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On the Spectral Radius of Convolution Dilation Operators

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Abstract. Convolution dilation operators with non-compactly supported kernels are considered and effective formulae for their spectral radii are found. The formulae depend on the behaviour of the eigenvalues of the dilation matrix.

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1. Introduction

Goodman, Micchelli and Ward [6] considered the continuous convolution dilation operator

$$(T_c f)(x) = \int_{\mathbb{R}^s} c(x - My) f(y)\, dy \qquad (x \in \mathbb{R}^s) \tag{1}$$

where c was supposed to be a compactly supported function from $L_\infty(\mathbb{R}^s)$, s denotes a natural number, and M refers to a non-singular matrix of real numbers with the property

$$\lim_{n\to\infty} M^{-n} = 0. \tag{2}$$

Under the above assumptions, the authors were able to establish formulae for the spectral radius $\rho_p(T_c)$ of the operator $T_c : L_p(\mathbb{R}^s) \to L_p(\mathbb{R}^s)$. It has been shown that

$$\rho_p(T_c) = \lim_{n\to\infty} \|c^n\|_p^{\frac{1}{n}} = \lim_{n\to\infty} \|\widehat{c^n}\|_q^{\frac{1}{n}}$$

where $\frac{1}{p} + \frac{1}{q} = 1, c^n = T_c^{n-1}c$ and $\widehat{c^n}$ denotes the Fourier transform of c^n.

Recently Gao, Sun and one of the authors of this paper have studied spectral properties of the operator

$$(W_{c,\alpha} f)(x) = \alpha \int_{\mathbb{R}} c(\alpha x - y) f(y)\, dy \qquad (x \in \mathbb{R}) \tag{3}$$

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assuming α is a real number strictly larger that one [5]. In that work the kernel c again was supposed to be compactly supported, as well as integrable on $\mathbb{R}$ with

$$\int_{\mathbb{R}} c(x)\,dx = 1. \tag{4}$$

They gave a description of the spectrum of the operator $W_{c,\alpha} : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ and derived a result for the spectral radius of (3) – viz.

$$\rho_p(W_{c,\alpha}) = \alpha^{1-\frac{1}{p}}. \tag{5}$$

The integral operator (3) is closely connected to the operator T_c of (1), such that in the case $s = 1$ their spectral radii coincide.

The operator $W_{c,\alpha}$ is also known as continuous refinement operator. Another kind of refinement operator is defined by the rule

$$(D^M_{c,b} f)(x) = \sum_{k\in\mathbb{Z}} c_k f(Mx - b_k) \qquad (x \in \mathbb{R}^s) \tag{6}$$

where $\{c_k\}$ and $\{b_k\}$ are sequences of complex numbers and vectors from $\mathbb{R}^s$, respectively. In contrast to (3) these operators will be called semi-continuous convolution dilation operators. They were studied in connection with the refinement equation

$$f(x) = \sum_{k\in\mathbb{Z}} c_k f(Mx - b_k) \qquad (x \in \mathbb{R}^s)$$

and mainly under the assumption that $\{c_k\}$ is a compactly supported sequence. For the results concerning properties of the operators (3) and (6) the reader can consult, for example, [1 - 4, 7 - 9, 11]. On the other hand, operators (6) are closely connected to a number of operators arising in various fields of mathematics. Prominent examples of such operators are the Ruelle operator in the theory of dynamical systems as well as subdivision and transfer operators in wavelet analysis and subdivision processes. Their spectral characteristics play an important role and were intensively studied during the last few decades (cf. [1, 2, 10, 12]).

