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A Unified Approach to Linear Differential Algebraic Equations and their Adjoints

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Abstract. Instead of a single matrix occurring in the standard setting, the leading term of the linear differential algebraic equation is composed of a pair of well matched matrices. An index notion is proposed for the equations. The coefficients are assumed to be continuous and only certain subspaces have to be continuously differentiable. The solvability of lower index problems is proved. The solution representations are based on the solutions of certain inherent regular ordinary differential equations that are uniquely determined by the problem data. The assumptions allow for a unified treatment of the original equation and its adjoint. Both equations have the same index and are solvable simultaneously. Their fundamental solution matrices satisfy a relation that generalizes the classical Lagrange identity.

Keywords: *Differential algebraic equations, adjoint equations, index, Lagrange identity*

AMS subject classification: 34A09, 34A30

1. Introduction

For the implicit regular linear ordinary differential equation

$$Ax' + Bx = 0 \tag{1.1}$$

with continuous coefficients $A, B : \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{C}^m)$, A non-singular, the adjoint equation reads

$$-(A^*y)' + B^*y = 0. \tag{1.2}$$

To obtain equation (1.2), equation (1.1) is transformed into the form $x' + A^{-1}Bx = 0$. Its adjoint is $-z' + B^*A^{-1*}z = 0$. Finally, we put $A^{-1*}z = y$. The standard theory mostly deals with *explicit* regular ordinary differential equations. For each solution pair of the explicit equations the Lagrange identity $z^*(t)x(t) \equiv z^*(t_0)x(t_0)$ is valid (see, e.g., [5]). In terms of equations (1.1) and (1.2), one obtains

$$y^*(t)A(t)x(t) \equiv y^*(t_0)A(t_0)x(t_0). \tag{1.3}$$

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This leads to the relation

$$\begin{aligned} Y(t) &= \{A(t_0)[X(t)]^{-1}[A(t)]^{-1}\}^* \\ &= \{[A(t)]^*\}^{-1}\{[X(t)]^{-1}\}^*[A(t_0)]^* \end{aligned} \tag{1.4}$$

which connects the fundamental solution matrices X and Y that are normalized at t_0. For equation (1.1), the natural solution regularity to be met is $x \in C^1(\mathcal{I})$ while for (1.2) a solution y has to be a continuous function with A^*y being differentiable (see, e.g., [6]).

In the case of differential algebraic equations, the leading coefficient A in equation (1.1) is singular everywhere on $\mathcal{I}$. The standard formulation (1.1) is incomplete until we fix a suitable solution regularity. In [9], a more precise reformulation

$$A(Px)' + (B - AP')x = 0 \tag{1.5}$$

is proposed instead of equation (1.1), where $P : \mathcal{I} \to L(\mathbb{C}^m)$ denotes an arbitrary C^1-projector function along the nullspace of A, say, $P = A^+A$. The continuous functions x with a continuously differentiable component Px form a natural solution space for equation (1.5), hence, for (1.1).

For differential algebraic equations (1.1), the adjoint is of form (1.2), too, as it was claimed first in [4]. However, (1.2) is not a standard form differential algebraic equation. The standard theory (index notions, etc., see [3]) does not apply unless we assume A to be smooth and change to $-A^*y' + (B^* - A^{*\prime})y = 0$ instead of (1.2). Similarly to the case of a non-singular coefficient A, there is no need for assuming A to be smooth. Certain subspaces, typically $\ker A$, may only be smooth (see [9, 12]). In [1] a solvability theorem for equation (1.2) is proved under the same conditions as used for (1.1) when the latter is an index-1 differential algebraic equation. Actually, a reformulation of the adjoint equation (1.2), namely equation

$$-P^*(A^*y)' + (B^* - P^{*\prime}A^*)y = 0, \tag{1.6}$$

stands behind the approach in [1]. One of the results of [1] establishes the connection between normalized maximal fundamental solution matrices of the index-1 equation (1.1) and its adjoint (1.2). With the notations of the present paper, the identity

$$Y(t) = [A(t)]_c^{*-}[X(t)]^{-*}[A(t_0)]^*$$

is stated. It generalizes formula (1.4). Here, $A(t), X(t)$ and $Y(t)$ are singular matrices and $[X(t)]^-$ and $[A(t)]_c^{*-}$ denote special reflexive generalized inverses.

An additional argument for reconsidering differential algebraic equations is delivered by optimal control. Both types of equations, (1.1) and (1.2) (or (1.5) and (1.6)), are coupled together into one large system (see [11]). The standard theory of differential algebraic equations does not apply to those mixed systems either.

In this paper we assign the term *linear differential algebraic equation* to the equation

$$A(Dx)' + Bx = q \tag{1.7}$$

with well matched continuous leading coefficients $A, D : \mathcal{I} \to L(\mathbb{C}^m)$. None of these coefficients has to be a projector. D is a kind of incidence matrix that figures out which derivatives are involved in fact. Only certain subspaces, typically $\operatorname{im} D$, are assumed to be spanned by C^1 functions. Moreover, the solution need not to be differentiable. A natural solution of equation (1.7) is a continuous function x having a continuously differentiable part Dx. We show that the new form brings more symmetry, transparency and beauty into the theory. In particular, the equation

$$-D^*(A^*y)' + B^*y = p \tag{1.8}$$

is of a similar structure as (1.7) and it proves to be the *adjoint* to (1.7). We focus on the simultaneous analysis of equations (1.7) and (1.8) under the lowest possible smoothness conditions. The results relying upon possible higher differentiability are out of the scope of this paper. Nevertheless, we touch the relation to some further index definitions that are applicable to smooth problems only. We stop at index 2. One could define equations with higher index (see [13]). As we shall show, each solution pair of the homogeneous versions of equations (1.7) and (1.8) satisfies the Lagrange identity

$$y^*(t)A(t)D(t)x(t) \equiv y^*(t_0)A(t_0)D(t_0)x(t_0).$$

In the case of equation (1.1), where A is non-singular, clearly, $D = I$. Formula (1.3) is a special case of the "new" Lagrange identity. In equations (1.5) and (1.6) we have $D = P$.

We begin our analysis with some basic notions in Section 2. The well matched matrices and the index will be defined there. In Section 3, solvability statements for index-1 and index-2 tractable equations are proved. The geometric solution spaces are described in detail. A uniquely determined *inherent regular ordinary differential equation* is shown to exist. The results on fundamental matrices are listed in Section 4. Section 5 deals with adjoint equations. We show that index-1 and index-2 tractability always appears simultaneously for the adjoint pairs. The section is accomplished with an explicit representation of the fundamental matrix for equation (1.8). In Section 6 we illustrate the role of the smoothness assumptions.

In this paper we concentrate on the presentation of basic results concerning the new index notion for linear differential algebraic equations that are appropriate for a unified treatment of the adjoint pairs. Some additional discussion on this material can be found in [2]. The proofs of theorems claimed in Section 4 are almost technical. Some claims presented and applied in Section 5 characterize the subspaces by their representations in different ways. For these proofs the interested reader is referred to [2]. Generalized inverses often occur in the paper. Additional material about them can be found, for example, in [13, 14].

2. Basic notions

We consider linear equations (1.7) with *continuous* matrix coefficients $A, D, B : \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{C}^m)$.

Definition 2.1. A *solution of equation* (1.7) is a continuous function $x : \mathcal{I} \to \mathbb{C}^m$ that has a continuously differentiable product Dx and satisfies (1.7) pointwise. Let

$$C^1_D := \{x \in C : Dx \in C^1\}$$

denote the respective function space.

The first argument for considering equations (1.7) and (1.8) simultaneously and nominating them to an adjoint pair is that a kind of Lagrange identity is valid. Namely, for each pair of solutions $x \in C^1_D,\ y \in C^1_{A^*}$ of the homogeneous equations (1.7), (1.8) $(p = q = 0)$,

$$y^*(t)A(t)D(t)x(t) = \text{const} \qquad (t \in \mathcal{I}) \tag{2.1}$$

holds. Indeed, $(y^*ADx)' = (D^*(A^*y)')^*x + y^*A(Dx)' = (B^*y)^*x - y^*Bx = 0$.

In order to obtain solvability and other statements, we assume that the leading term in (1.7) is properly formed so that condition C1 below be satisfied.

