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On Oscillation of a Differential Equation with Infinite Number of Delays

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Abstract. For a scalar delay differential equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) = 0 \qquad (h_k(t) \le t)$$

a connection between the following four properties is established:

- non-oscillation of this equation
- non-oscillation of the corresponding differential inequality
- positiveness of the fundamental function
- existence of a non-negative solution for a certain explicitly constructed nonlinear integral inequality.

Explicit non-oscillation and oscillation conditions, comparison theorems and a criterion of the existence of a positive solution are presented for this equation.

Keywords: *Delay differential equations, infinite number of delays, oscillation, non-oscillation*

AMS subject classification: 34K11

1. Introduction

This paper deals with a scalar differential equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) = 0 \qquad (t \ge t_0) \tag{1}$$

with infinite number of delays $h_k(t) \le t$. This equation is a natural generalization of equations with a finite number of delays which are well studied now (see, for example, monographs [5, 6, 8] and references therein, where various non-oscillation and oscillation conditions are presented).

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In our recent paper [3] we considered an equation with distributed delay. Equation (1) with bounded delays, i.e. satisfying

$$\inf_k h_k(t) > -\infty \qquad (t \geq 0), \tag{2}$$

is a particular case of the equation with distributed delay studied in [3]. However, condition (2) does not hold for important classes of equation (1), such as equations with constant delays $h_k(t) = t - \tau_k$, where $\lim \tau_k = \infty$, for instance, $\tau_k = k\tau$ with $\tau > 0$.

Later on we will employ oscillation properties of equation (1) for the investigation of a neutral equation. After some transformations a neutral equation can be rewritten in the form of equation (1). However, generally condition (2) does not hold for the latter equation. Hence for the investigation of a neutral equation we cannot apply the results of [3].

The purpose of the present paper is to study equation (1) without assumption (2). To the best of our knowledge such equations have not been considered. The main result is the equivalence of the following four properties for equation (1):

- non-oscillation of this equation
- existence of an eventually positive solution of the corresponding differential inequality
- existence of a non-negative solution of some nonlinear integral inequality which is explicitly constructed by the differential equation
- positiveness of the fundamental function of the differential equation.

The paper is organized as follows. Section 2 contains relevant definitions, notations and a "variation of constants formula". In Section 3 the equivalence of the four properties mentioned above is established. In Section 4 we obtain some comparison results. As a corollary, sufficient conditions on equation parameters and the initial function are established providing that the solution of the initial value problem is positive. At last, in Section 5 we suggest some explicit non-oscillation and oscillation conditions for equation (1).

2. Preliminaries

We consider equation (1) under the following conditions:

(a1) $a_k : [t_0, \infty) \to \mathbb{R}$ $(k \in \mathbb{N})$ are Lebesgue measurable functions, $a(t) = \sum_{k=1}^{\infty} |a_k(t)|$ is a locally essentially bounded function, where the series converges uniformly on any bounded interval $[t_0, b]$.

(a2) $h_k : [t_0, \infty) \to \mathbb{R}$ $(k \in \mathbb{N})$ are Lebesgue measurable functions, $h_k(t) \leq t$, $\lim_{t\to\infty} h_k(t) = \infty$, and

$$\text{for all } t, \text{ there exists } n \text{ such that } h_k(t) \leq t_0 \text{ for } k \geq n. \tag{3}$$

Together with equation (1) we consider for each $t_1 \geq t_0$ the initial value problem

$$\left.\begin{array}{r} \dot{x}(t) + \displaystyle\sum_{k=1}^{\infty} a_k(t) x(h_k(t)) = f(t) \quad (t \geq t_1) \\ x(t) = \varphi(t) \quad (t < t_1), \quad x(t_1) = x_0 \end{array}\right\} . \tag{4}$$

We also assume that the following hypothesis holds.

(a3) $f : [t_1, \infty) \to \mathbb{R}$ is a Lebesgue measurable locally essentially bounded function and $\varphi : (-\infty, t_1) \to \mathbb{R}$ is a Borel measurable bounded function.

Definition. A locally absolutely continuous on $[t_0, \infty)$ function is called a *solution of problem* (4) if it satisfies equation $(4)_1$ for almost all $t \geq t_1$ and equation $(4)_2$ for $t \leq t_1$.