The aim of this paper is to present spectral radius formulae for the continuous plus semi-continuous convolution dilation operator $W^M_{c,b} : L_2(\mathbb{R}^s) \to L_2(\mathbb{R}^s)$ defined by

$$(W^M_{c,b} f)(x) = \int_{\mathbb{R}^s} c(Mx - y) f(y)\,dy + \sum_{k\in\mathbb{Z}^s}^{\infty} c_k f(Mx - b_k) \qquad (x \in \mathbb{R}^s). \tag{7}$$

For the sake of simplicity we pay main attention to the case $s = 1$, i.e. to the operator $W^\alpha_{c,b} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ where $0 < \alpha \neq 1$. However, we also show which adjustments are to be done if $s > 1$. In the present work neither the function c nor the sequence $\{c_n\}$ are assumed to be compactly supported, only that $c \in L_1(\mathbb{R}^s)$ and $\{c_n\} \in l_1$. We also note that our proofs do not require either condition (2) or the similar condition $\alpha > 1$. Moreover, the approach used here allows us to establish spectral radius formulae for operators which are, in some sense, more general than the operators $W^M_{c,b}$.

2. Spectral radius of the operator $W^{\alpha}_{c,b}$

Let $f \in L_2(\mathbb{R})$, and let F and F^{-1} denote the direct and inverse Fourier transforms, respectively, i.e.

$$\begin{aligned} (Ff)(x) &= \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-iyx} f(y)\, dy \\ (F^{-1}f)(x) &= \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{iyx} f(y)\, dy \end{aligned} \qquad (x \in \mathbb{R}).$$

It is well known that if $a \in L_\infty(\mathbb{R})$, then the operator $C(a) = FaF^{-1}$ is bounded on the space $L_2(\mathbb{R})$ and

$$\|FaF^{-1}\|_2 = \|a\|_\infty. \tag{8}$$

Given a number $\alpha > 0$, the symbol B_α refers to the operator defined on the space $L_2(\mathbb{R})$ by

$$(B_\alpha f)(x) = f(\alpha x) \qquad (x \in \mathbb{R}).$$

We denote by $W_\alpha(a)$ the operator $B_\alpha C(a)$.

Note that under some restrictions on the kernel c and on the sequence $\{c_n\}$ the operator $W^{\alpha}_{c,b}$ admits the representation $W^{\alpha}_{c,b} = W_\alpha(a)$ with an a belonging to a special subset of L_∞. The afore-mentioned representation essentially simplifies the calculation of the spectral radius of the operator $W^{\alpha}_{c,b}$. To be more precise we consider a functional class $C_0 + AP$, where the symbol C_0 denotes the set of all functions f continuous on $\mathbb{R}$ and vanishing at infinity, i.e. $\lim_{x\to\pm\infty} f(x) = 0$, and AP stands for the closure of the span of the set

$$S = \left\{ e^{ib_k x} \,|\, b_k \in \mathbb{R} \right\}$$

in the norm of L_∞. The elements of AP are usually called almost periodic functions.

Lemma 1. *Let $c \in L_1(\mathbb{R})$ and $\{c_k\} \in l_1$. Then the operator $W^{\alpha}_{c,b}$ of (7) can be represented in the form*

$$W^{\alpha}_{c,b} = W_\alpha(a)$$

where

$$a(x) = \sqrt{2\pi}(F^{-1}c)(x) + \sum_{k=1}^{\infty} c_k e^{ib_k x} \qquad (x \in \mathbb{R}). \tag{9}$$

Proof. If $c \in L_1(\mathbb{R})$, then the function

$$d(x) = \sqrt{2\pi}(F^{-1}c)(x) = \int_{\mathbb{R}} e^{iyx} c(y)\, dy$$

belongs to C_0. Using the convolution theorem, one has

$$\int_{\mathbb{R}} c(\alpha x - y) f(y)\, dy = (B_\alpha C(d) f)(x).$$

On the other hand, the translation operator U_{b_k} $(b_k \in \mathbb{R})$ defined by

$$(U_{b_k} f)(x) = f(x - b_k)$$

admits the representation $U_{b_k} = C(u_{b_k})$ with $u_{b_k} = e^{ib_k x}$. The condition $\{c_k\} \in l_1$ implies that the function $b(x) = \sum_{k=1}^{\infty} c_k e^{ib_k x}$ is in AP. Moreover,

