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On Summands of Closed Bounded Convex Sets

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Abstract. In this paper properties of the Minkowski-Pontryagin subtraction of closed bounded convex sets are investigated (see Propositions 1 - 3) and four criteria for summands of closed bounded convex sets are given (see Theorems 1 - 4).

Keywords: *Minkowski sum, summands of convex sets*

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Let $X = (X, \tau)$ be a Hausdorff topological vector space, and let $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$) be the family of all non-empty, closed and bounded (resp. compact) convex subsets of X. If $A, B \in \mathcal{B}(X)$, then let $A \dot{+} B = \overline{A + B}$, where $\overline{C}$ denotes the closure of C and $C = A + B$ is the usual algebraic Minkowski sum of A and B.

The family $\mathcal{B}(X)$ plays an important role in multi-valued analysis. The algebraic structure of $\mathcal{B}(X)$ is far from being completely clarified. The family $\mathcal{B}(X)$ satisfies the order cancellation law, i.e. for $A, B, C \in \mathcal{B}(X)$ the inclusion $A \dot{+} B \subset B \dot{+} C$ implies $A \subset C$ (see [7]). The commutative semigroup $(\mathcal{B}(X), \dot{+})$ satisfies the law of cancellation.

The lattice of quotient classes in $\mathcal{B}(X) \times \mathcal{B}(X)$ (or $\mathcal{K}(X) \times \mathcal{K}(X)$) forms a vector space that was studied by Rådström and Hörmander. The lattice found an important application in quasidifferential calculus.

For $A \in \mathcal{B}(X)$, by $\operatorname{ext}(A)$ we denote the set of A's extremal points and by $\exp(A)$ the set of its exposed points. A set $B \in \mathcal{B}(X)$ is called a *summand* of $A \in \mathcal{B}(X)$ if there exists $C \in \mathcal{B}(X)$ such that $B \dot{+} C = A$. If $A, B \in \mathcal{B}(X)$, then let $A \dot{-} B = \{x \in X : x + B \subset A\}$ be the Minkowski-Pontryagin subtraction of A and B.

Summands of compact convex sets were studied by Schneider, Shephard, Weil and others (see, e.g., [5, 6, 10]). These summands found applications in the study of minimal representatives of quotient classes in $\mathcal{B}(X)^2$ and $\mathcal{K}(X)^2$. The Minkowski-Pontryagin subtraction was investigated for compact convex sets of finite-dimensional spaces.

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Lemma 1. *Let X be a topological vector space and let $A, B \in \mathcal{B}(X)$. Then*

$$A \dot{-} B = \bigcap_{b \in B} (A - b).$$

Proof. The proof is easy and we omit it ∎

From Lemma 1 it follows that if $A, B \in \mathcal{B}(X)$ and $A \dot{-} B \neq \emptyset$, then $A \dot{-} B \in \mathcal{B}(X)$. Indeed, $A \dot{-} B$ is an intersection of members of $\mathcal{B}(X)$.

Now, we prove several algebraic properties of the Minkowski-Pontryagin subtraction.

Proposition 1. *Let X be a Hausdorff topological vector space and let $A, B, C \in \mathcal{B}(X)$ and $0 < \alpha, \beta \in \mathbb{R}$ with $\alpha \geq \beta$. Then:*

- **(i)** *If $A \dot{-} B \neq \emptyset$, then $(A \dot{-} B) \dot{+} B \subset A$.*
- **(ii)** *If $A = B \dot{+} C$, then $C = A \dot{-} B$.*
- **(iii)** $(A \dot{-} B) \dot{-} C = A \dot{-} (B \dot{+} C)$.
- **(iv)** $(A \dot{+} C) \dot{-} (B \dot{+} C) = A \dot{-} B$.
- **(v)** *If $B \dot{-} C \neq \emptyset$, then $(A \dot{+} C) \dot{-} B \subset A \dot{-} (B \dot{-} C)$.*
- **(vi)** *If $B \dot{-} C \neq \emptyset$, then $(A \dot{-} B) \dot{+} C \subset A \dot{-} (B \dot{-} C)$.*
- **(vii)** $(A \dot{-} B) \dot{+} (B \dot{-} C) \subset A \dot{-} C$.
- **(viii)** $\alpha(A \dot{-} B) = \alpha A \dot{-} \alpha B$.
- **(ix)** $\alpha A \dot{-} \beta A = (\alpha - \beta) A$.