Condition C1: The decomposition

$$\ker A(t) \oplus \operatorname{im} D(t) = \mathbb{C}^m \qquad (t \in \mathcal{I}) \tag{2.2}$$

holds and there are functions $\eta_i \in C^1(\mathcal{I}, \mathbb{C}^m)\ \ (i = 1, \dots, m)$ spanning $\operatorname{im} D$ and $\ker A$ pointwise so that

$$\operatorname{im} D = \operatorname{span}\{\eta_1, \dots, \eta_r\} \qquad \text{and} \qquad \ker A = \operatorname{span}\{\eta_{r+1}, \dots, \eta_m\}.$$

If condition C1 is satisfied, the matrices $A(t)$ and $D(t)$ will be called *well matched.* Thus, the well matched matrices $A(t)$ and $D(t)$ are of constant rank r. Due to condition C1, there is a uniquely determined continuously differentiable matrix function $R : \mathcal{I} \to L(\mathbb{C}^m)$ such that

$$[R(t)]^2 = R(t), \quad \operatorname{im} R(t) = \operatorname{im} D(t), \quad \ker R(t) = \ker A(t) \qquad (t \in \mathcal{I}).$$

We say shortly that the ordered pair $\{A, D\}$ *provides a smooth* $\mathbb{C}^m$*-decomposition* if condition C1 is valid. The projector function R realizes this smooth decomposition.

Note. At this place it must be stressed that condition C1 is not a restriction but an extension with respect to the former assumptions on equation (1.5). The differential algebraic equation of standard form

$$\tilde{A}(t)x'(t) + \tilde{B}(t)x(t) = q(t), \tag{2.3}$$

where the leading coefficient has constant rank r, can be reformulated to yield equation (1.7) with $A = \tilde{A}$, $D = \tilde{P}$ and $B = \tilde{B} - \tilde{A}\tilde{P}'$ supposed that there is a C^1-projector P with $\ker \tilde{P} = \ker \tilde{A}$. Another problem setting is possible if $\tilde{A}$ itself is continuously differentiable. Namely, let $\tilde{R}$ be a C^1-projector with $\operatorname{im} \tilde{R} = \operatorname{im} \tilde{A}$, say $\tilde{R} = \tilde{A}\tilde{A}^+$. Set $A = \tilde{R}$, $D = \tilde{A}$ and $B = \tilde{B} - \tilde{R}\tilde{A}'$. The condition C1 is fulfilled in both cases.

The unified treatment of the leading terms in equations (1.7) and (1.8) is supported by the next assertion.

Lemma 2.1. *The matrix pair $\{A, D\}$ provides a smooth $\mathbb{C}^m$-decomposition if and only if the pair $\{D^*, A^*\}$ does so. If R realizes the decomposition for the pair $\{A, D\}$, then R^* does so for the pair $\{D^*, A^*\}$.*

Proof. Let condition C1 be true for $\{A, D\}$ and R be the projector realizing the decomposition. Since $R^{*2} = R^*$, R^* is also a projector;

$$\operatorname{im} R^* = \ker R^{\perp} = \ker A^{\perp} = \operatorname{im} A^*$$
$$\ker R^* = \operatorname{im} R^{\perp} = \operatorname{im} D^{\perp} = \ker D^*.$$

Thus, the decomposition $\mathbb{C}^m = \operatorname{im} R^* \oplus \ker R^* = \operatorname{im} A^* \oplus \ker D^*$ holds. Since R is continuously differentiable, so is R^*. Then $\operatorname{im} R^*$ and $\ker R^*$ are C^1-subspaces. Starting with condition C1 being valid for $\{D^*, A^*\}$ we apply similar arguments ∎

Due to the next simple claim, decomposition (2.2) may be replaced by an equivalent triple of conditions if needed.

Lemma 2.2. *For a pair of arbitrary matrices $H, J \in L(\mathbb{C}^m)$, $\ker H \oplus \operatorname{im} J = \mathbb{C}^m$ is true if and only if the relations*

$$\ker H \cap \operatorname{im} J = \{0\}, \quad \ker HJ = \ker J, \quad \operatorname{im} H = \operatorname{im} HJ$$

hold simultaneously.

We associate a chain of matrix functions and time-varying subspaces with equation (1.7) for utilization in this paper. The argument t is omitted everywhere.

$$\begin{aligned}
&G_0 := AD, \quad B_0 := B \\
&Q_i \text{ is a projector onto } \ker G_i, \quad P_i := I - Q_i \\
&W_i \text{ is a projector}, \quad \ker W_i = \operatorname{im} G_i \\
&G_{i+1} := G_i + B_i Q_i, \quad B_{i+1} := B_i P_i \\
&\mathbf{N}_i := \ker G_i = \operatorname{im} Q_i \\
&\mathbf{S}_i := \{z \in \mathbb{C}^m : B_i z \in \operatorname{im} G_i\} = \ker W_i B_i \qquad (i = 0, 1).
\end{aligned} \tag{2.4}$$

By construction,

$$\begin{aligned}
&\operatorname{im} G_0 \subseteq \operatorname{im} G_1, \quad \mathbf{S}_0 \subseteq \mathbf{S}_1, \quad \mathbf{N}_0 \subseteq \mathbf{S}_1, \\
&\mathbf{S}_1 = P_0 \mathbf{S}_1 \oplus \mathbf{N}_0, \quad \dim \mathbf{N}_1 = \dim(\mathbf{N}_0 \cap \mathbf{S}_0).
\end{aligned}$$

For given projectors Q_0 and W_0, we denote by D^- and A^- the reflexive generalized inverses of D and A such that

$$DD^- = R, \quad D^- D = P_0, \quad A^- A = R, \quad AA^- = I - W_0.$$

Thus, D^- and A^- are uniquely determined and depend only on the choice of P_0 and W_0. It is possible to choose projectors Q_0 and W_0 to be continuous and we will do so. Then, D^- and A^- are also continuous.

Now we fix one further assumption for certain subspaces of $\operatorname{im} D$.

Condition C2: Both subspaces $D\mathbf{S}_1$ and $D\mathbf{N}_1$ are spanned by continuously differentiable functions.

Definition 2.2. Let conditions C1 and C2 be valid. Equation (1.7) is said to be

1. an *index*-1 *tractable* differential algebraic equation if

$$\mathbf{N}_0(t) \cap \mathbf{S}_0(t) = \{0\} \qquad (t \in \mathcal{I}), \tag{2.5}$$

2. an *index*-2 *tractable* differential algebraic equation if

$$\dim \mathbf{N}_0(t) \cap \mathbf{S}_0(t) = \text{const} > 0 \tag{2.6}$$

$$\mathbf{N}_1(t) \cap \mathbf{S}_1(t) = \{0\} \quad (t \in \mathcal{I}). \tag{2.7}$$

Remark 2.1. The matrix chain used in [12] is a special case of (2.4). Thus, for equations of form (2.3) the reformulations mentioned in the above Note provide the same tractability-index μ ($\mu = 1, 2$) as the former definitions (see [1, 12]).

If all matrices are sufficiently smooth for defining the differentiation index of equation (2.3), then the tractability index μ ($\mu = 1, 2$) leads to the same differentiation index.

The solvability assertions in the next section will show that an index-μ tractable equation ($\mu = 1, 2$) also has perturbation index μ without the assumption that the coefficients or some of them be smooth. The continuity of the coefficients and a C^1-basis of certain subspaces will be sufficient.

If the coefficients A, D, B are time-invariant, it can be checked [8] that the matrix pencil $\{AD, B\}$ is regular with Kronecker index $\mu = 1$ or $\mu = 2$ if and only if relations (2.5) or (2.6) and (2.7), respectively, hold.

Decomposition (2.5) is valid if and only if G_1 is non-singular [9: Appendix A/Theorem 13]. In this case, $G_2 = G_1$. In the index-2 case, G_1 is singular and has constant rank, say, r_1. G_2 becomes non-singular. It also follows that the index definition is independent of the choice of Q_0 and Q_1 in the matrix chain (2.4).

Differential algebraic equations of index-2 are characterized by hidden constraints and a lower degree of freedom. The index-2 tractability ensures the decomposition $\mathbf{N}_1(t) \oplus \mathbf{S}_1(t) = \mathbb{C}^m$. In Subsections 3.1 and 3.2 we show that the subspace $D\mathbf{S}_1$ is relevant to the *inherent regular ordinary differential equation* (3.5) or to (3.17), while in the index-2 case, the non-empty set $D\mathbf{N}_1$ is responsible for the *hidden constraint* (3.18). This is the reason for the refinement of condition C1 in the form of condition C2.

Lemma 2.3. *For an index-*2 *tractable equation the decomposition*

$$D\mathbf{S}_1(t) \oplus D\mathbf{N}_1(t) \oplus \ker A(t) = \mathbb{C}^m \qquad (t \in \mathcal{I}) \tag{2.8}$$

holds. The dimensions of these subspaces are $r_1 + r - m$, $m - r_1$ *and* $m - r$, *respectively. If* $\hat{Q}_1(t)$ *is the special projector onto* $\mathbf{N}_1(t)$ *along* $\mathbf{S}_1(t)$ $(t \in \mathcal{I})$, *then*

$$D\hat{P}_1 D^-, \quad D\hat{Q}_1 D^-, \quad I - R \tag{2.9}$$

are continuously differentiable projectors that realize decomposition (2.8).