$$\sum_{k=1}^{\infty} c_k f(\alpha x - b_k) = (B_\alpha C(b) f)(x)$$

which proves the statement of the lemma with $a = d + b$ ∎

Lemma 2. *Let $a \in L_\infty(\mathbb{R})$ and $a_\alpha(x) = a(\frac{x}{\alpha})$ $(x \in \mathbb{R})$. Then*

$$B_\alpha C(a) = C(a_\alpha) B_\alpha. \tag{10}$$

Proof. Let $f \in L_2(\mathbb{R})$. Then

$$\begin{aligned}
(B_\alpha C(a) f)(x) &= \tfrac{1}{2\pi} \int_{\mathbb{R}} e^{-i\alpha xy} a(y) \int_{\mathbb{R}} e^{ity} f(t)\, dtdy \\
&= \tfrac{1}{2\pi\alpha} \int_{\mathbb{R}} e^{-ixu} a\Big(\frac{u}{\alpha}\Big) \int_{\mathbb{R}} e^{i\frac{tu}{\alpha}} f(t)\, dtdu \\
&= \tfrac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} a\Big(\frac{u}{\alpha}\Big) \int_{\mathbb{R}} e^{ipu} f(\alpha p)\, dpdu \\
&= (C(a_\alpha) B_\alpha f)(x)
\end{aligned}$$

and the lemma is proved ∎

Now we can obtain formulae for the spectral radius.

Theorem 3. *Let $a \in L_\infty(\mathbb{R})$. Then the spectral radius $\rho(W_\alpha(a))$ of the operator $W_\alpha(a)$ can be found either by the formula*

$$\rho(W_\alpha(a)) = \tfrac{1}{\sqrt{\alpha}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left(\prod_{j=1}^{n} \left| a\Big(\frac{x}{\alpha^j}\Big) \right| \right)^{\frac{1}{n}} \tag{11}$$

or

$$\rho(W_\alpha(a)) = \tfrac{1}{\sqrt{\alpha}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left(\prod_{j=0}^{n-1} \left| a\Big(\alpha^j x\Big) \right| \right)^{\frac{1}{n}}. \tag{12}$$

Let us recall that the spectral radius $\rho(A)$ of any continuous linear operator A can be calculated as the limit

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}. \tag{13}$$

Note that the exact computation or even finding satisfactory estimates for the norms $\|A^n\|$ usually represents a difficult task. Such an approach has been used in [6] for $A = T_c$, but corresponding estimates were obtained under the additional assumptions that c is a compactly supported function and that M satisfies condition (2). In contrast to [6], we use the so-called C^*-algebra property of the norm on $\mathcal{L}(L_2(\mathbb{R}))$ instead of directly estimating the norms of A^n. More precisely, for any operator $A \in \mathcal{L}(L_2(\mathbb{R}))$ one has

$$\|A\|^2 = \|AA^*\|. \tag{14}$$

Application of this equality to the operator $(W_\alpha(a))^n$ gives us the exact value for the norm $\|(W_\alpha(a))^n\|$.

Proof of Theorem 3. Let a be defined by (9). Taking into account the representation $B_\alpha^* = \frac{1}{\alpha} B_{\frac{1}{\alpha}}$ and relations (10) and (14) one has

$$\begin{aligned}
&\|(W_\alpha(a))^n\|^2 \\
&= (W_\alpha(a))^n((W_\alpha(a))^n)^* \\
&= (B_\alpha C(a))\cdots(B_\alpha C(a)) \;\cdot\; (B_\alpha C(a))^*\cdots(B_\alpha C(a))^* \\
&= \underbrace{(B_\alpha F a F^{-1})\cdots(B_\alpha F a F^{-1})}_{(n-1)-\text{times}} \cdot (B_\alpha F|a|^2F^{-1}B_\alpha^*)\cdot \underbrace{(F\bar{a}F^{-1}B_\alpha^*)\cdots(F\bar{a}F^{-1}B_\alpha^*)}_{(n-1)-\text{times}} \\
&= \frac{1}{\alpha}\underbrace{(B_\alpha F a F^{-1})\cdots(B_\alpha F a F^{-1})}_{(n-1)-\text{times}} \cdot (F|a_\alpha|^2F^{-1})\cdot \underbrace{(F\bar{a}F^{-1}B_\alpha^*)\cdots(F\bar{a}F^{-1}B_\alpha^*)}_{(n-1)-\text{times}} \\
&\;\vdots \\
&= \frac{1}{\alpha^n}\; F\; |a_{\alpha^n}|^2\cdots|a_\alpha|^2\; F^{-1}\;.
\end{aligned}$$

The latter equality along with (8) and (13) implies (11).