Proof. Assertion (i) follows immediately from the definition of $A \dot{-} B$.

Assertion (ii): Let $A = B \dot{+} C$. From (i) we have $(A \dot{-} B) \dot{+} B \subset A = B \dot{+} C$. Hence by the order cancellation law $A \dot{-} B \subset C$. On the other hand, for every $c \in C$ we have $c + B \subset C + B \subset C \dot{+} B = A$, hence $c \in A \dot{-} B$. Therefore, $C \subset A \dot{-} B$.

Assertion (iii): Let $x \in (A \dot{-} B) \dot{-} C$. Then $x + C \subset A \dot{-} B$. Hence, by (i), $x + C \dot{+} B \subset (A \dot{-} B) \dot{+} B \subset A$. Therefore, $x \in A \dot{-} (B \dot{+} C)$ and $(A \dot{-} B) \dot{-} C \subset A \dot{-} (B \dot{+} C)$. To prove the inverse inclusion, take any $x \in A \dot{-} (B \dot{+} C)$. Then $x + B \dot{+} C \subset A$ and $(x + B \dot{+} C) \dot{-} B \subset A \dot{-} B$. But $(x + B \dot{+} C) \dot{-} B = [(x + C) \dot{+} B] \dot{-} B = x + C$ by (ii). Therefore, $x + C \subset A \dot{-} B$ and $x \in (A \dot{-} B) \dot{-} C$.

Assertion (iv): Applying (iii) and (ii) we have

$$(A \dot{+} C) \dot{-} (B \dot{+} C) = [(A \dot{+} C) \dot{-} C] \dot{-} B = A \dot{-} B.$$

Assertion (v): Let $x \in (A \dot{+} C) \dot{-} B$. Then $x + B \subset A \dot{+} C$. Hence

$$(x + B) \dot{-} C \subset (A \dot{+} C) \dot{-} C = A.$$

But

$$\begin{aligned}(x + B) \dot{-} C &= \{y : \quad y + C \subset x + B\} \\ &= \{y : y - x + C \subset B\} \\ &= \{v + x : v + C \subset B\} \\ &= x + \{v : v + C \subset B\} \\ &= x + B \dot{-} C.\end{aligned}$$

Thus $x + (B\dot{-}C) \subset A$ and $x \in A\dot{-}(B\dot{-}C)$.

Assertion (vi): Let $x \in A\dot{-}B$ and $y \in C$. Then

$$x + y + (B\dot{-}C) \subset x + C + (B\dot{-}C) \subset x + C\dotplus(B\dot{-}C) \subset x + B \subset A.$$

Hence $(A\dot{-}B) + C \subset A\dot{-}(B\dot{-}C)$. Since $A\dot{-}(B\dot{-}C)$ is closed, then $(A\dot{-}B)\dotplus C \subset A\dot{-}(B\dot{-}C)$.

Assertion (vii): Let $x \in A\dot{-}B$ and $y \in B\dot{-}C$. Then

$$x + y + C \subset x + (B\dot{-}C)\dotplus C \subset x + B \subset A.$$

Hence $A\dot{-}B + B\dot{-}C \subset A\dot{-}C$ and, consequently, also $(A\dot{-}B)\dotplus(B\dot{-}C) \subset A\dot{-}C$.

Assertion (viii): Let $x \in \alpha(A\dot{-}B)$. Then $\frac{x}{\alpha} \in A\dot{-}B$. Thus $\frac{x}{\alpha} + B \subset A$ and $x + \alpha B \subset \alpha A$. Therefore, $x \in \alpha A\dot{-}\alpha B$. On other hand, if $x \in \alpha A\dot{-}\alpha B$, then $x + \alpha B \subset \alpha A$ and $\frac{x}{\alpha} + B \subset A$. Thus $\frac{x}{\alpha} \in A\dot{-}B$. Therefore, $x \in \alpha(A\dot{-}B)$.