Lemma 1. *Let conditions* (a1) - (a3) *hold. Then there exists one and only one solution of problem* (4).

Proof. Consider together with (4) the problem

$$\left.\begin{aligned} \dot{y}(t) + \sum_{k=1}^{\infty} a_k(t) y(h_k(t)) = g(t) \quad (t \geq t_1) \\ y(t) = 0 \qquad (t < t_1), \quad y(t_1) = x_0 \end{aligned}\right\} \tag{5}$$

where

$$g(t) = f(t) - \sum_{k=1}^{\infty} a_k(t)\varphi(h_k(t)) \qquad \text{with } \varphi(t) = 0 \text{ for } t \geq t_1.$$

If y is a solution of problem (5), then

$$x(t) = \begin{cases} y(t) & \text{if } t \geq t_1 \\ \varphi(t) & \text{if } t < t_1 \end{cases}$$

is a solution of problem (4). After substituting $y(t) = y(t_1) + \int_{t_1}^{t} z(s)\,ds$ with $z(t) = \dot{y}(t)$ into $(5)_1$ we obtain the operator equation

$$z(t) + \sum_{k=1}^{\infty} a_k(t) \int_{t_1}^{h_k(t)} z(s)\,ds = g(t) - \sum_{k=1}^{\infty} a_k(t) y(t_1) \tag{6}$$

where the sum in the left-hand side contains only such terms for which $h_k(t) \geq t_1$. Condition (3) implies that for every t the number of such terms is finite.

Suppose $t_2 > t_1$ is an arbitrary number and the integer n is such that

$$(t_2 - t_1) \sup_{t_1 < t < t_2} \sum_{k=n+1}^{\infty} |a_k(t)| < 1. \tag{7}$$

Consider the operator $H = H_1 + H_2$ in the space $L_{[t_1, t_2]}$ of all integrable on $[t_1, t_2]$ functions with the usual norm, where the summands are defined by

$$(H_1 z)(t) = \sum_{k=1}^{n} a_k(t) \int_{t_1}^{h_k(t)} z(s)\,ds$$

$$(H_2 z)(t) = \sum_{k=n+1}^{\infty} a_k(t) \int_{t_1}^{h_k(t)} z(s)\,ds.$$

Inequality (7) implies that in the space $L_{[t_1,t_2]}$ the inequality $r(H_2) \le \|H_2\| < 1$ holds, where $r(H_2)$ and $\|H_2\|$ are the spectral radius and the norm of the operator H_2 in this space, respectively. The operator H_1 is a finite sum of integral compact Volterra operators, hence (see [4: p. 519]) $r(H_1) = 0$ and then (see [1: p. 56]) $r(H) = r(H_1 + H_2) = r(H_2) < 1$. Consequently, equation (6) has a unique solution, hence problem (5) also has a unique solution and therefore the same is true for problem (4) ∎

Definition. For each $s \ge t_0$ the solution $X(t,s)$ of the problem

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) = 0, \quad x(t) = 0 \quad (t < s), \quad x(s) = 1 \tag{8}$$

is called a *fundamental function of equation* (1).

We assume $X(t,s) = 0$ for $t_0 \le t < s$. Lemma 1 implies that $X(t,s)$ exists.

Lemma 2. *Let conditions* (a1) - (a2) *hold. Then for the fundamental function* $X(t,s)$ *of equation* (1) *we have the estimate*

$$|X(t,s)| \le \exp\left\{ \sum_{k=1}^{\infty} \int_s^t |a_k(\tau)|\, d\tau \right\}.$$

Proof. Let $x(t) = X(t,s)$ for $t \ge s$. Equalities (8) imply

$$x(t) = 1 - \sum_{k=1}^{\infty} \int_s^t a_k(\tau)x(h_k(\tau))\, d\tau.$$

Denote $y(t) = \sup_{s \le \xi \le t} |x(\xi)|$. Then

$$y(t) \le 1 + \sum_{k=1}^{\infty} \int_s^t |a_k(\tau)| y(\tau)\, d\tau.$$

The Gronwall-Bellman inequality implies the inequality in question ∎

Theorem 1. *Let conditions* (a1) - (a3) *hold. Then there exists one and only one solution* x *of problem* (4) *that with the fundamental function* $X(t,s)$ *of equation* (1) *can be presented in the form*

$$\begin{aligned} x(t) = X(t,t_1)x_0 + \int_{t_1}^t X(t,s)f(s)\, ds \\ - \sum_{k=1}^{\infty} \int_{t_1}^t X(t,s)a_k(s)\varphi(h_k(s))\, ds \end{aligned} \tag{9}$$

where $\varphi(h_k(s)) = 0$ *if* $h_k(s) > t_1$.

