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Exact Solution of a System of Generalized Hopf Equations

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Abstract. In this paper we construct explicit solutions for initial value problem for a system of first order equations. When $n = 1$, this system is just the standard Hopf equation in conservative form. When $n > 1$, the system is non-conservative. We use the vanishing viscosity method to construct solutions. As the system is non-conservative we use Volpert product and the algebra of generalized Colombeau functions to make sense of the products which appear in the equations.

Keywords: *Generalized Hopf equations, Hopf-Cole transformation, explicit formula for the solutions of generalized Hopf equations*

AMS subject classification: 35L, 35D05

1. Introduction

In this paper we consider the system of first order equations for $u_j \quad (j = 1, ..., n)$ in the domain $\Omega = \mathbb{R}^1 \times [0, \infty)$

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k \right) (u_j)_x = 0 \tag{1.1}$$

with initial condition

$$u_j(x, 0) = u_{0j}(x) \tag{1.2}$$

where c_k are real constants. We assume that at least one $c_k \neq 0$, without loss of generality we assume $c_1 \neq 0$. When $n = 1$ and $c_1 = 1$, (1.1) is just the Hopf equation

$$u_t + \tfrac{1}{2}(u^2)_x = 0 \tag{1.3}$$

which was studied by Hopf [8]. He considered the initial value problem

$$\left.\begin{aligned} u_t + \tfrac{1}{2}(u^2)_x &= \tfrac{\nu}{2} u_{xx} \\ u(x, 0) &= u_0(x) \end{aligned}\right\}$$

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and linearized it by the transformation

$$u = -\nu \frac{v_x}{v} \tag{1.4}$$

known as Hopf-Cole transformation and obtained the formula

$$u^\nu(x,t) = \int_{\mathbb{R}} \frac{x-y}{t}\, d\mu^\nu_{(x,t)}(y)$$

for its solution where $d\mu^\nu_{(x,t)}(y)\quad (\nu > 0)$, a family of probability measures on $\mathbb{R}$ parametrized by (x,t), is given by

$$d\mu^\nu_{(x,t)}(y) = \frac{e^{-\frac{1}{\nu}\left[\int_0^y u_0(z)\,dz + \frac{(x-y)^2}{2t}\right]}dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\nu}\left[\int_0^y u_0(z)dz + \frac{(x-y)^2}{2t}\right]}dy}.$$

Further, Hopf studied its limit as $\nu \to 0$ and constructed the solution to equation (1.3) in the sense of distributions with initial condition $u(x,0) = u_0(x)$ in the class of bounded measurable functions.

Following this method we will construct the solution of system (1.1) with initial data (1.2) in the class of bounded measurable functions. There are two difficulties to overcome:

- Firstly, we need a generalized version of transformation (1.4).
- Secondly, for $n > 1$ system (1.1) is not conservative and then the product $\sum_{k=1}^n c_k u_k)(u_j)_x$ does not make sense in the standard theory of distributions.

We use the Volpert product [15] in the sense of measures and the Colombeau theory of generalized functions [3 - 5] to overcome these difficulties. In Section 2 we construct an explicit formula for the solution of the initial value problem with a viscous term with viscous parameter $\nu > 0$. In Section 3 we construct the exact solution of system (1.1) with Riemann data where the Volpert product [15] is used to define the non-conservative product. Finally, in Section 4 the case is studied when the initial data are in a class of the Colombeau algebra and the product is understood in the sense of Colombeau [3, 5].

2. Explicit solution with viscous term

In this section we consider the viscous system for $u_j\quad (j = 1, ..., n)$ in the domain $\Omega = \mathbb{R} \times [0, \infty)$

$$(u_j)_t + \left(\sum_{k=1}^n c_k u_k\right)(u_j)_x = \tfrac{\nu}{2}(u_j)_{xx} \tag{2.1}$$

with initial conditions

$$u_j(x,0) = u_{0j}(x) \tag{2.2}$$

where u_{0j} are bounded measurable functions. We use a generalised Hopf-Cole transformation to linearize system (2.1) and solve it in terms of a family of probability measures $d\mu^{\nu}_{(x,t)}(y)$ defined by

$$d\mu^{\nu}_{(x,t)}(y) = \frac{e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]} dy}.$$

More presisely, we shall prove the following result.