Proof. Choose $\hat{Q}_1(t)$ to be the projector onto $\mathbf{N}_1(t)$ along $\mathbf{S}_1(t)$. $\hat{Q}_1$ has a representation $\hat{Q}_1 = Q_1 G_2^{-1} BP_0$. Here Q_1 is an arbitrary projector onto $\mathbf{N}_1$ and G_2 is associated with Q_1 as (2.4) dictates. By construction, $\hat{Q}_1 = \hat{Q}_1 \hat{G}_2^{-1} BP_0$ [9: Lemma 14], where $\hat{G}_2 = G_1 + BP_0\hat{Q}_1$. Thus, $\hat{Q}_1 Q_0 = 0$; if $\hat{P}_1 = I - \hat{Q}_1$, then $P_0\hat{P}_1 Q_0$ vanishes. Therefore, formulas (2.9) define projectors and their pairwise products vanish, too. This results in $\operatorname{im} D\hat{P}_1 \subseteq D\mathbf{S}_1$ and $\operatorname{im} D\hat{Q}_1 \subseteq D\mathbf{N}_1$. Finally, because of the relation $P_0\hat{Q}_1 = (I - Q_0\hat{Q}_1)\hat{Q}_1$ where $(I - Q_0\hat{Q}_1)$ is non-singular, $P_0\mathbf{N}_1$ is of the same dimension $m - r_1$ as $\mathbf{N}_1$. Hence, $\dim D\mathbf{N}_1 = \dim P_0\mathbf{N}_1 = \dim \mathbf{N}_1 = m - r_1$. For reasons of dimensions, it follows that $\operatorname{im} D\hat{P}_1 D^- = D\mathbf{S}_1$ and $\operatorname{im} D\hat{Q}_1 D^- = D\mathbf{N}_1$.

It remains to show the continuous differentiability of the projectors. Let

$$\Gamma := \big(Ds_1, \ldots, Ds_{r_1-(m-r)}, Dn_1 \ldots, Dn_{m-r_1}, \eta_{r+1}, \ldots, \eta_m\big)$$

be the matrix function composed of continuously differentiable functions that span the subspaces $D\mathbf{S}_1$, $D\mathbf{N}_1$ and $\ker A$, respectively. Γ is non-singular. Set

$$\begin{aligned} I^{DS} &= \operatorname{diag}(I_{r_1+r-m}, 0_{m-r_1}, 0_{m-r}) \\ I^{DN} &= \operatorname{diag}(0_{r_1+r-m}, I_{m-r_1}, 0_{m-r}) \\ I^{A} &= \operatorname{diag}(0_{r_1+r-m}, 0_{m-r_1}, I_{m-r}). \end{aligned}$$

The indices show the dimensions of the unit and zero matrices. Obviously,

$$\Gamma I^{DS}\Gamma^{-1}, \quad \Gamma I^{DN}\Gamma^{-1}, \quad \Gamma I^{A}\Gamma^{-1} \tag{2.10}$$

are continuously differentiable projectors that are uniquely defined by the decomposition. Thus, they coincide with $D\hat{P}_1 D^-$, $D\hat{Q}_1 D^-$ and $I - R$, respectively. In particular, $D\hat{P}_1 D^-$ and $D\hat{Q}_1 D^-$ are continuously differentiable ∎

3. Solvability of index-1 and index-2 tractable differential algebraic equations

3.1 Initial value problem for index-1 equations. Let us rewrite the index-1 equation (1.7) as

$$\underbrace{(AD + BQ_0)}_{G_1}\{D^-(Dx)' + Q_0 x\} + BP_0 x = q.$$

The inverse of G_1 exists and scaling by G_1^{-1} leads to

$$D^-(Dx)' + Q_0 x + G_1^{-1} BP_0 x = G_1^{-1} q. \tag{3.1}$$

Multiplication by P_0 and Q_0 decouples equation (3.1) into

$$R(Dx)' + DG_1^{-1}BP_0x = DG_1^{-1}q \tag{3.2}$$
$$Q_0x + Q_0G_1^{-1}BP_0x = Q_0G_1^{-1}q. \tag{3.3}$$

Thus, a solution $x \in C_D^1$, if it exists, may be represented as

$$x = P_0x + Q_0x = D^-Dx + Q_0x = (I - Q_0G_1^{-1}B)D^-Dx + Q_0G_1^{-1}q$$

where Dx satisfies the regular ordinary differential equation

$$(Dx)' - R'Dx + DG_1^{-1}BD^-Dx = DG_1^{-1}q. \tag{3.4}$$

The latter equation is an equivalent of (3.2) since $RD = D$. If the equation

$$u' - R'u + DG_1^{-1}BD^-u = DG_1^{-1}q \tag{3.5}$$

is multiplied by $(I - R)$, we obtain $((I - R)u)' - (I - R)'(I - R)u = 0$. Hence, any solution u of equation (3.5) satisfies $u = Ru$ if $u(\tilde{t}) = R(\tilde{t})u(\tilde{t})$ holds at some $\tilde{t} \in \mathcal{I}$. In other words, $\operatorname{im} R = \operatorname{im} D$ represents a (time varying) invariant subspace for (3.5).

Definition 3.1. Equation (3.5) is called *the inherent regular ordinary differential equation* of the index-1 equation (1.7).

For purposes of the next theorem we recall that the index-1 property $\mathbf{N}_0 \cap \mathbf{S}_0 = \{0\}$ is equivalent to $\mathbf{N}_0 \oplus \mathbf{S}_0 = \mathbb{C}^m$ [9: Appendix A/Theorem 13]. This allows considering a special (canonical) projector P_{0c} onto $\mathbf{S}_0$ along $\mathbf{N}_0$. If P_0 is an arbitrary but fixed projector along $\mathbf{N}_0$, a possible representation is

$$P_{0c} = I - Q_0G_1^{-1}B \tag{3.6}$$

and, clearly, P_{0c} is continuous.

Theorem 3.1. *Let equation* (1.7) *be index*-1 *tractable.*

1. *For each $q \in C(\mathcal{I}, \mathbb{C}^m)$, $d \in \operatorname{im} D(t_0)$, $t_0 \in \mathcal{I}$, the initial value problem*

$$A(Dx)' + Bx = q, \quad D(t_0)x(t_0) = d \tag{3.7}$$

is uniquely solvable in C_D^1.

2. *Equation* (1.7) *has perturbation index* 1.

3. *Exactly one solution of the homogeneous equation passes through each pair (t_0, x_0), $t_0 \in \mathcal{I}$, $x_0 \in \mathbf{S}_0(t_0)$.*

Proof.

1. First we find the uniquely determined solution $u \in C^1$ of the inherent regular ordinary differential equation with the initial condition $u(t_0) = d$. Then we construct the function

$$x := P_{0c}D^-u + Q_0G_1^{-1}q \in C. \tag{3.8}$$

Observe that $Dx = DP_{0c}D^- u = DD^- u = Ru = u \in C^1$ and $D(t_0)x(t_0) = u(t_0) = d$. Now, the decoupling (3.2)-(3.3) shows that x satisfies the differential algebraic equation.

Assume that $\tilde{x} \in C^1_D$ is also a solution of problem (3.7) different from the solution x constructed above. Then, $\hat{x} = \tilde{x} - x$ satisfies (3.7) with $q = 0$ and $d = 0$. Due to equation (3.4), we obtain $D\hat{x} = 0$. Thus, $P_0\hat{x} = D^- D\hat{x} = 0$. Equation (3.3) turns into the relation $Q_0\hat{x} = 0$, i.e., $\tilde{x} = x + \hat{x} = x + P_0\hat{x} + Q_0\hat{x} = 0$ in contrast to the assumption.

2. Let $\mathcal{I}$ be a compact interval. Let us compare the solution x_q of problem (3.7) and the solution $x \in C^1_D$ of the homogeneous equation with the same initial condition. For the corresponding solutions u_q and u of the inherent regular ordinary differential equations, the inequality

$$\|u_q - u\|_\infty \leq K_1 \|DG_1^{-1} q\|_\infty$$

holds with some constant K_1, hence,

$$\|x_q - x\|_\infty = \|P_{0c}D^-(u_q - u) + Q_0 G_1^{-1} q\|_\infty \leq K_2 \|q\|_\infty$$

holds with some constant K_2.