Proof. Lemmas 1 and 2 imply that the solution of problem (4) exists, it is unique and the sum in (9) is well defined. By direct computation one can see that function (9) is the solution of problem (4) ∎

3. Non-oscillation criteria

In this section we investigate conditions providing the existence of a solution of problem (4) which is positive together with the initial function. Theorem 2 is the main result of the present paper.

Definition. We will say that equation (1) has a *non-oscillatory solution* if for some $t_1 \geq t_0, \varphi(t) \geq 0, x_0 > 0$ the solution of problem (4) is positive. Otherwise all solutions of equation (1) are *oscillatory*.

Remark. For the case of finite number of delays, if equation (1) has an eventually positive solution, then it has a positive solution with a non-negative initial function. Hence our definition of non-oscillation and the usual one for equations with finite number of delays are equivalent. This is also valid for equations with infinite number of delays satisfying inequality (2).

Consider together with equation (1) the delay differential inequality

$$\dot{y}(t) + \sum_{k=1}^{\infty} a_k(t) y(h_k(t)) \leq 0. \tag{10}$$

For this inequality the definition of non-oscillation is the same as for equation (1). The following theorem establishes non-oscillation criteria.

Theorem 2. *Suppose conditions* (a1) - (a3) *hold and* $a_k(t) \geq 0 \quad (k \in \mathbb{N})$. *Then the following hypotheses are equivalent:*

1) *Inequality* (10) *has a non-oscillatory solution.*

2) *There exists* $t_1 \geq t_0$ *such that the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} a_k(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)\, ds \right\} \qquad (t \geq t_1) \tag{11}$$

has a non-negative locally integrable solution, where the sum contains only terms for which $h_k(t) \geq t_1$.

3) *There exists* $t_1 \geq t_0$ *such that for the fundamental function of equation* (1) *we have* $X(t,s) > 0$ *for* $t \geq s \geq t_1$.

4) *Equation* (1) *has a non-oscillatory solution.*

Remark. Condition (3) implies that for every t the sum in (11) contains only finite numbers of terms. We will suppose, without loss of generality, that for solutions u of inequality (11) we have $u(t) = 0$ for $t \leq t_1$.

Proof of Theorem 2. In [2: Theorem 1] the equivalence of statements 1) - 4) was proved for an equation with a finite number of delays. The proof of the present theorem is similar to one of [2: Theorem 1]. Therefore we will give only a scheme of it.

1) $\Rightarrow$ 2): Let y be a positive solution of inequality (10) for $t \geq t_1$ with non-negative initial function $\varphi(t) \geq 0$. Denote

$$u(t) = -\frac{d}{dt} \ln \frac{y(t)}{y(t_1)} \qquad (t \geq t_1).$$

Then

$$y(t) = y(t_1) \exp\Big\{ - \int_{t_1}^{t} u(s)\, ds \Big\} \qquad (t \geq t_1).$$

We substitute this into (10) and obtain by carrying the exponent out of the brackets

$$\begin{aligned} -\exp\Big\{ - \int_{t_1}^{t} u(s)\, ds\Big\} y(t_1)\Big[u(t) - \sum_{k=1}^{\infty} a_k(t) \exp\Big\{ \int_{h_k(t)}^{t} u(s\,)ds \Big\}\Big] \\ + {\sum_{k=1}^{\infty}}' a_k(t)\varphi(h_k(t)) \leq 0 \end{aligned} \tag{12}$$

where the sum $\sum'$ contains such terms that $h_k(t) < t_1$. Since $y(t) > 0$ and $a_k(t) \geq 0$, then (12) implies (11).

