_0^1 \big\langle \nabla f(\gamma(t)), \dot\gamma(t) \big\rangle \, dt.$$

Suppose that the points $x_1, x_2 \in \Omega_1$ are sufficiently near in the following sense: for every $\varepsilon > 0$ there is a path $\gamma : [0,1] \to \Omega_1$ with $\gamma(0) = x_1$ and $\gamma(1) = x_2$ for which

$$\int_0^1 H\big(\gamma(t), \dot\gamma(t)\big) dt < \rho(x_2, x_1) + \varepsilon.$$

If $\nabla f(\gamma(t))$ exists almost everywhere on γ, then

$$\begin{aligned} f(x_2) - f(x_1) &= \int_0^1 \big\langle \nabla f(\gamma(t)), \dot\gamma(t) \big\rangle \, dt \\ &\le \int_0^1 \Phi\big(\gamma(t), \nabla f(\gamma(t))\big) \, H\big(\gamma(t), \dot\gamma(t)\big) dt \\ &\le \operatorname*{ess\,sup}_{\Omega_1} \Phi(x, \nabla f(x)) \int_0^1 H(\gamma(t), \dot\gamma(t)) \, dt \\ &\le \operatorname*{ess\,sup}_{\Omega_1} \Phi(x, \nabla f(x)) \big(\rho(x_2, x_1) + \varepsilon\big). \end{aligned} \tag{21}$$

Suppose that γ does not have the described property. Without loss of generality we may assume that the path γ is piecewise linear. We choose a unit vector θ such that for sufficiently small $\delta > 0$ the parallel translation γ_δ of the path γ on a vector $\delta\theta$ does not have intersections with each other. Using the Rademacher theorem about differentiability of Lipschitz functions almost everywhere, it is not hard to see that the function f has a total differential at almost every point $x \in \gamma_\delta$ on almost all γ_δ. Let γ_{δ_m} with $\delta_m \to 0$ be a sequence of curves with this property, and let $x_{2,m}$ and $x_{1,m}$ be their end points. Arguing as for the proof of (21), we find

$$f(x_{2,m}) - f(x_{1,m}) \le \operatorname*{ess\,sup}_{\Omega_1} \Phi\big(x, \nabla f(x)\big) \big(\rho(x_{2,m}, x_{1,m}) + \varepsilon\big).$$

Going over to the limit for $m \to \infty$ and $\varepsilon \to 0$, we obtain

$$f(x_2) - f(x_1) \le \operatorname*{ess\,sup}_{\Omega_1} \Phi(x, \nabla f(x)) \, \rho(x_2, x_1). \tag{22}$$

Similarly we get the inequality

$$f(x_2) - f(x_1) \ge -\operatorname*{ess\,sup}_{\Omega_1} \Phi(x, \nabla f(x)) \, \rho(x_1, x_2). \tag{23}$$

Relations (22) and (23) imply (20).

Suppose that (20) holds. Let $U \subset \Omega_1 \subset\subset \Omega$ and let $h < 1$ be a constant for which

$$\limsup_{y\to x} \frac{f(x)-f(y)}{\rho(x,y)} \le h$$
$$\liminf_{y\to x} \frac{f(x)-f(y)}{\rho(y,x)} \ge -h$$

for all $x \in \Omega_1$. Let $l \subset \Omega_1$ be an arbitrary segment and let θ be a unit directing vector of the segment l such that

$$l = \big\{ y : y = x + t\theta \ \ (0 \le t \le 1) \big\}.$$

Since the function f is ρ-Lipschitz, the derivative $\frac{\partial f}{\partial \theta}$ exists almost everywhere on l. In each point t_0, where $\frac{\partial f}{\partial \theta}(x+t_0\theta) > 0$, we have

$$\begin{aligned}\frac{\partial f}{\partial \theta}(x+t_0\theta) &= \lim_{t\to t_0+0} \frac{f(x+t\theta)-f(x+t_0\theta)}{t-t_0} \\ &\le \limsup_{t\to t_0+0} \frac{f(x+t\theta)-f(x+t_0\theta)}{\rho(x+t\theta, x+t_0\theta)} \liminf_{t\to t_0+0} \frac{\rho(x+t\theta, x+t_0\theta)}{t-t_0} \\ &\le h \lim_{t\to t_0+0} \tfrac{1}{t-t_0} \int_{t_0}^{t} H(x+s\theta,\theta)\, ds.\end{aligned}$$

If at a point t_0 for the derivative we have $\frac{\partial f}{\partial \theta}(x+t_0\theta) \le 0$, then the given inequality is obvious. So, for almost all $t \in [0,1]$,

$$\frac{\partial f}{\partial \theta}(x+t\theta) \le h \liminf_{t\to t_0+0} \tfrac{1}{t-t_0} \int_{t_0}^{t} H(x+s\theta,\theta)\, ds \le h\, H(x+t\theta,\theta).$$

Since the choice of segment $l \subset \Omega_1$ was arbitrary, we have $\frac{\partial f}{\partial \theta}(x) \le h\, H(x,\theta)$ for θ almost everywhere in Ω_1. As $\frac{\partial f}{\partial \theta}(x) = \langle \nabla f(x), \theta \rangle$ almost everywhere, we have

$$\Phi(x, \nabla f(x)) = \sup_{\theta\neq 0} \frac{\langle \nabla f(x), \theta\rangle}{H(x,\theta)} \le h$$

and the theorem is proved ∎

We say that a set K is *ρ-compact in the pseudometric space* (Ω_ρ, ρ) if for any sequence of points $\{x_m\}_{m=1}^{+\infty} \subset K$ there is subsequence $\{x_{m_k}\}_{k=1}^{+\infty}$ such that for some point $x \in K$

$$\rho(x_{m_k}, x) + \rho(x, x_{m_k}) \to 0 \qquad (k \to +\infty).$$

For an arbitrary pair of points $x_1, x_2 \in \Omega_\rho$ we set

$$\Gamma(x_1,x_2) = \Big\{ x \in \Omega_\rho : \ \rho(x_1,x_2) = \rho(x_1,x) + \rho(x,x_2) \Big\}.$$

Note that the set $\Gamma(x_1,x_2)$ is non-empty, since the points x_1, x_2 lie in $\Gamma(x_1,x_2)$ at least. But, in contrast to a metric, the equality $\Gamma(x,x) = x$ can be broken for the pseudometric ρ.

In the case when the extension of a function ϕ takes place from a compact set, the extension conditions may be essentially simplified.

Theorem 3. *Let $K \subset \partial\Omega_\rho$ be a ρ-compact set. The function $\phi : K \to \mathbb{R}$ is the trace on K of a function $f : \Omega \to \mathbb{R}$ satisfying the conditions*

$$\operatorname{ess\,sup} \Phi(x, \nabla f(x)) < 1$$

for any set $U \subset\subset \Omega$ if and only if ϕ satisfies the conditions

$$-\rho(x_2, x_1) \le \phi(x_1) - \phi(x_2) \le \rho(x_1, x_2) \quad (x_1, x_2 \in K) \tag{24}$$
$$-\rho(x_2, x_1) < \phi(x_1) - \phi(x_2) < \rho(x_1, x_2) \quad \text{if } \rho(x_1, x_2) > 0, \Gamma(x_1, x_2) \cap \Omega \ne \emptyset. \tag{25}$$

Proof. Condition (24) implies that ϕ satisfies the ρ-Lipschitz condition on K. Therefore, for the proof of the theorem it is sufficient to establish the equivalence of (25) and (20). Suppose that (25) holds out. We show that for an arbitrary set $U \subset\subset \Omega$ and for every $\delta > 0$ there is a number m_0 for which

$$\sup \left\{ L(x, y, z) : (x, y) \in A_\delta^{\frac{1}{m_0}} \text{ and } z \in U \right\} < 1.$$

Assume the opposite. Then there are $\delta > 0$, points $x_m, y_m \in K$ and $z_m \in U$ such that $\rho(x_m, y_m) \ge \delta$ and

$$\frac{\rho(x_m, y_m)}{\rho(x_m, z_m) + \rho(z_m, y_m)} \to 1 \qquad (m \to +\infty). \tag{26}$$

By virtue of the assumptions on the sets K and U, there are points $x_0, y_0 \in K$ and $z_0 \in U$ for which

$$\left.\begin{aligned} \rho(x_m, x_0) + \rho(x_0, x_m) \to 0 \\ \rho(y_m, y_0) + \rho(y_0, y_m) \to 0 \\ \rho(z_m, z_0) + \rho(z_0, z_m) \to 0 \end{aligned}\right\} \qquad (m \to +\infty).$$

From this by (26) we obtain $\rho(x_0, y_0) = \rho(x_0, z_0) + \rho(z_0, y_0)$ which implies $z_0 \in \Gamma(x_0, y_0)$.