Theorem 2.1. *Let* u_{0j} $(j = 1, ..., n)$ *be bounded measurable functions. Then the functions*

$$u^{\nu}_j(x,t) = \int_{R^n} u_{0j}(y)\, d\mu^{\nu}_{(x,t)}(y) \quad (j = 1, ..., n) \tag{2.3}$$

are infinitely differentiable in the variables (x,t) *and they are an exact solution of initial value problem* (2.1) – (2.2).

Proof. To prove the result first we introduce $\sigma = \sum_{k=1}^n c_k u_k$ as new unknown variable. It follows that problem (2.1) - (2.2) is equivalent to the problem

$$\left.\begin{aligned} (u_j)_t + \sigma(u_j)_x &= \tfrac{\nu}{2}(u_j)_{xx} \\ u_j(x,0) &= u_{0j}(x) \end{aligned}\right\} \qquad (j = 1, ..., n) \tag{2.4}$$

where σ is the solution to the problem

$$\left.\begin{aligned} \sigma_t + \frac{1}{2}(\sigma^2)_x &= \frac{\nu}{2}\sigma_{xx} \\ \sigma(x,0) &= \sum_{k=1}^n c_k u_{0k}(x). \end{aligned}\right\} \tag{2.5}$$

Let $w(x,t)$ be the solution of the problem

$$\left.\begin{aligned} w_t + \frac{(w_x)^2}{2} &= \frac{\nu}{2} w_{xx} \\ w(x,0) &= \sum_{k=1}^n c_k \int_0^x u_{0k}(y)\, dy. \end{aligned}\right\} \tag{2.6}$$

Then

$$\sigma(x,t) = w_x(x,t) \tag{2.7}$$

is a solution of problem (2.5). We introduce new unkown variables v and v_j $(j = 1, ..., n)$. Namely, v is defined by the usual Hopf-Cole transformation and v_j by a modified version of it as

$$\begin{aligned} v &= e^{-\frac{w}{\nu}} \\ v_j &= u_j e^{-\frac{w}{\nu}} \quad (j = 1, ..., n). \end{aligned} \tag{2.8}$$

An easy calculation shows that

$$\begin{aligned} v_t &= -\frac{1}{\nu}(w)_t e^{-\frac{w}{\nu}} \\ v_{xx} &= \frac{1}{\nu}\Big[\frac{(w_x)^2}{\nu} - (w)_{xx}\Big]e^{-\frac{w}{\nu}} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} (v_j)_t &= \Big[(u_j)_t - \frac{u_j}{\nu}w_t\Big]e^{-\frac{w}{\nu}} \\ (v_j)_{xx} &= \frac{1}{\nu}\Big[\nu(u_j)_{xx} - 2(u_j)_x w_x + \frac{u_j w_x^2}{\nu} - u_j w_{xx}\Big]e^{-\frac{w}{\nu}}. \end{aligned} \tag{2.10}$$

From (2.9) we get

$$v_t - \frac{\nu}{2}v_{xx} = -\frac{1}{\nu}\Big[w_t + \frac{(w_x)^2}{2} - \frac{\nu}{2}w_{xx}\Big]e^{-\frac{w}{\nu}}$$

and from (2.10) and (2.7) we get

$$\begin{aligned} (v_j)_t - \frac{\nu}{2}(v_j)_{xx} = {} & \Big[(u_j)_t + \sigma(u_j)_x - \frac{\nu}{2}w_{xx}\Big]\exp\Big(-\frac{w}{\nu}\Big) \\ & - \frac{1}{\nu}\Big[w_t + \frac{(w_x)^2}{2} - \frac{\nu}{2}w_{xx}\Big]u_j \exp\Big(-\frac{w}{\nu}\Big). \end{aligned}$$

From (2.4) - (2.7) and (2.9) - (2.10) it follows that v and v_j $(j = 1, ..., n)$ are solutions of the problems

$$\left.\begin{aligned} v_t &= \frac{\nu}{2}v_{xx} \\ v(x,0) &= e^{-\frac{1}{\nu}\sum_{k=1}^n c_k \int_0^x u_{0k}(y)\,dy} \end{aligned}\right\} \tag{2.11}$$

and

$$\left.\begin{aligned} (v_j)_t &= \frac{\nu}{2}(v_j)_{xx} \\ v_j(x,0) &= u_{0j}(x)e^{-\frac{1}{\nu}\sum_{k=1}^n c_k \int_0^x u_{0k}(y)dy} \end{aligned}\right\} \tag{2.12}$$

if and only if w is a solution of problem (2.6) and u_j $(j = 2, ..., n)$ is a solution of problem (2.4). Solving (2.11) and (2.12) explicitly we get

$$\begin{aligned} v(x,t) &= \tfrac{1}{(2\pi t\nu)^{1/2}} \int_{\mathbb{R}} e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]}dy \\ v_j(x,t) &= \tfrac{1}{(2\pi t\nu)^{1/2}} \int_{\mathbb{R}} u_{0j}(y)e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]}dy \end{aligned} \tag{2.13}$$