Formula (12) can be obtained in the same way as (11). The only difference is that instead of (14) one uses the equality

$$\left\|(W_\alpha(a))^n\right\|^2 = \left\|((W_\alpha(a))^n)^*(W_\alpha(a))^n\right\|.$$

Alternatively, one can get (12) from (11) by using the substitution $x = y\alpha^n$ ∎

Corollary 4. *Let $c \in L_1(\mathbb{R})$ and $\{c_k\} \in l_1$. Then*

$$\rho(W_{c,b}^\alpha) = \tfrac{1}{\sqrt{\alpha}} \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left(\prod_{j=1}^{n} |a(\alpha^j x)|\right)^{\frac{1}{n}} \tag{15}$$

where a is defined by (9).

Indeed, the conditions of Corollary 4 imply the continuity of the function a on $\mathbb{R}$, so the essential supremum in (12) can be replaced by supremum.

Remark 5. The case $s > 1$ can be considered analogously. Corresponding formulae for the spectral radius of the operator $W_{c,b}^M$ are established in the proof of Theorem 13.

Next we consider the operator T_c of (1).

Theorem 6. *Let $c \in L_1(\mathbb{R}^s)$ and let M be a non-singular $s \times s$ matrix of real numbers. Then the spectral radius $\rho(T_c)$ of the operator $T_c : L_2(\mathbb{R}^s) \to L_2(\mathbb{R}^s)$ can be found by the formula*

$$\rho(T_c) = \tfrac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \sup_{x\in\mathbb{R}^s} \left(\prod_{j=1}^{n} |a(N^j x)|\right)^{\frac{1}{n}} \tag{16}$$

where $N = (M^{-1})^T$ *and* $a(x) = \int_{\mathbb{R}^s} e^{ixt} c(t)\,dt$.

Proof. Let A be an $s \times s$ matrix of real numbers and let $B_A : L_2(\mathbb{R}^s) \to L_2(\mathbb{R}^s)$ denote the left dilation operator $(B_A f)(x) = f(Ax)$. The operator T_c of (1) admits the representation

$$T_c = \tfrac{1}{|\det M|} C(a) B_{M^{-1}} \tag{17}$$

and it is easily seen that its adjoint operator T_c^* has the form $T_c^* = B_M C(\overline{a})$. On the other hand, elementary calculations lead to the identity

$$B_M C(a) = C(a_M) B_M \tag{18}$$

with $a_M(x) = a((M^{-1})^T x) = a(Nx)$. Using equality (14) with $A = T_c^n$, as well as (17), (18) and (13), one arrives at (16) ∎

3. Consequences of general formulae

Representations (11), (12) and (15), (16) open a way to finding effective formulae for the spectral radius of continuous and semi-continuous convolution dilation operators. In this section we consider cases when limits (11) and (15) can be found explicitly. Note that formulae (11) and (12) allow us to guess that the only behaviour of the function a at the origin and at infinity can be relevant for the spectral radius of the operator $W_\alpha(a)$.

The following corollaries contain results which immediately follow from the representations mentioned.