On the other hand, as $(x_m, y_m) \in A_\delta^{\frac{1}{m}}$, then

$$\phi(x_m) - \phi(y_m) \ge (1 - \tfrac{1}{m})\rho(x_m, y_m).$$

Taking the limit, we establish that $\phi(x_0) - \phi(y_0) \ge \rho(x_0, y_0) \ge \delta$. Thus, as ϕ is a ρ-Lipschitz function, $\phi(x_0) - \phi(y_0) = \rho(x_0, y_0)$ and $z_0 \in \Gamma(x_0, y_0) \cap \Omega$ what contradicts to (25). Hence, condition (20) holds.

Conversely, suppose that (20) is true. Let us show that this implies (25). Suppose the opposite. Then there are points $x_0, y_0 \in K$ and $z_0 \in \Gamma(x_0, y_0) \cap \Omega$ for which $\phi(x_0) - \phi(y_0) = \rho(x_0, y_0) > 0$. We set $U = \{z_0\}$. For $\delta = \rho(x_0, y_0)$ it follows from (20) that $L(x_0, y_0, z_0) < 1$. Since $z_0 \notin \Gamma(x_0, y_0)$, we have a contradiction. The theorem is proved ∎

Let M be a Riemannian manifold and let g be a metric on M. Let $\delta(x) > 0$ be a function of class $C^1(M)$. Let L be Minkowski space with metric l. According to [2: Section 2.6] we shall call the manifold with Lorentzian metric $\overline{g}$ defined by the rule

$$\overline{g}(u, v) = g(\pi u, \pi v) + \delta(\pi(p))l(\eta u, \eta v) \qquad \big(u, v \in T_p(M \times_\delta L)\big)$$

where $p \in M \times_\delta L$, π and η are natural projections on M and L, respectively, by *Lorentzian warped product* $M \times_\delta L$. It is clear that the tangent spaces $T_{\pi(p)}M$ and $T_{\eta(p)}L$ are orthogonal.

A vector $u \in T_p(M \times_\delta L)$ is called *space-like* if $\overline{g}(u, u) > 0$. We shall consider Lorentzian warped spaces of the form $M \times_\delta \hat{\mathbb{R}}$, where $\hat{\mathbb{R}}$ is the real line provided by a negative definite metric. Suppose that the hypersurface F in $M \times_\delta \hat{\mathbb{R}}$ is defined as the graph of a function f over a domain $\Omega \subset M$. We give the condition under which it is space-like.

We put

$$r(m_1, m_2) = \inf \int_\gamma \delta^{-\frac{1}{2}}(m)$$

where the infimum is taken over all arcs $\gamma \subset \Omega$ joining points $m_1, m_2 \in \Omega$.

Lemma 6. *The surface F is space-like if and only if*

$$\limsup_{m' \to m} \frac{|f(m') - f(m)|}{r(m', m)} < 1$$

for all $m \in \Omega$.

The proof can be found in [4: Section 3.4].

Assuming that the completion Ω_r of the domain Ω by the metric r is compact, we obtain the following statement proved in [4].

Theorem 4. *A function $\phi : \partial\Omega_r \to \mathbb{R}$ is the trace of a locally Lipschitz function $f : \Omega \to \mathbb{R}$ with space-like graph if and only if ϕ satisfies conditions* (16) – (17) *in the metric r on $\partial\Omega_r$.*

4. Comparison with Euclidean boundary

The boundary data $\phi : \partial\Omega_\rho \to \mathbb{R}$ of a function f defined in a domain $\Omega \subset \mathbb{R}^n$ were understood above as limits of $f(x)$ with respect to the pseudometric ρ. In the general case there are no relations between limited data $f|_{\partial\Omega}$ and $f|_{\partial\Omega_\rho}$. So, a very important problem is to find conditions on the distribution of convex sets $\Xi(x)$, under realization of which the boundaries $\partial\Omega$ and $\partial\Omega_\rho$ can be compared. In this section we obtain some results in this direction.

We consider the intrinsic metric on Ω

$$d_\Omega(x, y) = \inf \int_0^1 |\dot{\gamma}(t)|\, dt$$

where the infimum is taken over all rectifiable curves $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. Let Ω_d be the completion of Ω by the metric d_Ω and let $\partial\Omega_d = \Omega_d \setminus \Omega$. Our purpose is to give the description of correlations between the boundaries $\partial\Omega_d$ and $\partial\Omega_\rho$.

Further we shall say that the pseudometric $\rho(x,y)$ is *uniformly continuous with respect to* $d_\Omega(x,y)$ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that, for any $x, y \in \Omega$ with $d_\Omega(x,y) < \delta$, $\rho(x,y) + \rho(y,x) < \varepsilon$. The intrinsic metric $d_\Omega(x,y)$ is uniformly continuous with respect to the pseudometric ρ if the uniform smallness of $\rho(x,y) + \rho(y,x)$ implies the uniform smallness of $d_\Omega(x,y)$.

We construct a mapping $j : \partial\Omega_d \to \partial\Omega_\rho$ by the following way. Let $\tilde{x} \in \partial\Omega_d$ be an arbitrary point and let $\{x_m\}_{m=1}^{+\infty}$ be a d-fundamental sequence of points $x_m \in \Omega$, convergent to $\tilde{x}$. Then

$$d_\Omega(x_m, x_n) \to 0 \qquad (m, n \to +\infty)$$

and by the supposition on uniform continuity that sequence is ρ-fundamental. Therefore, the sequence defines some point $\overline{x} \in \partial\Omega_\rho$. Set $j(\tilde{x}) = \overline{x}$. It is clear that the mapping $j : \partial\Omega_d \to \partial\Omega_\rho$ is one-valued. Similarly we may define a single-valued mapping $\hat{j} : \partial\Omega_\rho \to \partial\Omega_d$ which puts $\overline{x} \in \partial\Omega_\rho$ into correspondence to a point $\tilde{x} \in \partial\Omega_d$.

Note the following simple statement.

Lemma 7. *Let $f : \Omega_\rho \to \mathbb{R}$ be a ρ-Lipschitz function and let $\phi : \partial\Omega_\rho \to \mathbb{R}$ be such that $f|_{\partial\Omega_\rho} = \phi$ in the sense of the pseudometric ρ. Then $f|_{\partial\Omega_d} = \phi \circ j$ in the sense of the intrinsic metric d. Inversely, if $f : \Omega \to \mathbb{R}$ is a d-Lipschitz function and for the function $\phi : \partial\Omega_d \to \mathbb{R}$ it holds $f|_{\partial\Omega_d} = \phi$, then $f|_{\partial\Omega_\rho} = \phi \circ \hat{j}$ in the sense of the pseudometric ρ.*

Proof. It is enough to restrict oneself to the following explanations. Suppose that the point $\tilde{x} \in \partial\Omega_d$ and the sequence $\{x_m\}$ with property $d_\Omega(x_m, \tilde{x}) \to 0$ $(m \to +\infty)$ are given. The uniform continuity of ρ implies that the sequence $\{x_m\}$ is ρ-fundamental. So $j(\tilde{x}) = \overline{x}$ and $\rho(x_m, \overline{x}) \to 0$ under $m \to +\infty$. On the other hand, since the function f is ρ-Lipschitz, then

$$f(x_m) - \phi(j(\tilde{x})) \le \rho(x_m, \overline{x}) \to 0$$
$$f(x_m) - \phi(j(\tilde{x})) \ge -\rho(\overline{x}, x_m) \to 0,$$

that is $f|_{\partial\Omega_d} = \phi \circ j$.