3. $x_0 \in \mathbf{S}(t_0)$ means that $x_0 = P_{0c}(t_0)x_0 = P_{0c}(t_0)D^-(t_0)D(t_0)x_0$. The solution of the homogeneous equation with the initial condition $D(t_0)x(t_0) = D(t_0)x_0$ is $x = P_{0c}D^- u$, therefore,

$$x(t_0) = P_{0c}(t_0)D^-(t_0)u(t_0) = P_{0c}(t_0)D^-(t_0)D(t_0)x_0 = x_0,$$

and the theorem is proved ∎

Remark 3.1. The initial condition $D(t_0)x(t_0) = d \in \operatorname{im} D(t_0)$ can be replaced by

$$D(t_0)x(t_0) = D(t_0)x^0 \qquad (x^0 \in \mathbb{C}^m). \tag{3.9}$$

This choice has the advantage that x^0 remains in $\mathbb{C}^m$. In particular, the variational equation for $X := x'_{x_0} \in C^1_D(\mathcal{I}, L(C^m))$ takes the form

$$A(DX)' + BX = 0, \quad D(t_0)(X(t_0) - I) = 0.$$

This matrix problem will be addressed in the next section. However, when the initial condition (3.9) is set, one has to take into account that, in general, $x(t_0) \neq x^0$ has to be expected. The coincidence $x(t_0) = x^0$ holds only if x^0 is consistent, i.e., if $Q_{0c}x_0 = Q_0 G_1^{-1} q(t_0)$.

Remark 3.2. One might think that the inherent regular ordinary differential equation (3.5) depends on how the projector P_0 and, consequently, D^- are chosen. On the contrary, the terms DG_1^{-1} and $DG_1^{-1}BD^-$ remain invariant when changing P_0.

Indeed, let $P_0, \tilde{P}_0$ be two projectors along $\mathbf{N}_0$, while D^- and $\tilde{D}^-$ be the respective generalized inverses of D, $G_1 = AD + BQ_0$, $\tilde{G}_1 = AD + B\tilde{Q}_0$. Since $\tilde{Q}_0 = Q_0\tilde{Q}_0$,

$\tilde{G}_1 = AD + BQ_0\tilde{Q}_0 = G_1(P_0 + \tilde{Q}_0)$ and $\tilde{G}_1^{-1} = (\tilde{P}_0 + Q_0)G_1^{-1}$ hold, the terms $D\tilde{G}_1^{-1}$ and DG_1^{-1} are identical. Further,

$$\begin{aligned} D\tilde{G}_1^{-1}B\tilde{D}^- &= D\tilde{G}_1^{-1}B\tilde{D}^-DD^- = D\tilde{G}_1^{-1}B\tilde{D}^-R = D\tilde{G}_1^{-1}B\tilde{D}^-DD^- \\ &= D\tilde{G}_1^{-1}B\tilde{P}_0D^- = D\tilde{G}_1^{-1}BD^- = DG_1^{-1}BD^- \end{aligned}$$

since $\tilde{G}_1^{-1}B\tilde{Q}_0 = \tilde{Q}_0$ holds. We can also prove the identities

$$\begin{aligned} &\tilde{Q}_0\tilde{G}_1^{-1} = \tilde{Q}_0(\tilde{P}_0 + Q_0)G_1^{-1} = Q_0G_1^{-1} \\ &P_{0c}\tilde{D}^- = P_{0c}\tilde{D}^-D\tilde{D}^- = P_{0c}\tilde{D}^-DD^- = P_{0c}\tilde{P}_0D^- = P_{0c}D^-. \end{aligned}$$

In the latter one we used that P_{0c} is defined geometrically and thus, that it is independent of P_0 ($\tilde{P}_0$). Hence, in addition to the uniqueness of the solution to problem (3.7), we conclude that each term in splitting (3.8) is independent of the choice of P_0 (Q_0).

3.2 Initial value problem for index-2 equations. In the index-2 case we scale equation (1.7) by $\hat{G}_2^{-1}$. Noting that $A = ADD^-$ and $\hat{Q}_1 = \hat{Q}_1\hat{G}_2^{-1}BP_0$, the identities

$$\hat{G}_2^{-1}AD = \hat{P}_1P_0, \quad \hat{G}_2^{-1}A = \hat{P}_1P_0D^-, \quad \hat{G}_2^{-1}B = \hat{G}_2^{-1}BP_0\hat{P}_1 + \hat{Q}_1 + Q_0$$

emerge from construction. The scaled equation reads

$$\hat{P}_1P_0D^-(Dx)' + Q_0x + \hat{Q}_1x + \hat{G}_2^{-1}BP_0\hat{P}_1x = \hat{G}_2^{-1}q. \tag{3.10}$$

If multiplied by $\hat{Q}_1, P_0\hat{P}_1$ and $Q_0\hat{P}_1$, equation (3.10) decouples into three parts:

$$\hat{Q}_1x = \hat{Q}_1\hat{G}_2^{-1}q \tag{3.11}$$

$$P_0\hat{P}_1D^-(Dx)' + P_0\hat{P}_1\hat{G}_2^{-1}BP_0\hat{P}_1x = P_0\hat{P}_1\hat{G}_2^{-1}q \tag{3.12}$$

$$-Q_0\hat{Q}_1D^-(Dx)' + Q_0x + Q_0\hat{P}_1\hat{G}_2^{-1}BP_0\hat{P}_1x = Q_0\hat{P}_1\hat{G}_2^{-1}q. \tag{3.13}$$

An additional multiplication of both (3.11) and (3.12) by D yields the couple

$$D\hat{Q}_1x = D\hat{Q}_1\hat{G}_2^{-1}q \tag{3.14}$$

$$(D\hat{P}_1x)' - (D\hat{P}_1D^-)'Dx + D\hat{P}_1\hat{G}_2^{-1}BD^-D\hat{P}_1x = D\hat{P}_1\hat{G}_2^{-1}q. \tag{3.15}$$

Consequently, each solution $x \in C_D^1$ of equation (1.7) has to satisfy (3.13) - (3.15). Recalling Lemma 2.3 we state that $D\hat{Q}_1\hat{G}_2^{-1}q = D\hat{Q}_1x = D\hat{Q}_1D^-Dx \in C^1$. Hence,

$$\begin{aligned} Q_0\hat{Q}_1D^-(Dx)' &= Q_0\hat{Q}_1D^-[(D\hat{P}_1x)' + (D\hat{Q}_1\hat{G}_2^{-1}q)'] \\ &= Q_0\hat{Q}_1D^-D\hat{Q}_1D^-(D\hat{P}_1x)' + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)' \\ &= -Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D\hat{P}_1x + Q_0\hat{Q}_1D^-(D\hat{Q}_1\hat{G}_2^{-1}q)'. \end{aligned}$$

Inserting this expression into equation (3.13) leads to

$$\begin{aligned} Q_0 x = &- Q_0 \hat{P}_1 \hat{G}_2^{-1} B D^- D \hat{P}_1 x - Q_0 \hat{Q}_1 D^- (D \hat{Q}_1 D^-)' D \hat{P}_1 x \\ &+ Q_0 \hat{Q}_1 D^- (D \hat{Q}_1 \hat{G}_2^{-1} q)' + Q_0 \hat{P}_1 \hat{G}_2^{-1} q. \end{aligned}$$

By combination of the above expressions each solution $x \in C_D^1$ of equation (1.7) may be represented as

$$\begin{aligned} x &= D^- D \hat{P}_1 x + D^- D \hat{Q}_1 x + Q_0 x \\ &= K D^- D \hat{P}_1 x + D^- D \hat{Q}_1 \hat{G}_2^{-1} q + Q_0 \hat{P}_1 \hat{G}_2^{-1} q + Q_0 \hat{Q}_1 D^- (D \hat{Q}_1 \hat{G}_2^{-1} q)' \end{aligned} \tag{3.16}$$

where the component $D\hat{P}_1 x$ satisfies the regular ordinary differential equation

$$\begin{aligned} (D\hat{P}_1 x)' - (D\hat{P}_1 D^-)' D\hat{P}_1 x + D\hat{P}_1 \hat{G}_2^{-1} B D^- D \hat{P}_1 x \\ = D\hat{P}_1 \hat{G}_2^{-1} q + (D\hat{P}_1 D^-)' D \hat{Q}_1 \hat{G}_2^{-1} q \end{aligned}$$

and

$$K := I - Q_0 \hat{P}_1 \hat{G}_2^{-1} B P_0 - Q_0 \hat{Q}_1 D^- (D\hat{Q}_1 D^-)' D$$

is a non-singular matrix function; $K^{-1} = I + Q_0 \hat{P}_1 \hat{G}_2^{-1} B P_0 + Q_0 \hat{Q}_1 D^- (D \hat{Q}_1 D^-)' D$. The solution representation gives the idea what is the so-called *inherent regular ordinary differential equation* of (1.7) like.