2) $\Rightarrow$ 3): *Step* 1. Consider the initial value problem

$$\left.\begin{aligned} \dot{x}(t) + \sum_{k=1}^{\infty} a_k(t) x(h_k(t)) = f(t) \quad (t \geq t_1) \\ x(t) = 0 \qquad (t \leq t_1) \end{aligned}\right\}. \tag{13}$$

Denote

$$z(t) = \dot{x}(t) + u(t)x(t) \qquad \text{with } z(t) = 0 \text{ for } t \leq t_1 \tag{14}$$

where x is the solution of problem (13) and u is a non-negative solution of inequality (11). Equality (14) implies

$$x(t) = \int_{t_1}^{t} \exp\Big\{ - \int_{s}^{t} u(\tau)\, d\tau \Big\} z(s)\, ds \qquad (t \geq t_1). \tag{15}$$

After substituting this into (13) and some transformations problem (13) can be rewritten in the form

$$z - Hz = f \tag{16}$$

where

$$\begin{aligned} (Hz)(t) = & \int_{t_1}^{t} \exp\Big\{ - \int_{s}^{t} u(\tau)\, d\tau\Big\} z(s)\, ds \\ & \times \Big[u(t) - \sum_{k=1}^{\infty} a_k(t) \exp\Big\{ \int_{h_k(t)}^{t} u(s)\, ds \Big\}\Big] \\ & + \sum_{k=1}^{\infty} a_k(t) \int_{h_k(t)}^{t} \exp\Big\{ - \int_{s}^{h_k(t)} u(\tau)\, d\tau \Big\} z(s)\, ds. \end{aligned}$$

Inequality (11) yields that if $z(t) \geq 0$, then $(Hz)(t) \geq 0$ (i.e. the operator H is positive).

We have

$$0 \leq u(t) - \sum_{k=1}^{\infty} a_k(t) \exp\left\{ \int_{h_k(t)}^{t} u(s)\, ds \right\} \leq u(t)$$

and

$$\begin{aligned} 0 &\leq \sum_{k=1}^{\infty} a_k(t) \int_{h_k(t)}^{t} \exp\left\{ -\int_{s}^{h_k(t)} u(\tau)\, d\tau \right\} z(s)\, ds \\ &\leq \int_{t_1}^{t} a(t) z(s)\, ds. \end{aligned}$$

Then for every $b > t_1$ the operator $H : L_{[t_1,b]} \to L_{[t_1,b]}$ is the sum of compact integral Volterra operators. Hence for its spectral radius we have $r(H) = 0 < 1$. Thus if in (16) $f(t) \geq 0$, then

$$z(t) = f(t) + (Hf)(t) + (H^2 f)(t) + \ldots \geq 0.$$

The solution of problem (13) has form (15), with z being a solution of equation (16). Hence if in (13) $f(t) \geq 0$, then for the solution x of this equation $x(t) \geq 0$. On the other hand, the solution of problem (13) can be presented in form (9): $x(t) = \int_{t_1}^{t} X(t,s) f(s)\, ds$. As was shown above, $f(t) \geq 0$ implies $x(t) \geq 0$. Consequently, the kernel of the integral operator is non-negative, i.e. $X(t,s) \geq 0$ for $t \geq s > t_1$.

Step 2. Let us prove that in fact the strict inequality $X(t,s) > 0$ holds. Denote

$$x(t) = X(t,t_1) - \exp\left\{ -\int_{t_1}^{t} u(s)\, ds \right\}, \qquad x(t) = 0 \text{ for } t < t_1.$$

After substitution one can see that this function is a solution of problem (13) with $f(t) \geq 0$. Hence as proved above, $x(t) \geq 0$. Consequently,

$$X(t,t_1) \geq \exp\left\{ -\int_{t_1}^{t} u(s)\, ds \right\} > 0.$$

For $s > t_1$ the inequality $X(t,s) > 0$ can be proved similarly.

3) $\Rightarrow$ 4): A function $x(t) = X(t,t_1)$ is a positive solution of equation (1) for $t \geq t_1$. The implication 4) $\Rightarrow$ 1) is evident ∎

4. Comparison theorems and existence of a positive solution

Theorem 2 can be employed for obtaining comparison results in oscillation theory. To this end consider together with equation (1) the following one

$$\dot{x}(t) + \sum_{k=1}^{\infty} b_k(t)x(h_k(t)) = 0 \qquad (t \geq t_0). \tag{17}$$

In this section we suppose that conditions (a1) - (a2) hold for equations (1) and (17). Denote by $Y(t,s)$ the fundamental function of equation (17).