Conversely, if the point $\widetilde{x} \in \partial\Omega_\rho$ and the sequence $\{x_m\}$ for which $\rho(x_m, \widetilde{x}) + \rho(\widetilde{x}, x_m) \to 0$ are given, then the sequence $\{x_m\}$ is d-fundamental. Therefore $\hat{j}(\widetilde{x}) = \overline{x}$ and $d(x_m, \overline{x}) \to 0$. As the function f is d-Lipschitz, then $|f(x_m) - \phi(\hat{j}(\widetilde{x}))| \le d(x_m, \overline{x}) \to 0$ which is required ∎

Clearly, a similar statement is true for a pseudometric ρ, uniformly continuous in the Euclidean metric ∎

We give now a simple criterion of uniform continuity for the pseudometric ρ in the metric d_Ω. For that we define the quantity

$$\underline{\Phi}(x) = \min_{|\xi|=1} \Phi(x,\xi).$$

Lemma 8. *If there is a constant $c > 0$ such that*

$$\underline{\Phi}(x) \geq c \qquad (x \in \Omega), \tag{27}$$

then

$$\rho(x,y) + \rho(y,x) \leq \tfrac{2}{c}\, d_\Omega(x,y) \qquad (x,y \in \Omega). \tag{28}$$

Conversely, if there is a constant $c > 0$ such that

$$\max_{|\xi|=1} \Phi(x,\xi) = \overline{\Phi}(x) \leq c,$$

then

$$d_\Omega(x,y) \leq \tfrac{c}{2}\big(\rho(x,y) + \rho(y,x)\big) \qquad (x,y \in \Omega).$$

Proof. First of all, we observe that condition (27) implies

$$|H(x,\eta)| = \sup_{\Phi(x,\xi)\neq 0} \frac{|\langle \eta, \xi\rangle|}{|\Phi(x,\xi)|} \leq \sup_{\Phi(x,\xi)\neq 0} \frac{|\xi|\,|\eta|}{c\,|\xi|} = \frac{1}{c}\,|\eta|.$$

Fix points $x, y \in \Omega$ and a locally Lipschitz path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. We have

$$\rho(x,y) + \rho(y,x) \leq 2\int_0^1 |H(\gamma(t), \dot\gamma(t))|\, dt \leq \frac{2}{c}\int_\gamma |\dot\gamma(t)|\, dt = \frac{2}{c}\,\mathrm{length}\,\gamma.$$

Taking the infimum over all paths γ, we get (28).

Conversely, for any vector $\eta \in \mathbb{R}^n$ we have

$$H(x,\eta) = \sup_{\Phi(x,\xi)\neq 0} \frac{\langle \eta,\xi\rangle}{\Phi(x,\xi)} \geq \frac{|\eta|}{\max_{|\eta|=1}\Phi(x,\eta)} = \frac{|\eta|}{\overline{\Phi}(x)} \geq \frac{1}{c}\,|\eta|.$$

For fixed points $x, y \in \Omega$, by the definition of the intrinsic distance d_Ω,

$$\begin{aligned} d_\Omega(x,y) &= \inf_\gamma \int_\gamma |\dot\gamma|\, dt \leq \int_{\gamma_0} |\dot\gamma_0|\, dt \\ &\leq c\int_{\gamma_0} H(\gamma_0, \dot\gamma_0)\, dt \leq \frac{c}{2}\big(\rho(x,y) + \rho(y,x) - \varepsilon\big) \end{aligned}$$

for all $\varepsilon > 0$ where $\gamma_0 : \ [0,1] \to \Omega$ with $\gamma_0(0) = x$ and $\gamma_0(1) = y$ is the locally Lipschitz path, for which

$$\int_{\gamma_0} H(\gamma_0, \dot\gamma_0)\, dt < \begin{cases} \rho(y,x) - \frac{\varepsilon}{2} \\ \rho(x,y) - \frac{\varepsilon}{2}. \end{cases}$$

Because of the arbitrary choice of ε we have what was needed ∎

For an effective description of correlations between the boundary of the domain Ω in the ρ-metric $\partial\Omega_\rho$ and the Euclidean boundary $\partial\Omega$ we need the concept of p-modulus for the family of curves (see, for example, [11: Section 5]). Let $\{\gamma\}$ be a family of locally rectifiable arcs γ situated in the domain $\Omega \subset \mathbb{R}^n$ and let $p > 1$ be a certain number. *p-modulus* of the family $\{\gamma\}$ (in Euclidean metric) is called the quantity

$$\mathrm{mod}_p\{\gamma\} = \inf \frac{\int_\Omega \varrho^p dx}{\left(\inf_\gamma \int_\gamma \varrho\,|dx|\right)^p} \tag{29}$$

where the infimum is taken over all non-negative Borel functions ϱ. For an arbitrary pair of points $x, y \in \Omega$ we define the family $G(x,y) = \{\gamma\}$ as the *family of the rectifiable arcs* $\gamma \subset \Omega$ joining the points x and y. If $p > n$ and $y \to x$, then $\mathrm{mod}_p G(x,y) \to +\infty$. However, for $x, y \to z \in \partial\Omega$ the p-modulus of the family of arcs $G(x,y)$ may be both bounded and unbounded. This is connected with the structure of the boundary $\partial\Omega$ near the point $z \in \partial\Omega$, namely with the presence of arbitrarily "narrow" places of the domain Ω at a neighborhood of z.

Let $p > n$. We shall say that the domain Ω is *p-uniform* if a $\delta > 0$ can be found for every sufficiently large $\varepsilon > 0$ so that, for all $x, y \in \Omega$ with $d_\Omega(x,y) < \delta$, $\mathrm{mod}_p G(x,y) > \varepsilon$.

The statement formulated later establishes a criterion of uniform continuity for the pseudometric $\rho(x,y)$ and as corollary it sets the existence and continuity of the boundary mapping $j : \partial\Omega_d \to \partial\Omega_\rho$.

Theorem 5. *If the domain Ω is p-uniform and the function $\underline{\Phi}$ satisfies the condition*

$$\int_\Omega \frac{dx}{\underline{\Phi}^p(x)} < \infty \qquad (dx = dx_1 dx_2 \cdots dx_n), \tag{30}$$

then the pseudometric ρ is uniform continuous with respect to the intrinsic metric d_Ω.

Proof. We choose the function $\underline{\Phi}^{-1}$ in (29) as metric ϱ. For any $x, y \in \Omega$ we have

$$\mathrm{mod}_p G(x,y) \le \frac{\int_\Omega \underline{\Phi}^{-p}(x)\,dx}{\left(\inf_\gamma \int_\gamma \underline{\Phi}^{-1}(x)\,|dx|\right)^p}.$$

Since $\frac{1}{\underline{\Phi}(x)} \ge H(x,\eta)$ for any $\eta \in \mathbb{R}^n$, thus

$$\inf_\gamma \int_\gamma \underline{\Phi}^{-1}(x)\,|dx| \ge \inf_\gamma \int_\gamma |H(\gamma(t),\dot\gamma(t))|\,dt \ge \max\big\{\rho(x,y), \rho(y,x)\big\}.$$

Hence

$$\rho^p(x,y) + \rho^p(y,x) \le \frac{2}{\mathrm{mod}_p G(x,y)} \int_\Omega \underline{\Phi}^{-p}(x)\,dx.$$

It follows from condition (30) that

$$(\rho(x,y) + \rho(y,x))^p \le \frac{\mathrm{const}}{\mathrm{mod}_p G(x,y)}$$

and we can make the necessary conclusion because of the requirement about p-uniformity of the domain Ω ∎

Now we give the concept of p-modulus for the family of curves $\{\gamma\}$ in Finsler space (also see [10]). Let $\{\gamma\}$ be a family of locally rectifiable arcs $\gamma \subset \Omega$ and $p > 1$ be a number. We call the quantity

$$\widetilde{\mathrm{mod}}_p\{\gamma\} = \inf_{\rho \geq 0} \frac{\int_\Omega \rho^p dx}{\left(\inf_\gamma \int_\gamma \rho \, H(x, dx)\right)^p}$$

by *p-modulus* of this family, where ρ is a measurable function.