Definition 3.2. The equation

$$u' - (D\hat{P}_1 D^-)' u + D\hat{P}_1 \hat{G}_2^{-1} B D^- u = D\hat{P}_1 \hat{G}_2^{-1} q + (D\hat{P}_1 D^-)' D\hat{Q}_1 \hat{G}_2^{-1} q \tag{3.17}$$

is called the *inherent regular ordinary differential equation* of the index-2 equation (1.7).

Multiplying equation (3.17) by $(I - D\hat{P}_1 D^-)$ we can check that $D\mathbf{S}_1 = \operatorname{im} D\hat{P}_1$ is an invariant subspace of the inherent regular ordinary differential equation (3.17).

Remark 3.2 has shown the geometric origin of the inherent regular ordinary differential equation for index-1 equations. The next remark will do the same for the index-2 case.

Remark 3.3. Direct computations show that none of the terms $D\hat{Q}_1 D^-$, $D\hat{P}_1 D^-$, $D\hat{Q}_1 \hat{G}_2^{-1}$, $D\hat{P}_1 \hat{G}_2^{-1}$ or $D\hat{P}_1 \hat{G}_2^{-1} B D^-$ depends on the choice of P_0. Thus, the inherent regular ordinary differential equation does not depend on the choice of P_0.

Let P_0 be fixed. It turns out that the expression $Q_1 G_2^{-1}$ is independent of the choice of P_1 (Q_1), too. Combining with the above, $DQ_1 G_2^{-1}$ becomes independent of the specific choices of both P_0 and P_1 and so does the function space $C^1_{DQ_1 G_2^{-1}} := \{x \in C : DQ_1 G_2^{-1} x \in C^1\}$. They appear in the hidden constraint equation and in the solvability theorem.

Looking for the hidden constraint we multiply equation (1.7) by $D\hat{Q}_1 D^- A^-$:

$$(D\hat{Q}_1 x)' - (D\hat{Q}_1 D^-)' D x + D\hat{Q}_1 D^- A^- B x = D\hat{Q}_1 D^- A^- q.$$

Taking equation (3.14) into account, we obtain the hidden constraint equation

$$(D\hat{Q}_1 G_2^{-1} q)' - (D\hat{Q}_1 D^-)' Dx + D\hat{Q}_1 D^- A^- Bx = D\hat{Q}_1 D^- A^- q. \tag{3.18}$$

We stress that equation (3.18) is independent of the specific choice of projectors P_0 (Q_0) and W_0. Indeed, due to equations (3.14) and (3.15), $Dx = D\hat{P}_1 x + D\hat{Q}_1 x$ is independent and $A^-(Bx - q) = -R(Dx)' = \tilde{A}^-(Bx - q)$ also holds if $A\tilde{A}^- = I - \tilde{W}_0$.

Returning to (3.16), we observe that

$$\Pi_{\text{can }2} := KD^- D\hat{P}_1 = KP_0\hat{P}_1 \tag{3.19}$$

is again a projector with $\ker \Pi_{\text{can}\,2} = \ker D^- D\hat{P}_1 = \mathbf{N}_1 \oplus \mathbf{N}_0 = P_0\mathbf{N}_1 \oplus \mathbf{N}_0$ and

$$D\Pi_{\text{can }2} = D\hat{P}_1, \qquad \operatorname{im} D\Pi_{\text{can }2} = D\mathbf{S}_1.$$

In the next theorem we prove that the geometric space $\mathbb{S}_{\text{ind }2}$ containing all solutions of the homogeneous equation is exactly the image of $\Pi_{\text{can }2}$, i. e., $\mathbb{S}_{\text{ind}\,2} = \operatorname{im} \Pi_{\text{can }2}$. For the index-1 tractable equations we proved that $\mathbb{S}_{\text{ind}\,1} := \mathbf{S}_0$ is the respective geometric solution space and $\Pi_{\text{can }1} := P_{0c} = (I - Q_0 G_1^{-1} BP_0)P_0$ projects onto $\mathbb{S}_{\text{ind}\,1}$. For an index-2 equation the geometric solution space is of lower dimension; $\mathbb{S}_{\text{ind}\,2} \subset \mathbb{S}_{\text{ind}\,1}$.

Theorem 3.2. *Let equation* (1.7) *be index-*2 *tractable.*

1. *For each $q \in C^1_{DQ_1G_2^{-1}}$, $d \in D(t_0)\mathbf{S}_1(t_0)$, $t_0 \in \mathcal{I}$, the initial value problem*

$$A(Dx)' + Bx = q, \quad D(t_0)\hat{P}_1(t_0)x(t_0) = d \tag{3.20}$$

is uniquely solvable in C^1_D.

2. *Equation* (1.7) *has perturbation index* 2*.*

3. *Exactly one solution of the homogeneous equation passes through each pair (t_0, x_0), $t_0 \in \mathcal{I}$, $x_0 \in \mathbb{S}_{\text{ind}\,2}(t_0)$.*

Proof.

1. First we solve the initial value problem for the inherent regular ordinary differential equation (3.17) with the initial condition $u(t_0) = d$. $D\mathbf{S}_1$ is an invariant subspace for this equation; $u = D\hat{P}_1 D^- u$. With this u the continuous function

$$x := KD^- u + D^- D\hat{Q}_1 \hat{G}_2^{-1} q + Q_0 \hat{P}_1 \hat{G}_2^{-1} q + Q_0 \hat{Q}_1 D^- (D\hat{Q}_1 \hat{G}_2^{-1} q)' \tag{3.21}$$

satisfies equation (1.7), since $Dx = Ru + DQ_1 G_2^{-1} q \in C^1$. Further,

$$\begin{aligned} D(t_0)\hat{P}_1(t_0)x(t_0) &= D(t_0)\hat{P}_1(t_0)[D(t_0)]^- D(t_0)x(t_0) \\ &= D(t_0)\hat{P}_1(t_0)[D(t_0)]^- u(t_0) \\ &= d. \end{aligned}$$

Assume that $\tilde{x} \in C^1_D$ is also a solution of problem (3.20) different from (3.21). Then $\hat{x} = \tilde{x} - x$ satisfies equation (3.20) with $q = 0$ and $d = 0$. Equation (3.16) yields $\hat{x} = KD^-D\hat{P}_1\hat{x}$. The solution $D\hat{P}_1\hat{x}$ of the regular ordinary differential equation vanishes identically, i. e., $\hat{x} = 0$, in contrast to the assumption. Note that, in problem formulation (3.20), the sign $\hat{}$ cannot be removed.

2. Let $\mathcal{I}$ be a compact interval. Let us compare the solutions x and x_q of the homogeneous and the inhomogeneous equations supplied with the same initial condition, i.e., $D(t_0)P_1(t_0)(x(t_0) - x_q(t_0)) = 0$. With some constants K_1, K_2 the inequality

$$\|u_q - u\|_\infty \le K_1 \|D\hat{P}_1\hat{G}_2^{-1}q + (D\hat{P}_1D^-)'D\hat{Q}_1\hat{G}_2^{-1}q\|_\infty \le K_2\|q\|_\infty$$

is valid for the pair of inherent regular differential equations, therefore there is a constant K_3 such that $\|x_q - x\|_\infty \le K_3\left(\|q\|_\infty + \|(DQ_1G_2^{-1}q)'\|_\infty\right)$.

3. If $x_0 \in \mathbb{S}_{\text{ind}\,2}(t_0)$, then $x_0 = \Pi_{\text{can }2}(t_0)x_0$. The homogeneous equation with the initial condition $D(t_0)\hat{P}_1(t_0)x(t_0) = D(t_0)\hat{P}_1(t_0)x_0$ has the solution $x = KD^-u = KD^-D\hat{P}_1D^-u$. Since

$$\begin{aligned} x(t_0) &= \Pi_{\text{can }2}(t_0)[D(t_0)]^-u(t_0) \\ &= \Pi_{\text{can }2}(t_0)[D(t_0)]^-D(t_0)\hat{P}_1(t_0)x_0 = \Pi_{\text{can }2}(t_0)x_0 = x_0 \end{aligned}$$

this proves the claim ∎

Remark 3.4. The initial condition $D(t_0)\hat{P}_1(t_0)x(t_0) = d$, $d \in D(t_0)\mathbf{S}_1(t_0)$ can be replaced by

$$D(t_0)\hat{P}_1(t_0)x(t_0) = D(t_0)\hat{P}_1(t_0)x^0 \qquad (x^0 \in \mathbb{C}^m).$$

Consequently, the variational problem for $X := x'_{x_0}$ is simply $A(DX)' + BX = 0$, $D(t_0)\hat{P}_1(t_0)(X(t_0) - I) = 0$.