Theorem 3. *Suppose $a_k(t) \geq 0$ and $a_k(t) \geq b_k(t)$ $(t \geq t_1)$ and equation (1) has a positive solution for $t \geq t_1$. Then equation (17) has a positive solution for $t \geq t_1$, and for its fundamental function $Y(t,s)$ we have $Y(t,s) > 0$ for $t \geq s \geq t_1$.*

Proof. Consider the problem

$$\left.\begin{aligned} \dot{x}(t) + \sum_{k=1}^{\infty} b_k(t)x(h_k(t)) = f(t) & \quad (t \geq t_1) \\ x(t) = 0 \qquad & \quad (t \leq t_1) \end{aligned}\right\}. \tag{18}$$

We will show that if $f(t) \geq 0$, then the solution of this problem is non-negative. To this end rewrite the problem in the form

$$\left.\begin{aligned} \dot{x}(t) + \sum_{k=1}^{\infty} a_k(t)x(h_k(t)) + \sum_{k=1}^{\infty}[b_k(t) - a_k(t)]x(h_k(t)) = f(t) & \quad (t \geq t_1) \\ x(t) = 0 \qquad & \quad (t \leq t_1) \end{aligned}\right\}.$$

Substitute herein

$$x(t) = \int_{t_1}^{t} X(t,s)z(s)\,ds$$

where X is the fundamental function of equation (1). Then problem (18) is equivalent to the equation

$$z - Tz = f \tag{19}$$

where

$$(Tz)(t) = \int_{t_1}^{t} X(t,s) \sum_{k=1}^{\infty}[a_k(t) - b_k(t)]\chi_k(t,s)z(s)\,ds$$

and

$$\chi_k(t,s) = \begin{cases} 1 & \text{if } t_1 \leq s \leq h_k(t) \\ 0 & \text{if } h_k(t) < s \text{ or } h_k(t) < t_1. \end{cases}$$

Lemma 2 yields that the integral Volterra operator T is a compact one acting in the space of integrable functions $L_{[t_1,b]}$ for every $b > t_1$. Then for the spectral radius of this operator we have $r(T) = 0 < 1$. Theorem 2 implies that $X(t,s) > 0$ for $t \geq s \geq t_1$, hence the operator T is positive. Therefore for the solution z of equation (19) we have

$$z(t) = f(t) + (Tf)(t) + \ldots \geq 0 \qquad \text{if } f(t) \geq 0.$$

Then, as in the proof of Theorem 2, we conclude $Y(t,s) > 0$ for $t \geq s \geq t_1$ and therefore $x(t) = Y(t,t_1)$ is a positive solution of equation (21) ∎

Corollary 1. *Suppose* $a_k(t) \geq 0$ *and* $a_k(t) \geq b_k(t)$, *and equation* (1) *has a non-oscillatory solution. Then equation* (21) *has a non-oscillatory solution.*

Denote $a^+ = \max\{a, 0\}$.

Corollary 2. *If the equation*

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k^+(t) x(h_k(t)) = 0$$

has a non-oscillatory solution, then equation (1) *has a non-oscillatory solution.*

Corollary 2 can be employed for obtaining a comparison result which improves the statement of Theorem 3.

Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} b_k(t) x(g_k(t)) = 0. \tag{20}$$

Suppose conditions (a1) - (a2) hold for it and denote by $Y(t,s)$ its fundamental function.

Theorem 4. *Suppose* $a_k(t) \geq 0$ *and equation* (1) *has a non-oscillatory solution. If*

$$b_k(t) \leq a_k(t) \qquad \text{and} \qquad h_k(t) \leq g_k(t), \tag{21}$$

then equation (20) *has a non-oscillatory solution, and for some* $t_1 \geq t_0$ *for its fundamental function* $Y(t,s)$ *we have* $Y(t,s) > 0$ *for* $t \geq s \geq t_1$.