Denote by $G(x, y) = \{\gamma\}$ the family of arcs $\gamma \subset \Omega$ joining the points x and y. We say that the domain Ω is *p-uniform* with respect to the pseudometric ρ if for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, y \in \Omega$ under the condition $\rho(x, y) + \rho(y, x) < \delta$ it follows that $\widetilde{\mathrm{mod}}_p G(x, y) > \varepsilon$.

The following theorem is true.

Theorem 6. *If the domain Ω is p-uniform with respect to the pseudometric ρ and the function $H(x) = \min_{|\eta|=1} H(x, \eta)$ has the property $\int_\Omega \frac{dx}{H^p(x)} < \infty$, then the metric d_Ω is uniformly continuous with respect to the pseudometric ρ.*

Proof. For any pair of points $x, y \in \Omega$ let us consider the curve $\gamma \subset \Omega$ joining these points. Take the function H^{-1} as admissible function ρ. Then

$$\widetilde{\mathrm{mod}}_p G(x, y) \leq \frac{\int_\Omega H^{-p}(x) dx}{\left(\inf_\gamma \int_\gamma H^{-1}(x) H(x, dx)\right)^p}.$$

Since for any vector $\eta \in \mathbb{R}^n$ it is true that $\frac{1}{H(x)} \geq \frac{|\eta|}{H(x,\eta)}$, we obtain

$$\widetilde{\mathrm{mod}}_p G(x, y) \leq \frac{\int_\Omega H^{-p}(x) dx}{\left(\inf_\gamma \int_\gamma |dx|\right)^p} \leq \frac{\int_\Omega H^{-p}(x) dx}{d^p_\Omega(x, y)}.$$

Hence,

$$d^p_\Omega(x, y) \leq \frac{1}{\widetilde{\mathrm{mod}}_p G(x, y)} \int_\Omega H^{-p}(x) dx$$

implies uniform continuity of the metric d_Ω with respect to the pseudometric ρ, by virtue of the supposition about p-uniformity of the domain Ω ∎

We shall say that a domain $\Omega \subset \mathbb{R}^n$ satisfies the *h-ball condition* if we can touch every boundary point by a ball of radius h completely lying into the domain.

Proposition 1. *If the domain $\Omega \subset \mathbb{R}^n$ satisfies the h-ball condition, then it is p-uniform for every $p > n$.*

Proof. We suppose the contrary. There is $\varepsilon_0 > 0$ so that for any natural number m under the condition $d(x_m, y_m) < \frac{1}{m}$ it follows that $\mathrm{mod}_p\{\gamma\} \leq \varepsilon_0$. As $\overline{\Omega}$ is compact, we may assume that sequences of points $\{x_m\}$ and $\{y_m\}$ converge and $\lim_{m\to\infty} y_m = \lim_{m\to\infty} x_m = a \in \overline{\Omega}$. For the point a there are two possibilities of location: either $a \in \Omega$ or $a \in \partial\Omega$.

First let us consider the case when a is a boundary point of the domain Ω. We construct h-balls B and B_{1m}, B_{2m} tangent to the boundary such that

$$B \cap \partial\Omega = \{a\}, \quad B_{1m} \cap \partial\Omega \neq \emptyset, \quad x_m \in B_{1m}, \quad B_{2m} \cap \partial\Omega \neq \emptyset, \quad y_m \in B_{2m}.$$

Clearly, beginning with a certain number m_0 we have $B \cap B_{1m} \neq \emptyset$ and $B \cap B_{2m} \neq \emptyset$. Denote by b_m and a_m points nearest to a and lying in $B \cap B_{1m}$ and $B \cap B_{2m}$, respectively. It is clear that $b_m \to a$ and $a_m \to a$ for $m \to +\infty$.

Further, let γ_m be a family of curves joining the points x_m and y_m in the domain Ω, and δ_m be a family of broken lines passing through the points x_m, a_m, b_m, y_m. By properties of the modulus,

$$\operatorname{mod}_p\{\gamma_m\} \geq \operatorname{mod}_p\{\delta_m\}.$$

We estimate $\operatorname{mod}_p\{\delta_m\}$. Define in $\mathbb{R}^n$ cylindrical coordinates (t, z, θ) where the axis z is directed along a line passing through the points x_m and b_m. Let $\Pi(\theta_0)$ be a plane corresponding to the point $\theta_0 \in S^{n-2}$. Next we define polar coordinates (r, φ) with the center at the point x_m in $\Omega \cap \Pi(\theta_0)$. For an admissible function ρ, by Hölder's inequality we have

$$\begin{aligned}\Big(\int_{r(\varphi)} \rho\, dr\Big)^p &\leq \Big(\int_{r(\varphi)} \rho^p r^{n-1} dr\Big)\Big(\int_{r(\varphi)} r^{-\frac{n-1}{p-1}} dr\Big)^{p-1} \\ &\leq \Big(\frac{p-1}{p-n}\Big)^{p-1} r^{p-n}\Big|_{r(\varphi)} \cdot \int_{r(\varphi)} \rho^p r^{n-1} dr \\ &\leq \Big(\int_{r(\varphi)} \rho^p r^{n-1} dr\Big)\Big(\frac{p-1}{p-n}\Big)^{p-1} d^{p-n}(x, b).\end{aligned}$$

Here $r(\varphi)$ is the intersection of the ray corresponding angle φ with the set

$$\Omega \cap B\big(x_m, d(x_m, b_m)\big).$$

By the mean value theorem, there is an $\widetilde{\varphi} \in (0, \frac{\pi}{4})$ so that

$$\int_0^{\frac{\pi}{4}} \sin\varphi \Big(\int_{r(\varphi)} \rho\, dr\Big)^p d\varphi = \Big(\int_{r(\widetilde{\varphi})} \rho\, dr\Big)^p \Big(1 - \frac{\sqrt{2}}{2}\Big).$$

Integrating by the variable $\theta \in S^{n-2}$ we obtain

$$\begin{aligned}&\omega_{n-2}\Big(1 - \frac{\sqrt{2}}{2}\Big)\Big(\int_{r(\widetilde{\varphi})} \rho\, dr\Big)^p \\ &\quad\leq C d^{p-n}(x, b) \int_{S^{n-2}} \int_0^{\frac{\pi}{4}} \int_{r(\varphi)} \rho^p r^{n-1} \sin\varphi\, dr d\varphi d\theta \\ &\quad\leq C d^{p-n}(x_m, b_m) \int_\Omega \rho^p(x)\, dx\end{aligned}$$

where $C = [\frac{p-1}{p-n}]^{p-1}$ and ω_{n-2} is the area of the unit $(n-2)$-dimensional sphere. Similarly, for the point b_m we find the angle $\widetilde{\psi}$ such that

$$\begin{aligned}\omega_{n-2}\Big(1-\frac{\sqrt{2}}{2}\Big)\Big(\int_{r(\widetilde{\psi})}\rho\,dr\Big)^p &\\ &\le C d^{p-n}(x,b)\int_{S^{n-2}}\int_0^{\frac{\pi}{4}}\int_{r(\psi)}\rho^p r^{n-1}\sin\varphi\,d\varphi dr d\theta\\ &\le C d^{p-n}(x_m,b_m)\int_\Omega \rho^p(x)\,dx.\end{aligned}$$

Joining these two inequalities, we find the broken line $\delta(x_m, b_m)$ for which

$$\Big(\int_{\delta(x_m,b_m)}\rho\,dr\Big)^p \le \frac{2^p C}{\omega_{n-2}\big(1-\frac{\sqrt{2}}{2}\big)} d^{p-n}(x_m,b_m)\int_\Omega \rho^p dx.$$