Corollary 7. *Let* $a \in L_\infty(\mathbb{R})$ *and let* $\mathcal{M}$ *denote the set of the symbols* $\pm\infty$ *and* $0\pm$. *If for some* $r_0 \in \mathcal{M}$ *there exists the limit* $|a(r_0)| = \lim_{x\to r_0} |a(x)|$ *and if* $|a(r_0)| \geq |a(x)|$ *for all* $x \in \mathbb{R}$, *then*

$$\rho(W_\alpha(a)) = \frac{|a(r_0)|}{\sqrt{\alpha}}.$$

Corollary 8. *Let the kernel* $c \in L_1(\mathbb{R})$ *and* $c(x) \geq 0$ *for all* $x \in \mathbb{R}$, *and let the sequence* $\{c_k\} \in l_1$ *and* $c_k \geq 0$ *for all* $k \geq 1$. *Then*

$$\rho(W_{c,b}^\alpha) = \frac{a(0)}{\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}}\left(\int_{\mathbb{R}} c(x)\,dx + \sum_{k=1}^{\infty} c_k\right).$$

Really, under the conditions of Corollary 8 one has

$$|a(x)| \leq \int_{\mathbb{R}} c(t)\,dt + \sum_{k=1}^{\infty} c_k = a(0) \qquad (x \in \mathbb{R}),$$

so the result immediately follows from Corollary 7 and from the continuity of the function a.

An analogous result for the operator T_c has been established in [6] – cf. Corollary 2.1 there. In that paper it was also assumed that c is compactly supported and M satisfies (2).

Though Corollaries 7 and 8 provide us with very simple expressions for the spectral radius of the operator $W_\alpha(a)$, they require rather restrictive conditions on the behaviour of the function a. Below we show that similar results are true for functions which do not necessarily attains their supremum at one of the points $-\infty, +\infty, 0-$ or $0+$. To do that we have to study the limit

$$R_a = \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}} \left(\prod_{j=1}^{n} \left| a\left(\frac{x}{\alpha^j}\right)\right| \right)^{\frac{1}{n}} \qquad (0 < \alpha \neq 1).$$

We start with an auxiliary result.

Lemma 9. *Let f be a bounded on $\mathbb{R}$ function such that there exist the limits $A = \lim_{x\to+\infty} f(x)$ and $B = \lim_{x\to-\infty} f(x)$. Then*

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \left(\frac{1}{n} \sum_{k=1}^{n} f(t+k) \right) = \max(A, B).$$

Proof. We only consider the case $A \le B$. Set

$$C = \sup_{t\in\mathbb{R}} |f(t)|, \quad S_n(t) = \frac{1}{n} \sum_{k=1}^{n} f(t+k), \quad P_n = \operatorname*{ess\,sup}_{t\in\mathbb{R}} (S_n(t))$$

and show that for any $\varepsilon > 0$ there exists a number N such that $-\varepsilon < P_n - B < \varepsilon$ for all $n \ge N$. Obviously, the inequality $-\varepsilon < P_n - B < \varepsilon$ is equivalent to the following two assertions:

1. There exists a subset $E_n \subset \mathbb{R}$ with Lebesgue measure $mE_n = 0$ such that $S_n(t) \le B + \varepsilon$ for all $t \in \mathbb{R} \setminus E_n$.
2. There exists a subset $E'_n \subset \mathbb{R}$ with Lebesgue measure $mE'_n > 0$ such that $S_n(t) > B - \varepsilon$ for all $t \in E'_n$.

Fix $\varepsilon > 0$ and find $k_0 \in \mathbb{N}$ such that $f(t) < B + \frac{\varepsilon}{2}$ for all $t \in \mathbb{R}$ with $|t| \ge k_0$ and $f(t) > B - \frac{\varepsilon}{2}$ for all $t \le -k_0$. Choose $N = \left[\frac{2C(2k_0+1)}{\varepsilon}\right] + 1$ with $[\cdot]$ denoting the integral part of the corresponding real number and set $n \ge N$. Then among the points $t + k$ $(k = 1, 2, \ldots, n)$ there are at most $2k_0 + 1$ ones which belong to the interval $[-k_0, k_0]$. Therefore, for any $t \in \mathbb{R}$ one has

$$S_n(t) \le \frac{1}{n} C(2k_0 + 1) + \frac{1}{n}\Big(B + \frac{\varepsilon}{2}\Big)n \le \frac{\varepsilon}{2} + \Big(B + \frac{\varepsilon}{2}\Big) = B + \varepsilon$$

which proves assertion 1. On the other hand, assertion 2 is true for $E'_n = (-k_0 - n - 1, -k_0 - n)$. The proof for the case $B < A$ is similar ∎