Proof. Theorem 2 implies that for some $t_1 \geq t_0$ there exists a non-negative solution u of inequality (11) for $t \geq t_1$. Inequalities (21) yield that this function is also a solution of the inequality

$$v(t) \geq \sum_{k=1}^{\infty} b_k^+(t) \exp\left\{ \int_{g_k(t)}^{t} v(s)\, ds \right\} \qquad (t \geq t_1)$$

where the sum contains only terms for which $g_k(t) \geq t_1$. Hence by Corollary 2 of Theorem 3 equation (20) has a positive solution for $t \geq t_1$ and the fundamental function of this equation is positive ∎

Corollary. *Suppose* $0 \leq a_k(t) \leq b_k(t)$ *and* $g_k(t) \leq h_k(t)$, *and all solutions of equation* (1) *are oscillatory. Then all solutions of equation* (20) *are oscillatory.*

Inequality $X(t,s) > 0$ can be employed for comparison of solutions. To this end consider together with problem (4) the initial value problem

$$\left.\begin{aligned} \dot{y}(t) + \sum_{k=1}^{\infty} b_k(t) y(h_k(t)) = g(t) \quad (t \geq t_1) \\ y(t) = \psi(t) \quad (t < t_1), \quad y(t_1) = y_0 \end{aligned}\right\}. \tag{22}$$

Suppose conditions (a1) - (a3) hold for this problem. Denote by $x(t), X(t,s)$ and $y(t), Y(t,s)$ the solution and fundamental function of problems (4) and (22), respectively.

Theorem 5. *Suppose there exists a non-negative solution of inequality* (11) *for* $t \geq t_1$, $x(t) > 0$,

$$a_k(t) \geq b_k(t) \geq 0, \quad g(t) \geq f(t), \quad \varphi(t) \geq \psi(t) \geq 0 \qquad (t < t_1)$$

and $y_0 \geq x_0 > 0$. *Then* $y(t) \geq x(t) > 0$.

Proof. Denote by u a non-negative solution of inequality (11). The inequality $a_k(t) \geq b_k(t)$ yields that the function $u(t)$ is also a solution of the inequality corresponding to inequality (11) for equation $(22)_1$. Hence, by Theorem 1, $X(t,s) > 0$ and $Y(t,s) > 0$ for $t_1 \leq s < t$. Rewrite equation $(4)_1$ as

$$\dot{x}(t) + \sum_{k=1}^{\infty} b_k(t)x(h_k(t)) = \sum_{k=1}^{\infty} [b_k(t) - a_k(t)]x(h_k(t)) + f(t) \qquad (t \geq t_1).$$

Hence

$$\begin{aligned} x(t) = Y(t,t_1)x_0 &- \sum_{k=1}^{\infty} \int_{t_1}^{t} Y(t,s)b_k(s)\varphi(h_k(s))\,ds \\ &+ \int_{t_1}^{t} Y(t,s)f(s)\,ds - \sum_{k=1}^{\infty} \int_{t_1}^{t} Y(t,s)[a_k(s) - b_k(s)]x(h_k(s))\,ds \end{aligned}$$

and

$$y(t) = Y(t,t_1)y_0 - \sum_{k=1}^{\infty} \int_{t_1}^{t} Y(t,s)b_k(s)\psi(h_k(s))\,ds + \int_{t_1}^{t} Y(t,s)g(s)\,ds$$

where $\varphi(h_k(s)) = \psi(h_k(s)) = 0$ if $h_k(s) \geq t_1$ and $x(h_k(s)) = 0$ if $h_k(s) < t_1$. Therefore $y(t) \geq x(t) > 0$ ∎

Corollary. *Suppose* $a_k(t) \geq 0$, x *and* y *are positive solutions of equation* (1) *and inequality* (10), *respectively, for* $t \geq t_1$, *with the same non-negative initial function and positive initial value. Then* $x(t) \geq y(t)$ *for* $t > t_1$.

Now we proceed to the existence of a positive solution for equation (1). We will show that if inequality (11) has a non-negative solution for $t \geq t_1$ and the condition

$$0 \leq \varphi(t) \leq x(t_1) = x_0 \qquad (t \leq t_1, x_0 > 0) \tag{23}$$

holds, then the solution of the initial value problem (4) is positive. This result supplements some statements in [2, 6, 7] obtained for equations with a finite number of delays.

Theorem 6. *Suppose* $a_k(t) \geq 0, f(t) \geq 0$, *there exists a non-negative solution of the inequality*

$$u(t) \geq \sum_{k=1}^{\infty} a_k(t) \int_{\max\{t_1,h_k(t)\}}^{t} u(s)\,ds \qquad (t \geq t_1) \tag{24}$$

for a certain $t_1 \geq t_0$*, and conditions* (23) *holds. Then the solution of problem* (4) *is positive for* $t \geq t_1$.