Making similar arguments for the pairs of points (b_m, a_m) and (a_m, y_m), we find the respective broken lines $\delta(b_m, a_m)$ and $\delta(a_m, y_m)$, for which the inequalities

$$\Big(\int_{\delta(a_m,b_m)}\rho\,dr\Big)^p \le \frac{2^p C\,d^{p-n}(a_m,b_m)}{\omega_{n-2}\big(1-\frac{\sqrt{2}}{2}\big)}\int_\Omega \rho^p dx$$
$$\Big(\int_{\delta(a_m,y_m)}\rho\,dr\Big)^p \le \frac{2^p C\,d^{p-n}(a_m,y_m)}{\omega_{n-2}\big(1-\frac{\sqrt{2}}{2}\big)}\int_\Omega \rho^p dx$$

are true. Joining these broken lines in one line

$$\delta_m = \delta_m(x_m,b_m)\cup\delta(b_m,a_m)\cup\delta(a_m,y_m)$$

and summarizing the obtained inequalities, we get for any admissible function $\rho \ge 0$

$$\begin{aligned}&\Big(\int_{\delta_m}\rho\,ds\Big)^p\\ &= \Big(\int_{\delta_m(x_m,b_m)}\rho\,ds + \int_{\delta_m(b_m,a_m)}\rho\,ds + \int_{\delta_m(a_m,y_m)}\rho\,ds\Big)^p\\ &\le 3^{p-1}\Big[\Big(\int_{\delta_m(x_m,b_m)}\rho\,ds\Big)^p + \Big(\int_{\delta_m(b_m,a_m)}\rho\,ds\Big)^p + \Big(\int_{\delta_m(a_m,y_m)}\rho\,ds\Big)^p\Big]\\ &\le \frac{2\cdot 6^{p-1}C\big(d^{p-n}(x_m,b_m)+d^{p-n}(a_m,b_m)+d^{p-n}(a_m,y_m)\big)}{\omega_{n-2}\big(1-\frac{\sqrt{2}}{2}\big)}\int_\Omega \rho^p dx.\end{aligned}$$

Note that

$$\begin{aligned}d^{p-n}(x_m,b_m)+d^{p-n}(a_m,b_m)+d^{p-n}(a_m,y_m)&\\ &\le \big(d(x_m,b_m)+d(a_m,b_m)+d(a_m,y_m)\big)^{p-n}\end{aligned}$$

and going over to the infimum by all functions ρ in the preceding inequality, we find

$$\mathrm{mod}_p\{\delta_m\} \geq \frac{C_1}{\big[d(x_m, b_m) + d(a_m, b_m) + d(a_m, y_m)\big]^{p-3}}.$$

Passing to the limit in this inequality for $m \to \infty$ we obtain

$$\lim_{m\to\infty} \mathrm{mod}_p\{\delta_m\} = +\infty$$

which contradicts to assumption $\mathrm{mod}_p\{\delta_m\} \leq \varepsilon_0$. In the case in which a is an intrinsic point of the domain Ω we do the same thing, but choose the balls not necessarily tangent to the boundary $\partial\Omega$ ∎

5. Arbitrary codimension

In this section we give applications of our results obtained above to existence problems for space-like surfaces of codim > 1 with the prescribed boundary in the Minkowski space $\mathbb{R}^{n+1}_1$.

Let $\mathbb{R}^{n+1}_1$ be a Minkowski space-time. Let $\Omega \subset \mathbb{R}^k$ $(k \leq n)$ be a domain and let

$$F(u) = \big(x_1(u), x_2(u), ..., x_n(u), t(u)\big) : \Omega \to \mathbb{R}^{n+1}_1$$

be a Lipschitz mapping, which gives a k-dimensional Lipschitz surface $\mathcal{M}$ with boundary L. The problem is to find conditions (necessary and sufficient) on the boundary L for the existence of a space-like surfrace with the same boundary. We shall find the solution of the problem in the form

$$R(u) = \big(x_1(u), ..., x_n(u), f(u)\big) \tag{31}$$

where x_i are coordinate functions for the surface F and the function f coincides with the function t on the boundary of the domain $\partial\Omega$. The last condition means that the surfaces R and F have the similar boundary L.

We put necessary notations for the partial derivatives of the vector function $F : \mathbb{R}^k \to \mathbb{R}^{n+1}_1$, which exist almost everywhere because the surface is Lipschitz

$$\frac{\partial F}{\partial u_i}(y) = \left(\frac{\partial x_1}{\partial u_i}, ..., \frac{\partial x_n}{\partial u_i}, \frac{\partial t}{\partial u_i}\right).$$

Then the matrixes

$$G = \{g_{ij}(u)\}_{k,k}, \qquad g_{ij}(u) = \sum_{l=1}^{n} \frac{\partial x_l}{\partial u_i}(u) \cdot \frac{\partial x_l}{\partial u_j}(u)$$

$$A = \{a_{ij}\}_{k,k}, \qquad a_{ij} = \frac{\partial t}{\partial u_i}(u)\,\frac{\partial t}{\partial u_j}(u)$$

are defined almost everywhere. Besides that, the matrix G is non-negative defined as Gram matrix. Further, we suppose that the condition

$$\operatorname*{ess\,inf}_{u\in K} \det G(u) > 0 \tag{32}$$

is true for every compactly embedded subdomain $K \subset \Omega$. The existence of the inversal matrix G^{-1} with coefficients $g^{ij}(u)$ almost everywhere follows from this condition. We note that surface (31) is space-like if and only if $\det[G - A] > 0$ or $\det[E - G^{-1}A] > 0$. We calculate this determinant. Namely, we show that

$$\det[E - G^{-1}A] = 1 - \operatorname{tr}(G^{-1}A).$$

By virtue of all mentioned above, we may suppose that the matrix G is diagonal. Then

$$G = \operatorname{diag}\{\lambda_1, \lambda_2, ..., \lambda_k\}, \quad G^{-1} = \operatorname{diag}\{\tfrac{1}{\lambda_1}, \tfrac{1}{\lambda_2}, ..., \tfrac{1}{\lambda_k}\}.$$

Hence

$$G^{-1}A = \{\tfrac{a_{ij}}{\lambda_i}\} \qquad \text{and} \qquad E - G^{-1}A = \{\delta_{ij} - \tfrac{a_{ij}}{\lambda_i}\}.$$

We need the following

Lemma 9. *For any numbers* a_i, b_i $(i = 1, 2, ..., k)$ *the formula* $\det[\delta_{ij} - a_i b_j] = 1 - \sum_{l=1}^{k} a_l b_l$ *holds.*

Proof. We proceed by induction. It is clear that the statement is true for $k = p - 1$. We calculate the determinant of the order p, decomposing it by the elements of the first string, and get

$$\Delta_p = (1 - a_1 b_1)\Delta_{p-1} + a_1 b_1 \cdot a_2 b_2 \begin{vmatrix} -1 & a_3 & \dots & a_p \\ b_3 & 1 - a_3 b_3 & \dots & -a_p b_3 \\ \vdots & \vdots & \vdots & \vdots \\ b_p & -a_3 b_p & \dots & 1 - a_p b_p \end{vmatrix}$$

$$+ \ldots + a_1 b_1 \cdot a_p b_p \begin{vmatrix} b_2 & 1 - a_2 b_2 & \dots & -b_2 a_{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1} & -b_{p-1} a_2 & \dots & 1 - a_{p-1} b_{p-1} \end{vmatrix}.$$

Note that the determinants are of the same type except Δ_{p-1} and therefore it is sufficient for us to calculate one of them, for example the last one. Multiplying the first column of this determinant consecutively by a_i and adding it to the column with number i we find that

$$\begin{vmatrix} b_2 & 1 - a_2 b_2 & \dots & -b_2 a_{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p-1} & -b_{p-1} a_2 & \dots & 1 - a_{p-1} b_{p-1} \end{vmatrix} = -1.$$

Substituting this in the expression for the determinant Δ_p and using the induction hypothesis for $k = p - 1$, we obtain

$$\begin{aligned} \Delta_p &= (1 - a_1 b_1)(1 - a_2 b_2 - \ldots - a_p b_p) - a_1 b_1 (a_2 b_2 + \ldots + a_p b_p) \\ &= 1 - \sum_{l=1}^{p} a_l b_l \end{aligned}$$

which was what we needed to prove ∎

Lemma 10. *Let Π be a space-like k-dimensional plane and let the vectors $r_1, r_2, ..., r_k$ form a base in Π. If $R_1, R_2, ..., R_k$ are the respective projections of these vectors to the hyperplane $\{(x,t) : t = 0\}$, then*

$$\operatorname{ch}\theta = \frac{\textit{volume of parallelepiped on vectors } R_1, ..., R_k}{\textit{volume of parallelepiped on vectors } r_1, ..., r_k}.$$

Here the term 'volume' means the volume of the parallelepiped spanned by vectors in a space-like plane, on which an Euclidean structure may be naturally induced.