Assertion (ix): Since $\alpha \geq \beta$, we have $\alpha A = \beta A\dotplus(\alpha - \beta)A$. Therefore,

$$\alpha A\dot{-}\beta A = [\beta A\dotplus(\alpha - \beta)A]\dot{-}\beta A = (\alpha - \beta)A.$$

Thus the theorem is proved ∎

Corollary 1. *Let X be a Hausdorff topological vector space and $A, B \in \mathcal{B}(X)$. Then B is a summand of A if and only if $A \subset (A\dot{-}B)\dotplus B$.*

Proof. *Necessity*: Let $A = B\dotplus C$ for some $C \in \mathcal{B}(X)$. Then, by Proposition 1/(ii), $C = A\dot{-}B$, thus $A \subset B\dotplus(A\dot{-}B)$. *Sufficiency*: It follows immediately from Proposition 1/(i) ∎

Similarly as in [3] (see Theorem 3) we can prove the following lemma.

Lemma 2. *Let $A, B, F \in \mathcal{B}(X)$ and $A \cup B \in \mathcal{B}(X)$. If $A \cap B \subset (A\dotplus F) \cup (B\dotplus F)$, then $A \cap B \subset A\dotplus F$ or $A \cap B \subset B\dotplus F$.*

From Lemma 2 we get the following equality for Minkowski-Pontryagin subtraction:

Proposition 2. *Let $A, B, F \in \mathcal{B}(X)$. If $A \cup B$ is convex, then*

$$\big[(A\dotplus F) \cup (B\dotplus F)\big]\dot{-}A \cap B = \big[(A\dotplus F)\dot{-}A \cap B\big] \cup \big[(B\dotplus F)\dot{-}A \cap B\big].$$

Proposition 3. *Let X be a Hausdorff topological vector space and let $A, B \in \mathcal{B}(X)$. The inclusion $A \subset (A\dot{-}B) + B$ holds if and only if for every $a \in A$ there exists $b \in B$ such that $B - b \subset A - a$.*

Proof. *Necessity*: Let $A \subset (A\dot{-}B) + B$ and take any $a \in A$. There exists $x \in A\dot{-}B$ and $b \in B$ such that $a = x + b$. Since $x \in A\dot{-}B$, then $x + B \subset A$ and we obtain $B - b \subset A - a$.

Sufficiency: Take any $a \in A$. There exists $b \in B$ such that $B - b \subset A - a$, hence $a - b \in A\dot{-}B$. Therefore, $a = a - b + b \in (A\dot{-}B) + B$. Thus $A \subset (A\dot{-}B) + B$ ∎

Theorem 1. *Let X be a Hausdorff topological vector space and let $A, B \in \mathcal{B}(X)$. Let $\mathcal{U}$ denote a basis of neighbourhoods of 0 in X. Then B is a summand of A if and only if the condition*

$$\forall_{a\in A}\ \forall_{V\in\mathcal{U}}\ \exists_{b\in B}\ \exists_{v\in V}: \quad B - b - v \subset A - a$$

holds true.

Proof. *Necessity*: Suppose that $A \subset (A \dot{-} B)\dot{+}B$. Then

$$A \subset (A\dot{-}B)\dot{+}B = \bigcap_{V\in\mathcal{U}} \big[(A\dot{-}B) + B + V\big],$$

hence $A \subset (A\dot{-}B) + B + V$ for every $V \in \mathcal{U}$. Take any $a \in A$ and $V \in \mathcal{U}$. Then from the above inclusion it follows that there exist $x \in A\dot{-}B, b \in B, v \in V$ such that $a = x + b + v$, hence $x = a - b - v$. Since $x \in A\dot{-}B$, $x + B \subset A$. Therefore, $B - b - v \subset A - a$.