Proof. First assume $f \equiv 0$. Consider the auxiliary problem

$$\left.\begin{aligned} \dot z(t) + \sum_{k=1}^{\infty} a_k(t) z(h_k(t)) = 0 \quad (t \geq t_1) \\ z(t) = x_0 \quad (t \leq t_1) \end{aligned}\right\}.$$

Let $u(t) \geq 0$ be a non-negative solution of inequality (24). Denote

$$v(t) = \begin{cases} x_0 \exp\{-\int_{t_1}^{t} u(s)ds\} & \text{if } t \geq t_1 \\ x_0 & \text{if } t < t_1 \end{cases}$$

and for a fixed $t \geq t_1$ define the sets

$$N_1(t) = \{k :\ h_k(t) \geq t_1\}$$
$$N_2(t) = \{k :\ h_k(t) < t_1\}.$$

We obtain

$$\begin{aligned}
&\dot v(t) + \sum_{k=1}^{\infty} a_k(t) v(h_k(t)) \\
&\quad = -x_0 u(t) \exp\Big\{ - \int_{t_1}^{t} u(s)\,ds \Big\} \\
&\qquad + x_0 \sum_{k \in N_1(t)} a_k(t) \exp\Big\{ - \int_{t_1}^{h_k(t)} u(s)\,ds \Big\} + x_0 \sum_{k \in N_2(t)} a_k(t) \\
&\quad = -x_0 \exp\Big\{ - \int_{t_1}^{t} u(s)\,ds \Big\} \Big[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\Big\{ \int_{h_k(t)}^{t} u(s)\,ds \Big\} \\
&\qquad - \sum_{k \in N_2(t)} a_k(t) \exp\Big\{ \int_{t_1}^{t} u(s)\,ds \Big\} \Big] \\
&\quad = -x_0 \exp\Big\{ - \int_{t_1}^{t} u(s)\,ds \Big\} \Big[u(t) - \sum_{k \in N_1(t)} a_k(t) \exp\Big\{ \int_{\max\{t_1, h_k(t)\}}^{t} u(s)\,ds \Big\} \\
&\qquad - \sum_{k \in N_2(t)} a_k(t) \exp\Big\{ \int_{\max\{t_1, h_k(t)\}}^{t} u(s)\,ds \Big\} \Big] \\
&\quad = -x_0 \exp\Big\{ - \int_{t_1}^{t} u(s)\,ds \Big\} \Big[u(t) - \sum_{k=1}^{\infty} a_k(t) \exp\Big\{ \int_{\max\{t_1, h_k(t)\}}^{t} u(s)\,ds \Big\} \Big] \\
&\quad \leq 0.
\end{aligned}$$

Obviously, inequality (24) implies inequality (11). Thus Corollary of Theorem 5 yields $z(t) \geq v(t) > 0$. Conditions (23) and Theorem 5 imply $x(t) \geq z(t) > 0$ for $t \geq t_1$. For the case $f \equiv 0$ the theorem is proved. The general case is a consequence of Theorem 5 since $f(t) \geq 0$ ∎

5. Explicit conditions of oscillation and non-oscillation

In this section we will generalize some well known results for differential equations with finite number of delays. Some of these results one can see in the monographs [5, 6, 8].

We will begin with non-oscillation conditions.

Theorem 7. *Let conditions* (a1) - (a2) *hold and let there exist* $\lambda > 0$ *and* $t_1 \geq t_0$ *such that at least one the two inequalities*

$$\begin{aligned} \sup_{t\geq t_1} \sum_{k=1}^{\infty} a_k^+(t) \exp\{\lambda(t-h_k(t))\} \leq \lambda \\ \sup_{t\geq t_1} \frac{\sum_{k=1}^{\infty} a_k^+(t) \exp\left\{\lambda \int_{h_k(t)}^t \sum_{k=1}^{\infty} a_k^+(s)\,ds\right\}}{\sum_{k=1}^{\infty} a_k^+(t)} \leq \lambda \end{aligned} \tag{25}$$

hold where the sums contains only terms for which $h_k(t) \geq t_1$. *Then equation* (1) *has a non-oscillatory solution.*