Proof of Lemma 10. Let e be a directing unit vector of a time axis and $e = e^T + e^N$ be its decomposition into a tangent and a normal components to the plane Π. Then the desired cosinus of the angle is equal to

$$\operatorname{ch}\theta = \left|\frac{\langle e, e^N\rangle}{|e^N|}\right| = |e^N| = \sqrt{1 + |e^T|^2}.$$

Denote by Π' the projection of the plane Π to the hyperplane $t = 0$ and by $\pi : \Pi \to \Pi'$ the respective projecting mapping. Since the ratio of squares in the lemma does not depend on the choice of vectors $r_1, r_2, ..., r_k$, then we shall suppose that the vectors form an ortonormal base in Π. Also, we assume that $r_1, r_2, ..., r_{k-1} \in \Pi \cap \Pi'$. It is clear that $\pi(r_i) = R_i = r_i \quad (i = 1, 2, ..., k-1)$ and $\pi(r_k) = r_k + \langle r_{k-1}, e\rangle e$. So

$$|\pi(r_k)|^2 = 1 + \langle r_k, e\rangle^2 = 1 + \sum_{i=1}^{k} \langle r_i, e\rangle^2 = 1 + |e^T|^2 = \operatorname{ch}^2\theta.$$

We note that $\pi(r_k)$ is normal to $r_1, r_2, ..., r_{k-1}$. The volume of the parallelepiped built on the vectors $R_1, ..., R_k$ is $\operatorname{ch}\theta$, but the volume of the parallelepiped built on the vectors $r_1, ..., r_k$ equals to 1. The lemma is proved ∎

Lemma 11. *The condition to be space-like for the surface $\mathcal{M}$, given by the radius vector $F : \mathbb{R}^k \to \mathbb{R}_1^{n+1}$ and satisfying (32), may be written in the form*

$$\operatorname*{ess\,inf}_{u\in K} \left(1 - \operatorname{tr} G^{-1}A\right) > 0$$

where inequality holds on every compactly embedded subdomain K in $\mathbb{R}^k$.

Proof. Let $u \in \mathbb{R}^k$ be a point, in which there is a tangent plane. Then at this point there are tangent vectors $\frac{\partial F}{\partial u_i} \quad (i = 1, 2, ..., k)$. We calculate the quantity $\operatorname{ch}\theta(u)$. For that note that the volume of the parallelepiped, built on the vectors $\frac{\partial F}{\partial u_i}$, equals $\sqrt{\det G}$, but the volume of the parallelepiped spanned on their projections equals $\sqrt{\det (G - A)}$. So using Lemmas 9 and 10 we get the equality

$$\operatorname{ch}\theta(u) = \frac{\sqrt{\det G}}{\sqrt{\det (G - A)}} = \frac{1}{\sqrt{1 - \operatorname{tr} G^{-1}A}}.$$

Thus, the condition for the surface $\mathcal{M}$ to be space-like is equivalent to the condition

$$\operatorname*{ess\,sup}_{u\in K} \sum_{i,j=1}^{k} g^{ij} \frac{\partial t}{\partial u_i} \frac{\partial t}{\partial u_j} < 1$$

for every compactly embedded subdomain $K \subset \Omega$ ∎

We define the quantity

$$\rho(w,v)=\inf_{\gamma}\int_0^1\sqrt{g_{ij}(u(\tau))\frac{du_i}{d\tau}\cdot\frac{du_j}{d\tau}}\,d\tau$$

where the infimum is taken over all rectifable paths $\gamma:\ [0,1]\to\Omega$ joining the points w and v. By condition (32) this quantity defines an intrinsic distance in the domain Ω. Further, we denote by Ω_ρ the completion of the domain Ω by the metric ρ. Suppose that it is compact. Finally, set

$$\Gamma(u,v)=\big\{w\in\Omega_\rho:\ \rho(u,v)=\rho(u,w)+\rho(w,u)\big\}$$

Using Theorem 4, we obtain the following

Theorem 7. *If there is some k-dimensional Lipschitz surface with boundary L and satisfying condition* (32)*, then for the existence of a k-dimensional space-like surface of form* (31) *with prescribed boundary L it is necessary and sufficient to realize the conditions*

$$|t(v)-t(w)|\le\rho(v,w)\quad(v,w\in\partial\Omega_\rho)$$
$$|t(v)-t(w)|<\rho(v,w)\quad\textit{if}\ \Gamma(v,w)\setminus\partial\Omega_\rho\neq\emptyset.$$

Now we consider the existence problem of C^1-smooth space-like surfaces with prescribed boundary. First, we study the case when the surface is given by the graph of a function. Namely, suppose that $t=t(x)\ \ (x\in\Omega\subset\mathbb{R}^n$ be a Lipschitz function such that

$$\overline{\lim_{x\to y}}\frac{|t(x)-t(y)|}{|x-y|}<1\quad(x,y\in\Omega)\tag{33}$$

$$t(x)=\psi(x)\quad(x\in\partial\Omega).\tag{34}$$

Note that from (33) it follows for the surface $t=t(x)$ to be space-like (see, for example, [4]). The Lipschitz function may be changed to a smooth one by smoothing procedure.

Defintion. The function η is called *smoothing*, if

1) $\eta\in C_0^\infty(\mathbb{R}^n)$

2) $\eta(x)=0$ on the compact $B_1=\{x\in\mathbb{R}^n:\ |x|\le1\}$.

3) $\int_{\mathbb{R}^n}\eta(x)\,dx=1$.

As an example of such a function there may be chosen the function η defined by

$$\eta(x)=\begin{cases}C\,e^{-\frac{1}{1-|x|^2}} & \text{for } |x|<1\\ 0 & \text{for } |x|\ge1\end{cases}$$

where the constant C is defined by $C=\big(\int_{|x|<1}e^{-\frac{1}{1-|x|^2}}\,dx\big)^{-1}$.

For every function $g \in L^1_{\text{loc}}(\mathbb{R}^n)$ and every $\varepsilon > 0$ we set

$$g_\varepsilon(x) = \varepsilon^{-n} \int_{\mathbb{R}^n} \eta\left(\frac{x-z}{\varepsilon}\right) g(z)\,dz = \int_{\mathbb{R}^n} \eta(w) g(x+\varepsilon w)\,dw. \tag{35}$$

For this function we have $g_\varepsilon \in C^\infty(\mathbb{R}^n)$. We may smooth our function by formula (35), but it does not provide a realization of condition (34) for some new function t_ε which will be defined below. Therefore, we present a modificated method whose main idea is following:

For a smooth function $\varepsilon : \mathbb{R}^n \to \mathbb{R}$ with $0 < \varepsilon(x) < \frac{1}{3}\text{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $\varepsilon(x) = 0$ for $x \in \partial\Omega$ we define the function t_ε similarly to (35) by

$$t_\varepsilon(x) = \varepsilon^{-n}(x) \int_{\mathbb{R}^n} \eta\left(\frac{x-z}{\varepsilon(x)}\right) t(z)\,dz = \int_{\mathbb{R}^n} \eta(w) t(x+\varepsilon(x)w)\,dw. \tag{36}$$

This function will be desired if we prove the smoothness of it and the realization of conditions (34) and (33) for it.