From (2.7) - (2.8) we have $\sigma(x,t) = w_x(x,t) = -\nu\frac{v_x}{v}$ and $u_j(x,t) = \frac{v_j}{v}$ $(j = 1, ..., n)$. Substituting herein (2.13) we get (2.3) ∎

3. The Riemann problem

In this section we consider a system of n first order equations for n unknowns u_j with Riemann-type initial data. Thus we have the system

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k\right)(u_j)_x = 0 \tag{3.1}$$

for unknowns $u_j \quad (j = 1, ..., n)$ whose characteristic speeds are all same, namely $\sigma = \sum_{k=1}^{n} c_k u_k$, and consider it with Riemann-type initial data

$$(u_j)(x,0) = \begin{cases} u_{jL} & \text{if } x < 0 \\ u_{jR} & \text{if } x > 0 \end{cases}. \tag{3.2}$$

where u_{jL} and u_{jR} are constants. Let

$$\sigma_L = \sum_{k=1}^{n} c_k u_{jL} \qquad \text{and} \qquad \sigma_R = \sum_{k=1}^{n} c_k u_{kR}.$$

If $\sigma_L' = \sigma_R$, the problem reduces to a linear one and its explicit solution is easy. So we assume $\sigma_{jL} \neq \sigma_{kR}$. As in the work of Lax [12] for conservation laws and that of DalMaso, LeFloch and Murat [7] for non-conservative strictly hyperbolic systems, we expect the structure of the solution to be constant states seperated by shocks or rarefaction. We observe that the equation of characteristics for system (3.1) are $\frac{dx}{dt} = \sigma$ with $x(0) = y$ and equations (3.1) say that in the region of smoothness $\frac{du}{dt} = 0$ along the characteristics. It follows that the characteristics starting at $(y, 0)$ is $x = \sigma_L t + y$ along which $u_j = u_{jL}$ when $y < 0$ and is $x = \sigma_R t$ along which $u_j = u_{jR}$ when $y > 0$. So when $\sigma_L > \sigma_R$, the characteristics starting at $y < 0$ intersect with the characteristics starting at $y > 0$, and at the point of intersection the solution is multi-valued. The only way to get a global solution is by introducing a shock. On the other hand, when $\sigma_L < \sigma_R$, the characteristics do not meet and there is a region $\sigma_L t < x < \sigma_R t$ where we can not get the solution by the characteristic method and in fact this has to be filled by rarefaction.

We do these constructions by studying the limit $\lim_{\nu \to 0} u_j^\nu$ where $u_j^\nu \quad (j = 1, \ldots, n)$ is the solution of the system

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k\right)(u_j)_x = \frac{\nu}{2}(u_j)_{xx} \tag*{$(3.1)_\nu$}$$

with Riemann-type initial data

$$(u_j)(x,0) = \begin{cases} u_{jL} & \text{if } x < 0 \\ u_{jR} & \text{if } x > 0 \end{cases}. \tag*{$(3.2)_\nu$}$$

We shall prove the following

Theorem 3.1. *Let* u_j^ν $(j=1,...,n)$ *be the solution of problem* $(3.1)_\nu - (3.2)_\nu$*. Then the limit* $u_j(x,t) = \lim_{\nu\to 0} u_j^\nu(x,t)$ *exists and is given by the formula*

$$u_j(x,t) = \begin{cases} u_{jL} & \text{if } x \le \sigma_L t \\ \frac{u_{jR}-u_{jL}}{\sigma_R-\sigma_L}\,\frac{x}{t} + \frac{u_{jL}\sigma_R - u_{jR}\sigma_L}{\sigma_R-\sigma_L} & \text{if } \sigma_L t < x < \sigma_R t \\ u_{jR} & \text{if } x \ge \sigma_R t \end{cases} \tag{3.3}$$

when $\sigma_L < \sigma_R$ *and by*

$$u_j(x,t) = \begin{cases} u_{jL} & \text{if } x < \frac{\sigma_L+\sigma_R}{2} t \\ \frac{u_{jL}+u_{jR}}{2} & \text{if } x = \frac{\sigma_L+\sigma_R}{2} t \\ u_{jR} & \text{if } x > \frac{\sigma_L+\sigma_R}{2} t \end{cases} . \tag{3.4}$$

when $\sigma_L > \sigma_R$*. Further, these limit functions solve problem* $(3.1) - (3.2)$ *where the non-conservative product is understood in the sense of Volpert.*