Remark 3.5. Formally, the case of index-1 tractability might be considered as index-2 tractability with $\dim \mathbf{N}_0 \cap \mathbf{S}_0 = \{0\}$. Thus, $\mathbf{N}_1 = \{0\}$, $P_1 = I$, $G_2 = G_1$. In this case, statements 1 and 3 of Theorem 3.2 confirm the results and expressions of Theorem 3.1 once more. In particular, the inherent ordinary differential equations (3.17) and (3.5) coincide and so do the geometric solution spaces $\mathbb{S}_{\text{ind}\,2}$ and $\mathbb{S}_{\text{ind}\,1}$.

3.3 Canonical projectors. We obtained the rather complicated expression (3.19) for the projector onto the geometrical solution space $\mathbb{S}_{\text{ind}\,2}$ of an index-2 equation. The next lemma shows how to simplify it by a proper construction.

Lemma 3.1. *For the index-2 tractable equation* (1.7) *there is a "canonical" projector Q_{0c} onto $\mathbf{N}_0$ such that $\Pi_{\text{can}\,2} = P_{0c}P_{1c}$.*

Proof. Let us construct chain (2.4) starting with an arbitrary Q_0 and the corresponding reflexive generalized inverse D^- of D. One can check that

$$Q_{0c} := Q_0\hat{P}_1\hat{G}_2^{-1}B + Q_0\hat{Q}_1D^-(D\hat{Q}_1D^-)'D \tag{3.22}$$

is a projector and $\operatorname{im} Q_{0c} = \mathbf{N}_0$ holds. Chain (2.4) can be built up starting with Q_{0c}, too. Let us mark the elements of the new chain as well as the related generalized inverse of D by the index c. We claim that Q_{0c} is the canonical projector appearing in the statement. In other words, the expression

$$\mathcal{A} := \left\{Q_{0c}P_{1c}G_{2c}^{-1}BP_{0c} + Q_{0c}Q_{1c}D_c^-(DQ_{1c}D_c^-)'D\right\}P_{0c}P_{1c}$$

is well-defined and vanishes. The verification requires a sophisticated combination of matrix relations (see [2]) ∎

For an index-1 equation the projector P_{0c} defined by formula (3.6) and the projector $P_{0c} = I - Q_{0c}$ obtained by formula (3.22) coincide; it is a projector onto $\mathbf{S}_0$. For index–2 equations one cannot claim that $\operatorname{im} P_{0c} = \mathbf{S}_0$ in general.

Due to Lemma 3.1, we may obtain a formally nicer solution representation with $D^- := D_c^-$. In particular, we obtain the next theorem.

Theorem 3.3. *For the index-μ tractable equation ($\mu = 1, 2$) there exists a canonical projector Q_{0c} onto $\mathbf{N}_0$ such that*

$$\Pi_{\mathrm{can}\,1} = P_{0c} \textit{ projects onto } \mathbb{S}_{\mathrm{ind}\,1}$$
$$\Pi_{\mathrm{can}\,2} = P_{0c}P_{1c} \textit{ projects onto } \mathbb{S}_{\mathrm{ind}\,2}.$$

The reflexive generalized inverse D^- of D can be chosen so that $D^-D = P_{0c}$ holds and the solutions of the homogeneous equation is $x = D^-u$, where the functions u satisfy the homogeneous inherent regular ordinary differential equations.

4. Fundamental matrices

In parallel to equation (1.7), one may consider matrix equations

$$A(DX)' + BX = 0 \tag{4.1}$$

with respect to matrices $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ for arbitrary k, $1 \le k \le m$, with coefficients as described in Section 2. We associate some notions with equation (4.1) and show their significance. For the technical details of the proofs we refer to [2].

Definition 4.1. A matrix function $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ will be called a *solution of equation* (4.1) if each of its columns is a solution of (1.7) with $q = 0$.

Thus, if a continuous matrix function $X : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$, for which the product DX is continuously differentiable, satisfies equation (4.1) pointwise, then it is called a solution of equation (4.1). The relevant function space will be denoted by $\mathcal{C}_D^{1,k}$,

$$\mathcal{C}_D^{1,k} := \left\{X \in C(\mathcal{I}, L(\mathbb{C}^k, \mathbb{C}^m)) : DX \in C^1(\mathcal{I}, L(\mathbb{C}^k, \mathbb{C}^m))\right\}.$$

Definition 4.2. A solution $X \in \mathcal{C}_D^{1,k}$ of an index-i equation (4.1) (i=1,2) will be called a *fundamental solution for equation* (1.7) if $\operatorname{im} X = \mathbb{S}_{\mathrm{ind}\,i}$.

Theorem 3.1 and decomposition (2.5) ensure that $m_1 := \dim \mathbb{S}_{\mathrm{ind}\,1} = r$. Similarly, Theorem 3.2 and decomposition (2.8) result in $m_2 := \dim \mathbb{S}_{\mathrm{ind}\,2} = r_1 - (m - r)$. Hence, for an index-$i$ equation ($i = 1, 2$) no fundamental matrix can exist with $k < m_i$.

Theorem 4.1. *Let equation* (1.7) *be of index i ($i = 1, 2$). For arbitrary $k \geq m_i$ and $\overline{P}_i \in L(\mathbb{C}^k, \mathbb{C}^m)$ the function $X_i : \mathcal{I} \to L(\mathbb{C}^k, \mathbb{C}^m)$ defined as*

$$X_i(t) = \Pi_{\mathrm{can}\, i}(t) D^-(t) U_i(t) D(t_0) \overline{P}_i$$

is a fundamental matrix provided that $\operatorname{im} \overline{P}_1 = \operatorname{im} P_0(t_0)$ $(i = 1)$ *or* $\operatorname{im} \overline{P}_2 = \operatorname{im} \hat{P}_1(t_0)$ $(i = 2)$*, respectively.*

Here U_i ($i = 1, 2$) are the fundamental matrices for the inherent regular ordinary differential equations (3.5) and (3.17) with $U_i(t_0) = I$. The set of fundamental matrices is described by the next assertion.

Corollary 4.1. *In Theorem* 4.1 *the matrix $\overline{P}_i$ may be replaced by $\overline{\Pi}_i$ such that* $\operatorname{im} \overline{\Pi}_i = \operatorname{im} \Pi_{\mathrm{can}\, i}(t_0) = \mathbb{S}_{\mathrm{ind}\, i}(t_0)$ $(i = 1, 2)$.

To prove the statements, the representation formula can be verified directly.

Remark 4.1. Although Theorem 4.1 allows arbitrary $k \geq m_i$, the minimal value $k = m_i$ would be sufficient. However, the reasonable choice is $k = m$. In some cases it is much easier to construct a projector ($k = m$) than either a basis ($k = m_i$) in $\mathbb{S}_{\mathrm{ind}\, i}$ or vectors spanning $S_{\mathrm{ind}\, i}$. Frequently, the projectors are given explicitly by the form of the equation. Statements claimed in [1] concerning transformations between different fundamental matrices including those of different sizes remain valid.

As in [1], a kind of normalization may be applied to fundamental matrices of "maximal size" m and a generalized reflexive inverse can be introduced for them.

Definition 4.3. A fundamental matrix X_i of maximal size m for equation (1.7) of index i ($i = 1, 2$) will be called *normalized at $t = t_0$* if $\Pi_{\mathrm{can}\, i}(t_0)(X_i(t_0) - I) = 0$.

Theorem 4.2. *For an index-i equation* (1.7) $(i = 1, 2)$ *and arbitrary $t_0 \in \mathcal{I}$ there exists a unique fundamental matrix X_i normalized at $t = t_0$, and $X_i(t_0) = \Pi_{\mathrm{can}\, i}(t_0)$ holds.*

For brevity, let us denote the fundamental matrix normalized at $t = \widehat{t}$ by $X_i(\cdot, \widehat{t})$, while the fundamental matrix U_i of the regular ordinary differential equation (3.5) or (3.17) normalized by $U_i(\widehat{t}) = I$ be denoted by $U_i(\cdot, \widehat{t})$. For the fundamental matrix $X_i(\cdot, \widehat{t})$ we introduce a generalized reflexive inverse $X_i^-(\cdot, \widehat{t})$ by

$$X_i(t, \widehat{t}) X_i^-(t, \widehat{t}) = \Pi_{\mathrm{can}\ i}(t) \quad \text{and} \quad X_i^-(t, \widehat{t}) X_i(t, \widehat{t}) = \Pi_{\mathrm{can}\ i}(\widehat{t}) \tag{4.2}$$

for all $t \in \mathcal{I}$ and some fixed $\widehat{t} \in \mathcal{I}$.