Although, we shall solve the problem in a more general case. Suppose that in the domain $\Omega \subset \mathbb{R}^n$ there is given a metric ρ locally equivalent to the Euclidean one, that is for every compactly embedded subdomain $K \subset \Omega$ there is a constant $\mu \leq 1$ such that, for any points $x, y \in K$, $\frac{1}{\mu}|x-y| \leq \rho(x,y) \leq \mu|x-y|$. The conditions for the function ε can be written as

$$0 < \varepsilon(x) < \frac{\text{dist}(x, \partial\Omega_\rho)}{3} \quad (x \in \Omega) \qquad \text{and} \qquad \varepsilon(x) = 0 \quad (x \in \partial\Omega_\rho).$$

Let a function $t : \Omega_\rho \to \mathbb{R}$ be given, satisfying the condition $\overline{\lim}_{x\to y} \frac{|t(x)-t(y)|}{\rho(x,y)} < 1$ in Ω. It is necessary to construct a functon t_ε such that

$$t_\varepsilon \in C^1(\Omega) \tag{37}$$

$$t_{\varepsilon(x)} = t(x) = \psi(x) \quad (x \in \partial\Omega_\rho) \tag{38}$$

$$|\nabla t_\varepsilon(x)|_\rho = \overline{\lim_{x\to y}} \frac{|t_\varepsilon(x) - t_\varepsilon(y)|}{\rho(x,y)} < 1. \tag{39}$$

We prove that the function t_ε given by (36) solves problem (37) - (39). In the beginning we check condition (38). For all $x \in \partial\Omega_\rho$ we have $\varepsilon(x) = 0$, $t(x) = \psi(x)$ and

$$t_\varepsilon(x) = \int_{\mathbb{R}^n} \eta(w) t(x)\,dw = \int_{|\omega|\leq 1} \eta(x) t(x+\varepsilon(x)\omega)\,d\omega.$$

The functions t and ε are uniformly continuous, and going to the limit for $x \to x_0 \in \Omega_\rho$ we get

$$\lim_{x\to x_0} t_{\varepsilon(x)} = \int_{|\omega|\leq 1} \eta(x) t(x_0)\,d\omega = \psi(x_0).$$

So (38) is really true. Since $\varepsilon(x) > 0$, $\frac{\partial\varepsilon}{\partial x_i} \in C(\Omega)$ and $\eta \in C^2(\mathbb{R}^n)$, the function $t = t_\varepsilon$ has continuous partial derivatives, i.e. Condition (37) is true. Finally, we find conditions on the function ε in order to provide the realization of condition (39). We have

$$\overline{\lim_{x\to y}}\frac{|t_\varepsilon(x)-t_\varepsilon(y)|}{\rho(x,y)} = \overline{\lim_{x\to y}}\int_{\mathbb{R}^n}\eta(w)\frac{\big|t(x+\varepsilon(x)w)-t(y+\varepsilon(y)w)\big|}{\rho(x,y)}\,dw.$$

Denote

$$g(x)=\max\left\{\frac{1}{2},\sup_{x',x''\in B_{3\varepsilon(y)}}\frac{|t(x')-t(x'')|}{\rho(x',x'')}\right\}$$

where $B_{3\varepsilon(y)}=\big\{\tilde{x}\in\Omega:\,|\tilde{x}-y|<3\varepsilon(x)\big\}$. Then

$$\begin{aligned}\overline{\lim_{x\to y}}\frac{|t_{\varepsilon(x)}-t_{\varepsilon(y)}|}{\rho(x,y)} &\le \int_{|\omega|\le 1}\eta(\omega)\frac{\big|t(x+\varepsilon(x)\omega-t(y+\varepsilon(x)\omega\big|}{\rho(x,y)}\,d\omega\\ &\quad+\int_{|\omega|\le 1}\eta(\omega)\frac{\big|t(y+\varepsilon(x)\omega)-t(y+\varepsilon(y)\omega\big|}{\rho(x,y)}\,d\omega\\ &\le g(y)\overline{\lim_{x\to y}}\int_{|\omega|\le 1}\eta(\omega)\left(1+\mu(y)\frac{|\varepsilon(x)-\varepsilon(y)|}{|x-y|}\right)d\omega\end{aligned}$$

where the function μ is defined by $\mu(y)=\sup_{x',x''\in B_{3\varepsilon(y)}}\frac{\rho(x',x'')}{|x'-x''|}$. Hence for the realization of condition (39) it is necessary and sufficient that $g(y)\big(1+\mu(y)|\nabla\varepsilon(y)|\big)<1$ in the domain Ω. Note that $g(y)<1$ on every compactly embedded subdomain of the domain Ω and the function μ is locally bounded. Then the function standing in the left hand of the inequality is locally positive. So, the function t_ε defined by formula (36) is desired if there is a C^1-smooth function $\varepsilon:\mathbb{R}^n\to\mathbb{R}$ such that

$$0<\varepsilon(x)<\frac{\operatorname{dist}(x,\partial\Omega_\rho)}{3}\ \text{and}\ \varepsilon(x)=0\ \ (x\in\partial\Omega_\rho) \tag{40}$$

$$|\nabla\varepsilon(x)|<\frac{1-g(x)}{\mu(x)g(x)}. \tag{41}$$

Note that by the mentioned theorem the function t has a total differential almost everywhere in Ω and, in general, the function $\frac{1-g(y)}{\mu(y)g(y)}$ is not continuous.

We use the following

Lemma 12. *There is a point $x_0\in\Omega$ and Ω_ρ $\varphi=\varphi(x)$ such that $\varphi|_{\partial\Omega_\rho}=1$, $\varphi\in C^1(\Omega)$, $\varphi(x_0)=0$ and, becides that, there is a constant $c>0$ such that $|\nabla\varphi(x)|\le c$ for all $x\in\Omega$.*

At this point we mean that the exausting function is a C^1-smooth function $\varphi:\Omega\to[0,1]$ such that, for an arbitrary sequence of points $x_k\in\Omega$ $(k\ge 1)$, $\varphi(x_k)\to 1$ if and only if $x_k\to\partial\Omega$.

Proof of Lemma 12. By [1: Theorem 4.1] there is a solution of the equality

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big)=1,\quad |\nabla u|<1\ \text{ in }\Omega$$

in the domain Ω with boundary condition $u|_{\partial\Omega} = 1$. By the maximum principle for the solution of the given equation, $u(x) < 1$ in Ω. We choose a point $x_0 \in \Omega$ such that $u(x_0) = \min_\Omega u(x)$. Then the desired function φ is equal to $\varphi(x) = \frac{u(x)-u(x_0)}{1-u(x_0)}$. So $|\nabla u| < 1$. Then $|\nabla\varphi(x)| < \frac{1}{1-u(x_0)}$. It is clear that the function φ may be continuously extended to the boundary $\partial\Omega_\rho$ setting $\varphi(x) = 1$ for $x \in \partial\Omega_\rho$. The lemma is proved ∎

Lemma 13. *There exists a function* ε *with prescribed properties* (40) – (41).

Proof. Define the function by setting

$$\Sigma(\tau) = \{x \in \Omega : \varphi(x) = \tau\}, \quad \delta(\tau) = \inf_{y\in\Sigma(\tau)} \frac{1-g(y)}{\mu(y)g(y)}, \quad a(\tau) = \sup_{y\in\Sigma(\tau)} |\nabla\varphi(x)|.$$

We find the function ε in the form $\varepsilon(x) = \varepsilon(\varphi(x))$ and require for it the execution of the more strong inequality than (41)

$$|\nabla\varepsilon(\varphi(x))| = |\varepsilon'(\varphi(x))| \cdot |\nabla\varphi(x)| \le a(\varphi(x)) \cdot |\varepsilon'(\varphi(x))| < \delta(\varphi(x)),$$

that is $|\varepsilon'(\tau)| < \frac{\delta(\tau)}{a(\tau)}$. In virtue of the remark done earlier and the definitions of the functions $\delta(\varphi(x))$ and $a(\varphi)$, we can not simply put $\varepsilon(\varphi(x)) = k\int_{\varphi(x)}^1 \frac{\delta(\varphi)}{a(\varphi)}\,d\varphi$ with some constant $k < 1$, as in this case the function $\varepsilon = \varepsilon(\varphi(x))$ may be not C^1-smooth.