Proof. To give the proof, first we rewrite formula (2.3) for initial data $(3.2)_\nu$ in a more convenient way as

$$u_j^\nu(x,t) = \frac{u_{jL}A_{jL}^\nu(x,t) + u_{jR}A_{jR}^\nu(x,t)}{A_{jL}^\nu(x,t) + A_{jR}^\nu(x,t)} \tag{3.5}$$

where

$$A_{jR}^\nu(x,t) = \int_0^\infty e^{-\frac{1}{\nu}[\frac{(x-y)^2}{2t} + \sigma_R y]} dy$$
$$A_{jL}^\nu(x,t) = \int_0^\infty e^{-\frac{1}{\nu}[\frac{(x+y)^2}{2t} - \sigma_L y]} dy.$$

Next we try to write the above formula for u_j^ν in terms of the standard 'erfc' function

$$\text{erfc}(y) = \int_y^\infty e^{-y^2}\, dy.$$

Namely, since

$$A_{jR}^\nu(x,t) = e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} \int_0^\infty e^{-\frac{(y-x+\sigma_R t)^2}{2t\nu}} dy = (2t\nu)^{\frac{1}{2}} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} \int_{\frac{-x+\sigma_R t}{(2t\nu)^{1/2}}}^\infty e^{-y^2} dy$$

we get

$$A_{jR}^\nu(x,t) = (2t\nu)^{1/2} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} \text{erfc}\Big(\frac{-x+\sigma_R t}{(2\nu t)^{1/2}}\Big)$$

and similarly

$$A_{jL}^\nu(x,t) = (2t\nu)^{1/2} e^{\frac{\sigma_L^2 t}{2\nu} - \frac{\sigma_L x}{\nu}} \text{erfc}\Big(\frac{x-\sigma_L t}{(2\nu t)^{1/2}}\Big).$$

Using the asymptotic expansions of the erfc-function

$$\begin{aligned} \text{erfc}(y) &= \Big(\frac{1}{2y} - \frac{1}{4y^3} + o\Big(\frac{1}{y^3}\Big)\Big)e^{-y^2} \\ \text{erfc}(-y) &= \sqrt{\pi} - \Big(\frac{1}{2y} - \frac{1}{4y^3} + o\Big(\frac{1}{y^3}\Big)\Big)e^{-y^2} \end{aligned} \qquad (y \to \infty).$$

we get for $\nu \to 0$

$$A^{\nu}_{jR}(x,t) \approx \begin{cases} \frac{t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}} & \text{if } -x+\sigma_R t > 0 \\ (\frac{\pi t\nu}{2})^{1/2} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} & \text{if } -x+\sigma_R t = 0 \\ (2\pi t\nu)^{1/2} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} + \frac{t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}} & \text{if } -x+\sigma_R t < 0 \end{cases} \tag{3.6}$$

and

$$A^{\nu}_{jL}(x,t) \approx \begin{cases} \frac{t\nu}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}} & \text{if } x-\sigma_L t > 0 \\ (\frac{\pi t\nu}{2})^{1/2} e^{\frac{\sigma_L^2 t}{2\nu} - \frac{\sigma_L x}{\nu}} & \text{if } x-\sigma_L t = 0 \\ (2\pi t\nu)^{1/2} e^{\frac{\sigma_L^2 t}{2\nu} - \frac{\sigma_L x}{\nu}} + \frac{t\nu}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}} & \text{if } x-\sigma_L t < 0 \end{cases} \tag{3.7}$$