The last statement of this section provides the group properties.

Theorem 4.3. *For the normalized fundamental solutions X_i the identities*

$$X_i^-(t_2, t_1) = X_i(t_1, t_2) \quad \textit{and} \quad X_i(t_1, t_3) = X_i(t_1, t_2) X_i(t_2, t_3)$$

hold for all $t_1, t_2, t_3 \in \mathcal{I}$ $(i = 1, 2)$.

The proof utilizes only the representations of the fundamental matrices and the group properties of the normalized fundamental matrices belonging to the inherent regular ordinary differential equations (3.5) and (3.17).

5. Adjoint equation

Lemma 2.1 ensures that in the adjoint equation (1.8), i. e., in the equation

$$A_*(D_*y)' + B_*y = -p \tag{5.1}$$

with $A_* := D^*$, $D_* := A^*$ and $B_* := -B^*$, the ordered matrix pair (A_*, D_*) gives rise to a smooth $\mathbb{C}^m$ decomposition provided that the pair (A, D) does so. The projector R_* realizing the decomposition satisfies the relation $R_* = R^*$. For equation (5.1) let us introduce subspaces and matrices as we did for equation (1.7) by chain (2.4):

$$\begin{aligned}
&G_{*0} := A_*D_*, \quad B_{*0} := B_* = -B^* \\
&Q_{*i} \text{ is a projector onto } \ker G_{*i}, \quad P_{*i} := I - Q_{*i} \\
&W_{*i} \text{ is a projector}, \quad \ker W_{*i} = \operatorname{im} G_{*i} \\
&G_{*i+1} := G_{*i} + B_{*i}Q_{*i}, \quad B_{*i+1} := B_{*i}P_{*i} \\
&\mathbf{N}_{*i} := \ker G_{*i} = \operatorname{im} Q_{*i} \\
&\mathbf{S}_{*i} := \{z \in \mathbb{C}^m : \ B_{*i}z \in \operatorname{im} G_{*i}\} = \ker W_{*i}B_{*i} \qquad (i = 0, 1).
\end{aligned} \tag{5.2}$$

We define the reflexive inverses $D_*^- = A^{*-}$ and $A_*^- = D^{*-}$ with products

$$A^*A^{*-} = R_* = R^*, \quad A^{*-}A^* = P_{*0}, \quad D^*D^{*-} = I - W_{*0}, \quad D^{*-}D^* = R_* = R^*.$$

Remark 5.1. In general, the identities

$$D^{-*} = D^{*-}P_0^*, \quad D^{*-} = D^{-*}(I - W_{*0}), \quad A^{-*} = (I - W_0^*)A^{*-}, \quad A^{*-} = P_{*0}A^{-*}$$

hold. However, if we put $W_0 = Q_{*0}^*$ ($W_{*0} = Q_0^*$), then $A^{*-} = A^{-*}$ ($D^{*-} = D^{-*}$).

Now we aim at presenting the common features of equations (1.7) and (1.8). For this purpose we point out first several auxiliary assertions connecting the characteristic subspaces of the pair of equations. To do so we need the auxiliary matrices

$$\mathcal{G}_{i+1} := G_i + W_iB_iQ_i \quad \text{and} \quad \mathcal{G}_{*i+1} := G_{*i} + W_{*i}B_{*i}Q_{*i} \qquad (i = 0, 1).$$

The matrices $\mathcal{G}_i$ and $\mathcal{G}_{*i}$ are related to G_i and G_{*i} by the formulas

$$G_i = \mathcal{G}_iF_{i-1}, \qquad G_{*i} = \mathcal{G}_{*i}F_{*i-1}, \tag{5.3}$$

where

$$\begin{aligned}
F_0 &= I + D^-A^-BQ_0 \qquad & F_{*0} &= I - A^{*-}D^{*-}B^*Q_{*0} \\
F_1 &= I + G_1^-BP_0Q_1 \qquad & F_{*1} &= I - G_{*1}^-B^*P_{*0}Q_{*1}.
\end{aligned}$$

Here the reflexive inverses of G_1 and G_{*1} are defined by

$$G_1G_1^- = I - W_1, \quad G_1^-G_1 = P_1, \quad G_{*1}G_{*1}^- = I - W_{*1}, \quad G_{*1}^-G_{*1} = P_{*1}.$$

All F-s are non-singular; in order to obtain the corresponding inverses one has to change the signs $+$ and $-$ for their opposite. Our basic assertion is

$$\mathbf{N}_i \cap \mathbf{S}_i = \ker \mathcal{G}_{i+1}, \quad \mathbf{N}_{*i} \cap \mathbf{S}_{*i} = \ker \mathcal{G}_{*i+1} \qquad (i = 0, 1). \tag{5.4}$$

It links up the matrix chains (2.4) and (5.2) and the algebraic conditions (2.5)-(2.7) occurring in the index definition. The connection between chains (2.4) and (5.2) is established by the relations

$$\begin{aligned} \ker \mathcal{G}_1 &= \ker \mathcal{G}^*_{*1} & \ker \mathcal{G}_{*1} &= \ker \mathcal{G}^*_1 \\ F_0 \ker \mathcal{G}_2 &= \ker G^*_{*2} & F_{*0} \ker \mathcal{G}_{*2} &= \ker G^*_2. \end{aligned} \tag{5.5}$$

Each of identities (5.4) and (5.5) as well as the identities

$$D\mathbf{S}_1 = R(A^*\mathbf{N}_{*1})^\perp, \quad A^*\mathbf{S}_{*1} = R^*(D\mathbf{N}_1)^\perp \tag{5.6}$$

can be checked by returning to the formal definition of the subspace under consideration and by utilizing its features (for details see [2]). Note that the subspaces occurring in claim (5.6) would be associated with equations (1.7) and (1.8) supposed that the equations were equipped with an index (see Section 3). We can also state the connection between the projectors onto these subspaces. Namely, if decomposition (2.2) in condition C1 and relations (2.6) and (2.7) hold, then

$$(D\hat{P}_1 D^-)^* = A^* \hat{P}_{*1} A^{*-}. \tag{5.7}$$

The main point in the verification is to show that

$$(D\mathbf{N}_1 \oplus \ker R)^\perp = R^*(D\mathbf{N}_1)^\perp, \quad (A^*\mathbf{N}_{*1} \oplus \ker R^*)^\perp = R(A^*\mathbf{N}_{*1})^\perp.$$

Now we can formulate the statement justifying the title of the paper.

Theorem 5.1. *Equation* (1.8) *is of index* 1 *or of index* 2 *if and only if equation* (1.7) *is so.*

Proof. By Lemma 2.1, condition C1 holds for equations (1.7) and (1.8) simultaneously. If equation (1.7) satisfies the algebraic condition (2.5), then, together with G_1, $\mathcal{G}_1$ is non-singular, which yields $\mathbf{N}_{*0} \cap \mathbf{S}_{*0} = \{0\}$. This is exactly condition (2.5) for the adjoint equation (1.8). The opposite direction can be proved similarly. Thus, we are done with the index-1 case.

For the index-2 case we first check the algebraic relations (2.6) and (2.7). Let relations (2.2), (2.6) and (2.7) be valid for equation (1.7). Then, the subspace $\ker \mathcal{G}_1 = \mathbf{N}_0 \cap \mathbf{S}_0$ has constant dimension $m - r_1$. Therefore $m - r_1 = \dim \ker \mathcal{G}^*_{*1}$; this yields $m - r_1 = \dim \ker \mathcal{G}_{*1}$. Now, applying formula (5.4) we obtain $\dim \mathbf{N}_{*0} \cap \mathbf{S}_{*0} = m - r_1$, i. e., condition (2.6) for the adjoint equation (1.8) is fulfilled. We derive

$$\begin{aligned} \dim(\mathbf{N}_{*1} \cap \mathbf{S}_{*1}) &= \dim \ker \mathcal{G}_{*2} = \dim \ker G^*_2 \\ &= \dim \ker G_2 = \dim \ker \mathcal{G}_2 = \dim(\mathbf{N}_1 \cap \mathbf{S}_1) \end{aligned}$$

that is, condition (2.7) is transferred from (1.7) to (1.8) and vice versa.