We change the function $\frac{\delta(\varphi)}{a(\varphi)}$ into a positive function denoted by $p(\tau)$, such that the new function shall be continuous and satisfy the inequality $p(\tau) < \frac{\delta(\tau)}{a(\tau)}$. Let

$$b(\tau) = \min_{\tau_0\le\tau} \frac{\delta(\tau_0)}{a(\tau_0)} \le \frac{\delta(\tau)}{a(\tau)} \qquad (0 \le \tau \le 1).$$

Then $b(0) = \frac{\delta(0)}{a(0)} > 0$ as $g(x_0) < 1$ and $|\nabla\varphi(x_0)| \le c < +\infty$. The function b is decreasing. Let τ_k be an increasing sequence of points converging to 1. We put

$$p(\tau) = \begin{cases} b(\tau_k) & \text{for } \left(\tau_{k-1} \le \tau < \frac{\tau_{k-1}+\tau_k}{2}\right) \\ 2\,\frac{b(\tau_{k+1})-b(\tau_k)}{\tau_k-\tau_{k-1}}(\tau-\tau_k) + b(\tau_{k+1}) & \text{for } \frac{\tau_{k-1}+\tau_k}{2} \le \tau < \tau_k. \end{cases}$$

If $\tau \in [\tau_{k-1}, \tau_k]$, then it is obvious that $p(\tau) \le b(\tau_k) \le b(\tau)$. From the construction of the function p it is clear that it is continuous and increasing. So we conclude that $0 < p(\tau) < b(\tau) \le \frac{\delta(\tau)}{a(\tau)}$. Now we give as required function ε

$$\varepsilon(\varphi(x)) = \tfrac{1}{4}\int_{\varphi(x)}^1 p(\varphi)\,d\varphi.$$

We calculate the derivative of this function. We have

$$\varepsilon'(\varphi) = -\frac{1}{4}p(\varphi) \in C(\Omega)$$

$$|\nabla\varepsilon(x)| = |\varepsilon'(\varphi)|\,|\nabla\varphi(x)| < \tfrac{1}{4}b(\varphi)|\nabla\varphi(x)| \le \tfrac{1}{4}b(\varphi)\,a(\varphi) < \tfrac{1}{4}\delta(\varphi) < \frac{1-g(x)}{4g(x)\mu(x)}$$

$$\varepsilon(\varphi(x)) = 0 \quad (x \in \partial\Omega_\rho).$$

Therefore, we obtain the following result:

Theorem 8. *Let $\psi : \partial\Omega_\rho \to \mathbb{R}$ be a function. If the Lipschitz function $t : \Omega_\rho \to \mathbb{R}$ is such that $\overline{\lim}_{x\to y}\frac{|t(x)-t(y)|}{\rho(x,y)} < 1$ in Ω and $t = \psi$ on $\partial\Omega_\rho$, then there exists a smooth function $t_\varepsilon : \Omega_\rho \to \mathbb{R}$ such that*

$$|\nabla t_\varepsilon(x)|_\rho < 1 \quad (x \in \Omega) \qquad \text{and} \qquad t_\varepsilon(x) = \psi(x) \quad (x \in \partial\Omega_\rho).$$

Theorem 8 claims that if a Lipschitz space-like hypersurface is given by the graph of a function, then there is a smooth space-like hypersurface with the same boundary. In the general case this is not true. Consider an example of a contour, which may be spanned by Lipschitz space-like surface, but there is no smooth space-like surface.

Example 2. Let

$$\Pi = \big\{(u_1, u_2) : -1 < u_1 < 1 \text{ and } 0 < u_2 < 1\big\}.$$

We consider the surface $\mathcal{M}$ given by the Lipschitz mapping

$$F(u_1,u_2) = \begin{cases} t = 0 & \\ x_1 = u_1 & \\ x_2 = u_2 & \text{for } 0 \le u_1 < 1 \\ t = \frac{|u_1|}{2} & \\ x_1 = |u_1| & \\ x_2 = u_2 & \text{for } -1 < u_1 \le 0. \end{cases}$$

The desired contour is the boundary of a surface $\mathcal{M}$. Clearly, $\mathcal{M}$ is a space-like surface, which consists of two parts of planes. If the obtained contour may be spanned by a smooth space-like surface $\mathcal{M}_1$, then a smooth curve joining the points $A(\frac{1}{3}, 0, 0)$ and $B(\frac{1}{3}, 0, \frac{2}{3})$ of this contour lying on the surface $\mathcal{M}_1$ and having tangent line, parallel to the time axis, may be constructed. The last contradicts to fact that the surface $\mathcal{M}_1$ is space-like.

We begin to study the case of surfaces with an arbitrary codimension.

Let L be a $(k-1)$-dimensional closed surface, which is the boundary of a C^1-smooth surface

$$F(u) = \big(x_1(u), ..., x_n(u), t(u)\big) : \Omega \subset \mathbb{R}^k \to \mathbb{R}^{n+1}_1$$

so that, for every compactly subdomain $K \subset \Omega$,

$$\operatorname*{ess\,inf}_K \det G > 0, \quad G = \{g_{ij}\}, \quad g_{ij} = \sum_{p=1}^{n} \frac{\partial x_p}{\partial u_i} \cdot \frac{\partial x_p}{\partial u_j}. \tag{42}$$

This condition means that vectors orthogonal to the plane of a projection can not belong to tangent planes. We introduce the Riemanian metric h with the element of length $ds^2 = \sum_{l=1}^{n} dx_l^2$ in the domain Ω. Denote by $\rho(w,v)$ an intrinsic distance between points $w, v \in \Omega$ in this metric and by Ω_ρ the completion of the domain Ω by the metric ρ. Suppose that in the intrinsic metric the conditions

$$\begin{aligned} &|t(w) - t(v)| \le \rho(w,v) \quad (w, v \in \partial\Omega_\rho) \\ &|t(w) - t(v)| < \rho(w,v) \quad \text{if } \Gamma(w,v) \setminus \partial\Omega_\rho \ne \emptyset \end{aligned} \tag{43}$$

hold, that is there is a Lipschitz function f for which

$$f(u) = t(u) \quad (u \in \partial\Omega_\rho) \qquad \text{and} \qquad \overline{\lim_{u\to v}} \frac{|f(u) - f(v)|}{\rho(u,v)} < 1 \quad (u, v \in \Omega_\rho).$$

From the last inequality, $1 - |\nabla f|_h^2 > 0$ follows almost everywhere in Ω_ρ. Let $\widetilde{f} \in C^1(\Omega_\rho)$ be the result of smoothing f with a smoothing function ε, $\widetilde{f} = t$ on Ω_ρ and

$$\overline{\lim_{u\to v}} \frac{|\widetilde{f}(u) - \widetilde{f}(v)|}{\rho(u,v)} < 1 \quad \Longleftrightarrow \quad 1 - |\nabla \widetilde{f}|_\rho^2 > 0.$$

Now we show that from this condition it follows that the surface

$$\tilde{R}(u) = \big(x_1(u), ..., x_n(u), \widetilde{f}(u)\big)$$

is space-like and the vectors $\frac{\partial \tilde{R}}{\partial u_i}(u) \;\; (i = 1, .., k)$ are linear independent. We denote

$$A = \{a_{ij}\} = \Big\{\frac{\partial \tilde{f}}{\partial u_i} \cdot \frac{\partial \tilde{f}}{\partial u_j}\Big\}.$$

The surface, given by the radius vector $\tilde{R}(u)$, is space-like if and only if for every vector $\xi = (\xi_1, .., \xi_k) \in \mathbb{R}^k$

$$\xi \cdot (G - A) \cdot \xi^T > 0 \tag{44}$$

is true. This condition follows from the inequality $\det(G - A) > 0$, or the equivalent inequality $1 - \mathrm{tr}(G^{-1}A) = 1 - |\nabla \widetilde{f}|_\rho^2 > 0$. So, the obtained surface $\tilde{R}(u)$ is space-like and C^1-smooth because the linear indepedence of vectors $\frac{\partial \tilde{R}}{\partial u_i} \;\; (i = 1, ..., k)$ follows from the fact that Gram's determinant for these vectors is not equal to zero.

Therefore the following theorem is correct:

Theorem 9. *If the contour L is such that assumptions* (42) – (43) *are true, then there is a C^1-smooth space-like surface spanned this contour.*

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