1. First we consider the case $\sigma_L < \sigma_R$ and prove (3.3) for which we have to treat three different regions.

Region 1: $x \le \sigma_L t$. Since $\sigma_L < \sigma_R$, in this region $x < \sigma_R t$ and so $-x + \sigma_R t > 0$ and $x - \sigma_L t \le 0$. First we treat the strong case $x - \sigma_L t < 0$. Using (3.6) - (3.7) in (3.5) we get

$$\begin{aligned} u^{\nu}_j(x,t) &\approx \frac{u_{jL}\Big[(2\pi t\nu)^{1/2} e^{\frac{\sigma_L^2 t}{2\nu} - \frac{\sigma_L x}{\nu}} + \frac{t\nu}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}}\Big] + \frac{u_{jR}(t\nu)}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}}{(2\pi t\nu)^{1/2} e^{\frac{\sigma_L^2 t}{2\nu} - \frac{\sigma_L x}{\nu}} + \frac{t\nu}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}} + \frac{t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}} \\ &= \frac{u_{jL}(2\pi)^{1/2} + \Big[\frac{(t\nu)^{1/2}}{x-\sigma_L t} - u_{jR}\frac{(t\nu)^{1/2}}{x-\sigma_R t}\Big] e^{-\frac{(x-\sigma_L t)^2}{2\nu t}}}{(2\pi)^{1/2} + \Big[\frac{(t\nu)^{1/2}}{x-\sigma_L t} - \frac{(t\nu)^{1/2}}{x-\sigma_R t}\Big] e^{-\frac{(x-\sigma_L t)^2}{2\nu t}}}. \end{aligned}$$

On the other hand, if $x - \sigma_L t = 0$, then using (3.6) - (3.7) and rearranging the terms we get

$$u^{\nu}_j(x,t) \approx \frac{u_{jL}(2\pi)^{1/2} - u_{jR}\frac{(t\nu)^{1/2}}{x-\sigma_R t}}{(2\pi)^{1/2} - \frac{(t\nu)^{1/2}}{x-\sigma_R t}}.$$

From both above expressions for u^{ν}_j we get

$$\lim_{\nu\to 0} u^{\nu}_j(x,t) = u_{jL} \qquad \text{if } x \le \sigma_L t. \tag{3.8}$$

Region 2: $\sigma_L t < x < \sigma_R t$. In this case $-x + \sigma_R t > 0$ and $x - \sigma_L t > 0$, and using (3.6) - (3.7) in (3.5) we get

$$u^{\nu}_j(x,t) \approx \frac{-\frac{u_{jL}t\nu}{-x+\sigma_L t} e^{-\frac{x^2}{2\nu t}} + \frac{u_{jR}t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}}{-\frac{t\nu}{-x+\sigma_L t} e^{-\frac{x^2}{2\nu t}} + \frac{t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}} = \frac{-\frac{u_{jL}}{-x+\sigma_L t} + \frac{u_{jR}}{-x+\sigma_R t}}{-\frac{1}{-x+\sigma_L t} + \frac{1}{-x+\sigma_R t}}.$$

Simplifying this we get

$$\lim_{\nu\to 0} u^{\nu}_j(x,t) = \frac{u_{jR} - u_j L}{\sigma_R - \sigma_L} \cdot \frac{x}{t} + \frac{u_{jL}\sigma_R - u_{jR}\sigma_L}{\sigma_R - \sigma_L} \qquad (\sigma_L t < x < \sigma_R t). \tag{3.9}$$

Region 3: $x \geq \sigma_R t$. First we take the strong case $x > \sigma_R t$. Then $-x + \sigma_R t < 0$ and $x - \sigma_L t > 0$, and using (3.6) - (3.7) in (3.5) we get

$$u_j^\nu(x,t) \approx \frac{\frac{u_{jL}(t\nu)}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}} + u_{jR}(2\pi t\nu)^{1/2} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} + \frac{u_{jR}(t\nu)}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}}{\frac{t\nu}{x-\sigma_L t} e^{-\frac{x^2}{2\nu t}} + (2\pi t\nu)^{1/2} e^{\frac{\sigma_R^2 t}{2\nu} - \frac{\sigma_R x}{\nu}} + \frac{t\nu}{-x+\sigma_R t} e^{-\frac{x^2}{2\nu t}}}$$

$$= \frac{\frac{u_{jL}(t\nu)^{1/2}}{x-\sigma_L t} e^{-\frac{(x-\sigma_R t)^2}{2\nu t}} + u_{jR}(2\pi)^{1/2} + \frac{u_{jR}(t\nu)^{1/2}}{-x+\sigma_R t} e^{-\frac{(x-\sigma_R t)^2}{2\nu t}}}{\frac{(t\nu)^{1/2}}{x-\sigma_L t} e^{-\frac{(x-\sigma_R t)^2}{2\nu t}} + (2\pi)^{1/2} + \frac{(t\nu)^{1/2}}{-x+\sigma_R t} e^{-\frac{(x-\sigma_R t)^2}{2\nu t}}}.$$