Theorem 10. *Let a bounded on $\mathbb{R}$ function $|a|$ have finite limits $|a(0\pm)|$ and $|a(\pm\infty)|$. Then*

$$\rho(W_\alpha(a)) = \tfrac{1}{\sqrt{\alpha}} \max\big(|a(\pm\infty)|, |a(0\pm)|\big).$$

Proof. Fix an $\varepsilon > 0$ and introduce a new function $\widetilde{a}$ by

$$\widetilde{a}(t) = \max(\varepsilon, |a(t)|).$$

If $|a|$ possesses the above mentioned finite limits, then the function $\widetilde{a}$ also does, and for $r \in \{\pm\infty, 0\pm\}$ one has

$$|a(r)| \le \widetilde{a}(r) \le |a(r)| + \varepsilon. \tag{19}$$

In addition,

$$\max\big(|a(\pm\infty)|, |a(0\pm)|\big) \le R_a \le R_{\widetilde{a}}. \tag{20}$$

Now, applying Lemma 9 to the function $f(t) = \ln \widetilde{a}(\alpha^{-t})$ $(t \in \mathbb{R})$ one obtains

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname*{ess\,sup}_{x>0} \Big(\prod_{j=1}^{n} \widetilde{a}\Big(\frac{x}{\alpha^j}\Big)\Big)^{\frac{1}{n}} \\
&\quad = \exp\Big(\lim_{n\to\infty} \operatorname*{ess\,sup}_{t\in\mathbb{R}} \Big(\frac{1}{n} \sum_{j=1}^{n} f(t+j)\Big)\Big) \\
&\quad = \exp\big(\max f(\pm\infty)\big) \\
&\quad = \max\big(\widetilde{a}(0+), \widetilde{a}(+\infty)\big).
\end{aligned}$$

Analogously,

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{x<0} \Big(\prod_{j=1}^{n} \widetilde{a}\Big(\frac{x}{\alpha^j}\Big)\Big)^{\frac{1}{n}} = \max\big(\widetilde{a}(-\infty), \widetilde{a}(0-)\big).$$

Hence, $R_{\widetilde{a}} = \max\big(\widetilde{a}(\pm\infty), \widetilde{a}(0\pm)\big)$ so that inequalities (19) and (20) imply

$$\max\big(|a(\pm\infty)|, |a(0\pm)|\big) \le R_a \le \max\big(|a(\pm\infty)|, |a(0\pm)|\big) + \varepsilon$$

and the result follows from Theorem 3 ∎

Corollary 11. *Let $c \in L_1(\mathbb{R})$. Then the spectral radius of the operator $W_{c,\alpha} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is*

$$\rho(W_{c,\alpha}) = \sqrt{\alpha} \Big| \int_{\mathbb{R}} c(t)\, dt \Big|. \tag{21}$$

Remark 12. It was mentioned in [5] that the spectral radius of $W_{c,\alpha}$ is independent of the kernel c (see (5)). However, as one can see from (21), such an impression arose because c was assumed to satisfy equality (4).

Next we consider a multi-dimensional version of Theorem 10. However, if we have a look at the symbol

$$a(x) = (2\pi)^{\frac{s}{2}} (F^{-1}c)(x) + \sum_{k\in\mathbb{Z}^s}^{\infty} c_k e^{ib_k x} \qquad (x \in \mathbb{R}^s)$$

of the convolution dilation operator (7) we discover that, in general, it does not possess a limit at infinity. Nevertheless, the condition of the existence of a limit at infinity can be replaced by another one.