Proof. Inequality $(25)_1$ implies that the function $u(t) = \lambda$ is a positive solution of inequality (11), where $a_k(t)$ are replaced by $a_k^+(t)$. Corollary 2 of Theorem 3 implies that equation (1) has a non-oscillatory solution. If inequality $(25)_2$ holds, then the function $u(t) = \lambda \sum_{k=1}^{\infty} a_k^+(t)$ is a non-negative solution of inequality (11) ∎

Example 1. Consider the autonomous equation with infinite number of delays

$$\dot{x}(t) + \sum_{k=1}^{\infty} b_k e^{-k} x(t-k) = 0 \tag{33}$$

where $b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < 1$. It is easy to see that $\lambda = 1$ is a solution of inequality $(25)_1$. Hence the equation has a non-oscillatory solution.

Now we proceed to oscillation conditions. The following statement is an immediate corollary of comparison Theorem 3.

Theorem 8. *Suppose conditions* (a1) - (a2) *hold,* $a_k(t) \geq 0$, *there exist indices* k_i $(i \in \mathbb{N})$ *such that all solutions of the equation*

$$\dot{x}(t) + \sum_{k_i} a_{k_i}(t) x(h_{k_i}(t)) = 0$$

are oscillatory. Then all solutions of equation (1) *are oscillatory.*

Corollary. *Suppose conditions* (a1) - (a2) *hold,* $a_k(t) \geq 0$ *and there exists an index* k *such that*

$$\limsup_{t\to\infty} \int_{h_k(t)}^t a_k(s)\,ds > \frac{1}{e}.$$

Then all solutions of equation (1) *are oscillatory.*

Theorem 9. *Suppose conditions* (a1) - (a2) *hold,* $a_k(t) \geq 0$*, there exist indices* k_i *such that*

$$\liminf_{t\to\infty} (t - h_{k_i}(t)) > 0 \text{ and } \liminf_{t\to\infty} \sum_{k_i} a_{k_i}(t) > 0$$
$$\liminf_{t\to\infty} \left[\inf_{\lambda>0} \left\{ \frac{1}{\lambda} \sum_{k=1}^{\infty} a_k(t) \exp\{\lambda(t - h_k(t))\} \right\} \right] > 1. \tag{26}$$

Then all solutions of equation (1) *are oscillatory.*

Corollary. *Suppose conditions* (a1) - (a2) *hold,* $a_k(t) \geq 0$*, there exist indices* k_i *such that* $(26)_1$ *holds and*

$$\liminf_{t\to\infty} \sum_{k=1}^{\infty} a_k(t)(t - h_k(t)) > \frac{1}{e}.$$

Then all solutions of equation (1) *are oscillatory.*

Proof. The proof of Theorem 9 and its corollary is the same as for [6: Theorem 3.4.2 and its corollary] for the case of finite number of delays ∎

Example 2. Consider the equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} \frac{a_k}{t} x(\mu_k t) = 0$$

where $a_k \geq 0$ and $1 > \mu_k > 0$. If $\sum_{k=1}^{\infty} a_k < \infty$ and $\sum_{k=1}^{\infty} a_k(1 - \mu_k) > \frac{1}{e}$, then all solutions of this equation are oscillatory.

For a general autonomous equation we will obtain well known criteria of non-oscillation. To this end consider the equation

$$\dot{x}(t) + \sum_{k=1}^{\infty} a_k x(t - \tau_k) = 0. \tag{27}$$

Theorem 10. *Suppose* $a_k \geq 0, \tau_k > 0, \sum_{k=1}^{\infty} a_k < \infty$ *and* $\lim_{t\to\infty} \tau_k = \infty$*. Equation* (27) *has a non-oscillatory solution if and only if its characteristic equation*

$$\lambda = \sum_{k=1}^{\infty} a_k \exp\{\lambda \tau_k\} \tag{28}$$

has a positive solution.

Proof. Suppose equation (28) has no positive solution. Then

$$\inf_{\lambda>0} \left\{ \frac{1}{\lambda} \sum_{k=1}^{\infty} a_k \exp\{\lambda \tau_k\} \right\} > 1.$$

Theorem 9 implies that all the solutions of equation (27) are oscillatory.

It is easy to see that if in assumptions of Theorem 10 $\sum_{k=1}^{\infty} a_k \tau_k > \frac{1}{e}$, then all solutions of equation (27) are oscillatory.

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