Sufficiency: Take any $a \in A$ and $V \in \mathcal{U}$. Then there exist $b \in B$ and $v \in V$ such that $a - b - v + B \subset A$ and $a - b - v \in A\dot{-}B$. Thus

$$a = a - b - v + b + v \in (A\dot{-}B) + B + V$$

for every $V \in \mathcal{U}$. Hence

$$a \in \bigcap_{V\in\mathcal{U}} \big[(A\dot{-}B) + B + V\big] = (A\dot{-}B)\dot{+}B.$$

Therefore, $A \subset (A\dot{-}B)\dot{+}B$ and B is a summand of A ∎

Remark. From Proposition 3 and Theorem 1 it follows that a summand of $A \in \mathcal{B}(X)$ does not need to satisfy the condition contained in Proposition 3. However, it has to satisfy this condition if we replace $\mathcal{B}(X)$ by $\mathcal{K}(X)$.

Indeed, if for example $X = c_0$ (the usual space of sequences coverging to zero), then there exist $B, C \in \mathcal{B}(X)$ such that $B + C \neq B\dot{+}C$ (see [2]). Let $A = B\dot{+}C$. Then B is a summand of A, and $C = A\dot{-}B$. Suppose that B satisfy the condition contained in Proposiotion 3. Then

$$B\dot{+}C = B\dot{+}(A\dot{-}B) = A \subset B + A\dot{-}B = B + C.$$

This is imposible because $B + C \neq B\dot{+}C$.

Now we give some criteria for being summand in terms of external and exposed points. We start with three short lemmas.

Lemma 3. *Let X be a vector space and let $a + b \in \operatorname{ext}(A + B)$ with $a \in A$ and $b \in B$. Then $a \in \operatorname{ext}(A)$ and $b \in \operatorname{ext}(B)$.*

Proof. Suppose there exist $a_1, a_2 \in A$ such that $a = \alpha a_1 + \beta a_2$ for $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then $a + b = \alpha a_1 + \beta a_2 + b = \alpha(a_1 + b) + \beta(a_2 + b)$, hence $a_1 = a$ and $a_2 = a$. Therefore, $a \in \operatorname{ext}(A)$. In the same way we can prove that $b \in \operatorname{ext}(B)$ ∎

Lemma 4. *Let X be a vector space and let $a+b \in \operatorname{ext}(A+B)$ with $a \in A$ and $b \in B$, where A and B are convex subsets of X. If for some $a_1 \in A$ and $b_1 \in B$ the equality $a+b = a_1+b_1$ holds true, then $a = a_1$ and $b = b_1$.*

Proof. Let $\alpha, \beta > 0$ with $\alpha+\beta = 1$. Then, by assumption,

$$a+b = \alpha(a+b)+\beta(a_1+b_1) = (\alpha a+\beta a_1)+(\alpha b+\beta b_1).$$

Since A and B are convex, then $\alpha a+\beta a_1 \in A$ and $\alpha b+\beta b_1 \in B$. Hence, by Lemma 3, $\alpha a+\beta a_1 \in \operatorname{ext}(A)$ and $\alpha b+\beta b_1 \in \operatorname{ext}(B)$. Thus $a = a_1$ and $b = b_1$ ∎

Lemma 5. *Let X be a Hausdorff topological vector space and let $A, B \in \mathcal{B}(X)$. If $a+b \in \operatorname{ext}(A+B)$ with $a \in A, b \in B$ and $a-b \in A \dot{-} B$, then $a-b \in \operatorname{ext}(A \dot{-} B)$.*

Proof. Suppose that $a-b = \alpha x+\beta y$, where $x, y \in A \dot{-} B$ and $\alpha, \beta > 0$ with $\alpha+\beta = 1$. Since $x+B \subset A$ and $y+B \subset A$, there exist $a_1, a_2 \in A$ such that $x+b = a_1$ and $y+b = a_2$. Hence

$$\alpha a_1+\beta a_2 = \alpha x+\beta y+b = a-b+b = a.$$

But from Lemma 3 we have that $a \in \operatorname{ext}(A)$. Thus $a = a_1 = a_2$. Therefore, $x = y$ and $a-b \in \operatorname{ext}(A \dot{-} B)$ ∎

Theorem 2. *Let X be a locally convex Hausdorff topological vector space and let $A, B \in \mathcal{K}(X)$. Then B is a summand of A if and only if for every sum $a+b \in \operatorname{ext}(A+B)$ with $a \in A$ and $b \in B$ we have $a-b \in \operatorname{ext}(A \dot{-} B)$.*