It remains to check whether the smoothness of the subspaces $D\mathbf{S}_1$ and $D\mathbf{N}_1$ implies that of $A^*\mathbf{S}_{*1}$ and $A^*\mathbf{N}_{*1}$. If equation (1.7) is index-2 tractable, the projectors $D\hat{P}_1 D^-$, $D\hat{Q}_1 D^-$ and $I - R$ are continuously differentiable due to Lemma 2.3. Because of relation (5.7), $A^*\hat{P}_{*1}A^{*-}$ and $A^*\hat{Q}_{*1}A^{*-} = R^* - A^*\hat{P}_{*1}A^{*-}$ are continuously differentiable, too, and so are their image spaces $A^*\mathbf{S}_{*1}$ and $A^*\mathbf{N}_{*1}$. The proof in the opposite direction is similar ∎

As a consequence, the adjoint equation is of the same structure and has the same properties as the original one. We formulate the next corollary for the index-2 equations. If the index equals 1, the modification is obvious. In the statements, $C^1_{A^*Q_{*1}G^{-1}_{*2}} := \{x \in C : A^*Q_{*1}G^{-1}_{*2}x \in C^1\}$, $\hat{P}_{*1}$ denotes the special projector satisfying $\hat{P}_{*1} = I - Q_{*1}G^{-1}_{*2}B^*P_{*0}$. $\Pi_{*\mathrm{can}\, i}$ is the canonical projector onto $\mathbb{S}_{*\mathrm{ind}\, i}$ constructed by the scheme for $\Pi_{\mathrm{can}\, i}$. For greater transparency, the fundamental matrix of the adjoint equation is denoted by X_* instead of Y as done in Section 1.

Corollary 5.1. *Let equation* (1.7) *be index-*2 *tractable.*

1. *For each $p \in C^1_{A^*Q_{*1}G^{-1}_{*2}}$, $a \in A^*(t_0)\mathbf{S}_{*1}(t_0)$, $t_0 \in \mathcal{I}$ the initial value problem for equation* (1.8) (*equivalently,* (5.1)) *with initial condition $A^*(t_0)\hat{P}_{*1}(t_0)y(t_0) = a$ is uniquely solvable in $C^1_{A^*}$.*

2. *Equation* (1.8) *has perturbation index* 2.

3. *Exactly one solution of the homogeneous equation passes through each pair $(t_0, y(t_0))$, $t_0 \in \mathcal{I}$, $y(t_0) \in \mathbb{S}_{*\mathrm{ind}\, 2}(t_0)$.*

Corollary 5.2. *Let equation* (1.7) *be of index i, $i = 1$ or $i = 2$. Then, for the adjoint equation* (1.8) *and arbitrary $t_0 \in \mathcal{I}$ there exists a unique fundamental matrix X_{*i} normalized at $t = t_0$, and it is of the form*

$$X_{*i}(t, t_0) = \Pi_{*\mathrm{can}\, i}(t)A^{*-}(t)U_{*i}(t)A^*(t_0)\Pi_{*\mathrm{can}\, i}(t_0),$$

where U_{*i} is the normalized fundamental matrix of the inherent regular ordinary differential equation for the adjoint equation (1.8).

If formula (2.1) is applied to solutions forming fundamental matrices of equations (1.7) and (1.8), then, for an arbitrary pair of fundamental matrices X_i and X_{*i} (even of different dimensions), the Lagrange identity takes the form $X^*_{*i}ADX_i = \mathrm{const}$.

If the fundamental matrices are normalized at the same point $t = t_0$, the value of the above constant can be computed. Indeed,

$$\begin{aligned} X^*_{*i}(t, t_0)A(t)D(t)X_i(t, t_0) &\equiv X^*_{*i}(t_0, t_0)A(t_0)D(t_0)X_i(t_0, t_0) \\ &= \Pi^*_{*can\ i}(t_0)A(t_0)D(t_0)\Pi_{can\ i}(t_0). \end{aligned} \tag{5.8}$$

This identity allows to state relationships between the normalized fundamental matrices of equations (1.7) and (1.8). For this purpose, we use the reflexive inverses defined by formula (4.6) at $\widehat{t} = t_0$ and, similarly, by

$$X_{*i}(t, t_0)X^-_{*i}(t, t_0) = \Pi_{*\mathrm{can}\, i}(t) \quad \text{and} \quad X^-_{*i}(t, t_0)X_{*i}(t, t_0) = \Pi_{*\mathrm{can}\, i}(t_0). \tag{5.9}$$

We multiply (5.8) from the right by $X_i^-(t, t_0)$ and take the adjoint

$$\Pi^*_{\mathrm{can}\, i}D^*A^*X_{*i} = X_i^{-*}D^*(t_0)A^*(t_0)\Pi_{*\mathrm{can}\ i}(t_0). \tag{5.10}$$

Here and further, the argument t and the pair (t, t_0) are omitted.

The explicit formula will be derived only for the index-2 case, which formally includes the index-1 case (see Remark 3.5). We multiply identity (5.10) by $A_c^{*-}D^{-*}$

from the left (see Subsection 3.3 for $A_c^{*-} = D_{*c}^{-}$). The term $A^*(t_0)\hat{P}_{*1}(t_0)$ on the right-hand side is replaced by $A^*(t_0)\hat{P}_{*1}(t_0)A^{*-}(t_0)A^*(t_0)$. On both sides there appear terms for which formula (5.7) applies (with t or t_0, respectively). Recalling that $A_c^{*-}A^* = P_{*0c}$ and $P_{*0c}X_{*2} = X_{*2}$ hold, we arrive at the explicit expression for X_{*2}. Observe that $D^{-*}X_2^{-*} = D^{*-}X_2^{-*}$ is valid. The formula for X_2 is the result of analogous computations and the use of the relation $A^{*-*}X_2^{-*} = A^-X_2^{-*}$. This possibility of changing A^{*-*} and D^{-*} for A^- and D^{*-} here makes clear that the final statement is independent of special choices of P_0, P_{*0} and W_0, W_{*0}.

Theorem 5.2. *The fundamental matrices of the index-i equations* (1.7) *and* (1.8) ($i = 1, 2$) *normalized at the same point* $t = t_0$ *are connected by the formulas*

$$X_{*i}(t) = A_c^{*-}(t)D^{*-}(t)X_i^{-*}(t)D^*(t_0)A^*(t_0)$$
$$X_i(t) = D_c^{-}(t)A^{-}(t)X_{*i}^{-*}(t)A(t_0)D(t_0)$$

provided that the reflexive inverses (4.2) *and* (5.9) *are taken and* D_c^- *and* A_c^{*-} *are constructed by means of* P_{0c} *and* P_{*0c} *of Lemma* 3.1.

6. Final remarks

In our paper we have aimed at giving a precise and detailed analysis of the extended class of low index differential algebraic equations with not very smooth data as they arise in lots of applications. As a complement to the present paper, [10] deals with nonlinear equations. Integration methods are addressed in [7, 9, 10].

One might ask whether it would be possible to allow rank changes in the matrices $A(t)$ and $D(t)$ and, thus, to weaken the basic condition C1. The following example shows that rank changes indicate extra-ordinary points. Therefore, they should be treated as such, even if they are harmless in the context of smooth problems, i. e., smooth coefficients and smooth solutions.

System (1.7) with $m = 2$,

$$A(t) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad D(t) = \begin{pmatrix} 0 & t^\delta \\ 0 & 0 \end{pmatrix}, \quad B(t) \equiv I, \quad q(t) = \begin{pmatrix} 0 \\ \gamma(t) \end{pmatrix}$$

for $t \in \mathbb{R}$ and $\delta > 0$ reads $(t^\delta x_2(t))' + x_1(t) = 0, \quad x_2(t) = \gamma(t)$. It is tractable with index 2 on both subintervals $\mathcal{I}_1 = (-\infty, 0)$ and $\mathcal{I}_2 = (0, \infty)$. Due to Theorem 3.2, for each $\sigma > 0$, the continuous function γ such that $\gamma(t) = t^\sigma$ if $t > 0$ and $\gamma(t) \equiv 0$ if $t \le 0$ gives rise to solutions in the sense of Definition 2.1 on $\mathcal{I}_1$ and $\mathcal{I}_2$ *separately.* Namely,

$$\begin{aligned} x_1(t) &= 0, & x_2(t) &= 0 \quad \text{for } t \in \mathcal{I}_1 \\ x_1(t) &= -(\delta + \sigma)t^{\delta+\sigma-1}, & x_2(t) &= t^\sigma \quad \text{for } t \in \mathcal{I}_2. \end{aligned}$$

If $\delta + \sigma < 1$, i. e., if the coefficients are continuous only, then there is no continuous extension of the solutions to the entire interval $(-\infty, \infty)$. The differentiation index is defined only on $\mathcal{I}_1$ and $\mathcal{I}_2$ and equals 2, separately. One cannot assign a differentiation index to the system on $(-\infty, \infty)$. The situation is quite different if one restricts oneself to smooth problems only, i.e., at least $\delta \geq 1$ and $\sigma \geq 1$ is supposed. In this smooth case the differentiation index is 2 on $(-\infty, \infty)$ and the solution is continuously differentiable, too.

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