On the other hand, if $x = \sigma_R t$, then $x - \sigma_L t > 0$ and using (3.6) - (3.7) in (3.5) we get

$$u_j^\nu(x,t) \approx \frac{\frac{u_{jL}(t\nu)^{1/2}}{x-\sigma_L t} + u_{jR}(2\pi)^{1/2}}{\frac{(t\nu)^{1/2}}{x-\sigma_L t} + u_{jR}(2\pi)^{1/2}}.$$

From both above expressions for u_j^ν we get

$$\lim_{\nu\to 0} u_j^\nu(x,t) = u_{jR} \qquad \text{if } x \geq \sigma_R t. \tag{3.10}$$

Combining (3.8) - (3.10) we get (3.3).

2. Now we shall take the case $\sigma_L > \sigma_R$ and prove (3.4). Based on (3.6) - (3.7) there are

Region 1: $x \leq \sigma_R t$
Region 2: $\sigma_R t < x < \sigma_L t$.
Region 3: $x \geq \sigma_L t$

to consider here. Regions 1 and 3 are can be treated exactly as Regions 1 and 3 in the case $\sigma_L < \sigma_R$ and we get

$$\lim_{\nu\to 0} u_j^\nu(x,t) = \begin{cases} u_{jL} & \text{if } x < \sigma_R t \\ u_{jR} & \text{if } x > \sigma_L t \end{cases}. \tag{3.11}$$

For the remaining Region 2, again using (3.6) - (3.7) in (3.5) and rearranging the terms we get

$$u_j^\nu(x,t) \approx$$

$$\frac{u_{jL}(2\pi)^{1/2} + \left[\frac{(t\nu)^{1/2}}{x-\sigma_L t} + \frac{(t\nu)^{1/2}}{-x+\sigma_R t}\right] e^{-\frac{(x-\sigma_L t)^2}{2\nu t}} + u_{jR}(2\pi)^{1/2} e^{\frac{(\sigma_L-\sigma_R)}{\nu}(x - \frac{\sigma_L+\sigma_R}{2} t)}}{(2\pi)^{1/2} + \left[\frac{(t\nu)^{1/2}}{x-\sigma_L t} + \frac{(t\nu)^{1/2}}{-x+\sigma_R t}\right] e^{-\frac{(x-\sigma_L t)^2}{2\nu t}} + (2\pi)^{1/2} e^{\frac{(\sigma_L-\sigma_R)}{\nu}(x - \frac{\sigma_L+\sigma_R}{2} t)}}.$$

From here, since $\sigma_L > \sigma_R$, we get

$$\lim_{\nu\to 0} u_j^\nu(x,t) = \begin{cases} u_{jL} & \text{if } \sigma_R t < x < \frac{\sigma_L+\sigma_R}{2} t \\ \frac{u_{jL}+u_{jR}}{2} & \text{if } x = \frac{\sigma_L+\sigma_R}{2} t \\ u_{jR} & \text{if } \frac{\sigma_L+\sigma_R}{2} t < x < \sigma_L t \end{cases}. \tag{3.12}$$

Combining (3.11) and (3.12) we get (3.4).

At last, the proof that the limit functions (3.3) - (3.4) solves Riemann problem (3.1) - (3.2) in the sense of Volpert [15] follows along the same way as by LeFloch [13] or Joseph [9] and is omitted ∎

4. Generalized solutions in the sense of Colombeau

In this section we consider system (1.1) with more general initial data and use the theory of Colombeau to construct the solution. First we describe the Colombeau algebra of generalized functions in $\Omega = \{(x,t) : x \in \mathbb{R} \text{ and } t > 0\}$ denoted by $\mathcal{G}(\Omega)$. Let $C^\infty(\Omega)$ be the class of infinitely differentiable functions in Ω and take the infinite product $\mathcal{E}(\Omega) = [C^\infty(\Omega)]^{(0,1)}$. Thus any element u of $\mathcal{E}(\Omega)$ is a map from $(0,1)$ to $C^\infty(\Omega)$ which we denote by $u = (u^\nu)_{0<\nu<1}$. Such an element is called