Proof. *Necessity*: Let us assume that B is a summand of A. Since A and B are compact, then $A = B+(A \dot{-} B)$. Now by Proposition 3 there exists $b_0 \in B$ such that $B-b_0 \subset A-a$. Hence there exists $a_0 \in A$ such that $b-b_0 = a_0-a$. Hence $a+b = a_0+b_0$. Thus, by Lemma 4, $a = a_0$ and $b = b_0$. Therefore, $a-b \in A \dot{-} B$ and, by Lemma 5, $a-b \in \operatorname{ext}(A \dot{-} B)$.

Sufficiency: Suppose that for every sum $a+b \in \operatorname{ext}(A+B)$ with $a \in A$ and $b \in B$ we have $a-b \in \operatorname{ext}(A \dot{-} B)$. Then

$$a+b = a-b+b+b \in (A \dot{-} B)+B+B.$$

Hence

$$\operatorname{ext}(A+B) \subset (A \dot{-} B)+B+B.$$

Thus, by the Krein-Milman theorem,

$$A+B = \overline{\operatorname{conv}}(\operatorname{ext}(A+B)) \subset \overline{\operatorname{conv}}((A \dot{-} B)+B+B) = (A \dot{-} B)+B+B.$$

Now, by the order cancellation law, $A \subset (A \dot{-} B)+B$. Thus B is a summand of A ∎

Corollary 2. *Let X be a locally convex Hausdorff topological vector space. A set $B \in \mathcal{K}(X)$ is summand of a set $A \in \mathcal{K}(X)$ if and only if for every sum $a+b \in \operatorname{ext}(A+B)$ with $a \in A$ and $b \in B$ we have $a-b \in A \dot{-} B$.*

Proof. *Necessity* follows immediately from Theorem 2, *sufficiency* from Lemma 5 and Theorem 2 ∎

Theorem 3. *Let X be a real Banach space and let $A, B \in \mathcal{K}(X)$. Then B is a summand of A if and only if for every sum $a+b \in \exp(A+B)$ with $a \in A$ and $b \in B$ we have $a - b \in A \dot{-} B$.*

Proof. Since $\exp(A+B) \subset \operatorname{ext}(A+B)$, *necessity* follows from Corollary 2. As to *sufficiency*, analogously as in Theorem 2 we have

$$\exp(A+B) \subset \operatorname{ext}(A+B) \subset (A \dot{-} B) + B + B.$$

Now, by the modification of Klee (see [4]) of the Krein-Milman theorem,

$$A + B \subset \overline{\operatorname{conv}}(\exp(A+B)) \subset \overline{\operatorname{conv}}((A \dot{-} B) + B + B) \subset (A \dot{-} B) + B + B.$$

Thus, by the order cancellation law, $A \subset (A \dot{-} B) + B$ ∎

Remark. If (X, τ) is a reflexive locally convex topological vector space, then $\mathcal{B}((X,\tau)) \subset \mathcal{K}((X,\tau^*))$ where τ^* denotes the weak topology in X. Thus, by the Krein-Milman theorem, $A \subset \overline{\operatorname{conv}^*}(\operatorname{ext}(A))$ for $A \in \mathcal{B}((X,\tau))$, where $\overline{\operatorname{conv}^*}(A)$ denotes the closed convex hull of A in (X, τ^*). But it is easy to observe that $\overline{\operatorname{conv}}(A) = \overline{\operatorname{conv}^*}(A)$. Therefore, $A = \overline{\operatorname{conv}}(\operatorname{ext}(A))$ for $A \in \mathcal{B}((X,\tau))$.

Applying this observation and Theorem 2 we get the following

Theorem 4. *Let X be a reflexive locally convex Hausdorff topological vector space and let $A, B \in \mathcal{B}(X)$. Then B is summand of A if and only if for every sum $a + b \in \operatorname{ext}(A+B)$ with $a \in A$ and $b \in B$ we have $a - b \in A \dot{-} B$.*

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