- *moderate* if, given a compact subset K of Ω and non-negative integers j and ℓ, there exists $N > 0$ such that
$$\|\partial_t^j \partial_x^\ell u^\nu\|_{L^\infty(K)} = \mathcal{O}(\nu^{-N}) \qquad (\nu \to 0)$$
- *null* if, for all compact subsets K of Ω, for all non-negative integers j and ℓ and for all $M > 0$,
$$\|\partial_t^j \partial_x^\ell u^\nu\|_{L^\infty(K)} = \mathcal{O}(\nu^{M}) \qquad (\nu \to 0).$$

The sets of all moderate and null elements are denoted by $\mathcal{E}_{\mathcal{M}}(\Omega)$ and $\mathcal{N}(\Omega)$, respectively. It is easy to see that $\mathcal{E}_{\mathcal{M}}(\Omega)$ is an algebra with partial derivatives, the operations being defined pointwise on representatives, and $\mathcal{N}(\Omega)$ is an ideal closed under differentiation. The quotient space denoted by
$$\mathcal{G}(\Omega) = \frac{\mathcal{E}_{\mathcal{M}}(\Omega)}{\mathcal{N}(\Omega)}$$
is an algebra with partial derivatives, the operations being defined on representatives. The algebra $\mathcal{G}(\Omega)$ is called the *Colombeau algebra of generalized functions*. Two elements u and v in $\mathcal{G}(\Omega)$ are said to be *associated* if, for some (and hence all) representatives $(u^\nu)_{0<\nu<1}$ and $(v^\nu)_{0<\nu<1}$ of u and v, $u_\nu - v_\nu \to 0$ as $\nu \to 0$ in the sense of distributions, and this fact is denoted by "$u \approx v$". We remark that this notion is different from that of equality in $\mathcal{G}(\Omega)$, which means that $u - v \in \mathcal{N}(\Omega)$ or, in other words,
$$\|\partial_t^j \partial_x^\ell (u^\nu - v^\nu)\|_{L^\infty(K)} = \mathcal{O}(\nu^{M})$$
for all M, all compact subsets K of Ω and all non-negative integers j and ℓ.

In the works [1 - 6, 9 - 11, 14] and those cited herein there was shown that the Colombeau algebra is a useful tool to find global solutions of initial value problems when non-conservative products appear. Thus we consider the coupled Hopf equation
$$(u_j)_t + \Big(\sum_{k=1}^n c_k u_k\Big)(u_j)_x \approx 0 \qquad (j = 1, ..., n) \tag{4.1}$$
with initial conditions
$$u_j(x,0) = u_{j0} \tag{4.2}$$
where $u_{j0} = (u_{j0}^\nu)_{0<\nu<1}$ are in the algebra of generalized functions $\mathcal{G}(\mathbb{R})$ and we assume that u_{j0}^ν are obtained by mollifying bounded measurable functions u_{j0} with Friedrichs mollifiers so that we have the estimates
$$\|\partial_x^\ell u_{j0}^\nu\|_{L^\infty(\mathbb{R}^n)} = \mathcal{O}(\nu^{-\ell}). \tag{4.3}$$

For each $(x,t) \in \Omega$ and $\nu > 0$ define the probability measures

$$d\mu^{\nu}_{(x,t)}(y) = \frac{e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u^{\nu}_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]} dy}{\int_{-\infty}^{\infty} e^{-\frac{1}{\nu}\left[\sum_1^n c_k \int_0^y u^{\nu}_{0k}(z)dz + \frac{(x-y)^2}{2t}\right]} dy}. \tag{4.4}$$

From Theorem 2.1 it follows that

$$u^{\nu}_j(x,t) = \int_{\mathbb{R}^n} u^{\nu}_{0j}(y) d\mu^{\nu}_{(x,t)}(y) \qquad (j = 1, ..., n) \tag{4.5}$$

are infinitely differentiable and bounded and satisfy

$$(u_j)_t + \left(\sum_{k=1}^{n} c_k u_k\right)(u_j)_x = \frac{\nu}{2}(u_j)_{xx} \tag{4.6}$$

in the domain $\Omega = \mathbb{R} \times [0, \infty)$ with initial condition

$$u_j(x,0) = u^{\nu}_{0j}(x).$$

These facts help us to prove the following result.

Theorm 4.1. *Let* $u = (u^{\nu}_1, u^{\nu}_2, ..., u^{\nu}_n,)_{0<\nu<1}$ *with* u^{ν}_j $(j = 1, ..., n)$ *given by* (4.4) – (4.5) *and with initial data* u^{ν}_{j0} *are as described above. Then* u *is in the Colombeau algebra of generalized functions* $\mathcal{G}(\Omega)$ *and solves problem* (4.1) – (4.2).

Proof. First we show that $u = (u^{\nu}_1, u^{\nu}_2, ..., u^{\nu}_n)$ is in $\mathcal{G}(\Omega)$. It is clear from formulas (4.4) - (4.5) that u^{ν}_j are in $C^{\infty}(\Omega)$. Further, u^{ν}_j can be written as $u^{\nu}_j = \frac{F^{\nu}_1}{F^{\nu}_2}$ where

$$F^{\nu}_1(x,t) = \int_{-\infty}^{+\infty} u^{\nu}_{0j}(y) e^{-\frac{1}{\nu}\left[\int_0^y \sigma^{\nu}_0(z)dz + \frac{(x-y)}{2t}\right]} dy$$

$$F^{\nu}_2(x,t) = \int_{-\infty}^{+\infty} e^{-\frac{1}{\nu}\left[\int_0^y \sigma^{\nu}_0(z)dz + \frac{(x-y)}{2t}\right]} dy.$$

By Leibinitz's rule, $\partial^{j_0}_x u^{\nu}_j$ is a finite linear combination of elements of the form

$$\frac{\partial^{j_k}_x F^{\nu}_1}{F^{\nu}_2} \cdot \frac{\partial^{(j_{k-1}-j_k)}_x F^{\nu}_2}{F^{\nu}_2} \cdots \frac{\partial^{j_0-j_1}_x F^{\nu}_2}{F^{\nu}_2}$$

where $j_k < j_{k-1} < j_1 < j_0$ and $k = 0, 1, ..., j_0$. Making the change of variable $y = x - \sqrt{2t\nu}\, z$ in the integrals of F^{ν}_1 and F^{ν}_2 and using (4.4) we get $\|\frac{\partial^j_x F^{\nu}_1}{F^{\nu}_2}\|_{L^{\infty}(\Omega)} = \mathcal{O}(\nu^{-j})$ and $\|\frac{\partial^j_x F^{\nu}_2}{F^{\nu}_2}\|_{L^{\infty}(\Omega)} = \mathcal{O}(\nu^{-j})$. These estimates together with our earlier observation on the form of $\partial^{j_0} u^{\nu}_j$ leads to the estimates

$$\|\partial^{j_0}_x u^{\nu}_j\|_{L^{\infty}(\Omega)} = \mathcal{O}(\nu^{-j_0}) \qquad (j = 1, ..., n)$$

from where with (4.6) we get

$$\|\partial_t u_j^\nu\|_{L^\infty(\Omega)} = \mathcal{O}(\nu^{-1}) \qquad (j = 1, ..., n).$$

Now we apply the differential operator ${\partial_t}^{j_0}{\partial_x}^{\ell}$ on both sides of (4.6), first for $\ell = 1, j_0 \geq 0$, then for $\ell = 2, j_0 \geq 0$, etc. Proceeding succesively we get for each non-negative integer ℓ the estimates $\|\partial_t^{j_0}\partial_x^{\ell} u_j^\nu\|_{L^\infty(\Omega)} = \mathcal{O}(\nu^{-(j_0+\ell)}), j_0 \geq 0$ showing that u is in $\mathcal{G}(\Omega)$.

Now to show that u satisfies equation (4.1) in the sense of association we multiply (4.6) by a test function ϕ and integrate by parts on the right-hand side to get

$$\int_0^\infty\int_{\mathbb{R}} \Big((u_j^\nu)_t\phi + \sum_{k=1}^n c_k u_k^\nu (u_j^\nu)_x\phi\Big) dxdt = \tfrac{\nu}{2}\int_0^\infty\int_{\mathbb{R}^n} u^\nu \phi_{xx} dxdt. \tag{4.7}$$

for $j = 1, ..., n$. By assumption (4.3) on the initial data and formula (4.5), u_j^ν are uniformly bounded. An application of the dominated convergence theorem shows that the right-hand side of (4.7) goes to 0 as $\nu \to 0$. This completes the proof of the theorem ∎

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