Theorem 13. *Let a be a bounded on $\mathbb{R}^s$ function such that there exists the limit $A = \lim_{x\to 0} |a(x)|$ and let $A \geq \limsup_{x\to\infty} |a(x)|$. If the eigenvalues of the matrix MM^* are either all greater or all less than* 1*, then*

$$\rho(W_M(a)) = \frac{A}{\sqrt{|\det M|}}. \tag{22}$$

Proof. We only treat the case that all the eigenvalues of the matrix MM^* are greater than one. Then the operator (spectral) norm $\|M^{-1}\|$ of the inverse matrix M^{-1} satisfies the inequality

$$\|M^{-1}\| < 1. \tag{23}$$

Following the proof of Theorem 3 we obtain

$$\rho(W_M(a)) = \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \left(\prod_{j=0}^{n-1} |a((M^T)^j x)|\right)^{\frac{1}{n}}. \tag{24}$$

One can exploit this formula, but in contrast to the proof of Theorem 10 there are difficulties with using results like Corollary 9 because of problems with the function $x \to M^x$ where $x \in \mathbb{R}^s$. However, the product in the right-hand side of (24) can be calculated directly. Thus, we show that

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \left(\prod_{j=0}^{n-1} |a((M^T)^j x)|\right)^{\frac{1}{n}} = A. \tag{25}$$

From now on we will use the notation

$$P_n(x) = \left(\prod_{j=0}^{n-1} |a((M^T)^j x)|\right)^{\frac{1}{n}}.$$

Relation (25) is equivalent to the existence for any $\varepsilon > 0$ of an $N > 0$ such that, for all $n \geq N$:

1. There exists a subset $E_n \subset \mathbb{R}^s$ with Lebesgue measure $mE_n = 0$ such that $P_n(x) \leq A + \varepsilon$ for all $x \in \mathbb{R}^s \setminus E_n$.
2. There exists a subset $E'_n \subset \mathbb{R}^s$ with Lebesgue measure $mE'_n > 0$ such that $P_n(x) \geq A - \varepsilon$ for all $x \in E'_n$.

We start with the first assertion. Let $\|x\|_2$ denote the Euclidean norm of $x \in \mathbb{R}^s$. Fix $\varepsilon > 0$ and find $k_0 > 0$ such that $|a(x)| < A + \frac{\varepsilon}{2}$ for all x with $\|x\|_2 \geq e^{k_0}$ and $A - \frac{\varepsilon}{2} < |a(x)| < A + \frac{\varepsilon}{2}$ for all x with $0 < \|x\|_2 \leq e^{-k_0}$. Then for any $x \in \mathbb{R}^s$ the number of the points of the form $(M^T)^j x$ ($j \in \mathbb{N}$) located in the spherical segment

$$S_{k_0} = \left\{y \in \mathbb{R}^s : e^{-k_0} \leq \|y\|_2 \leq e^{k_0}\right\} \tag{26}$$

is bounded. Indeed, a well-known property of the norms of inverse operators yields

$$\|(M^T)^j x\|_2 \geq \frac{1}{\|(M^T)^{-1}\|^j}\|x\|_2 = \frac{1}{\|M^{-1}\|^j}\|x\|_2 \qquad (x \in \mathbb{R}^s).$$

Let $[r]$ refers to the integral part of $r \in \mathbb{R}$. Now it follows from (23) that for any $x \in \mathbb{R}^s$ there are at most $q = \left[-\frac{2k_0}{\ln \|M^{-1}\|} \right] + 1$ points of the form $(M^T)^j x$ $(j \in \mathbb{N})$ which belong to S_{k_0}. The latter estimate is independent of $x \in \mathbb{R}^s$, so our claim is proved.

Without loss of generality we may assume that $\varepsilon < \frac{4}{3A}$. Let C denotes the maximum of 1 and $\sup_{x \in \mathbb{R}^s} |a(x)|$. Since $\lim_{k\to\infty} C^{q/k} = 1$, there exists $k_1 \in \mathbb{N}$ such that, for any $k \geq k_1$, $0 \leq |C|^{q/k} < 1 + \frac{\varepsilon}{2A}$. On the other hand, $\lim_{n\to\infty} \left(A + \frac{\varepsilon}{2}\right)^{1-q/n} = A + \frac{\varepsilon}{2}$. Hence, there exists $n_2 \in \mathbb{N}$ such that, for any $n > n_2$, $\left(A + \frac{\varepsilon}{2}\right)^{1-q/n} < A + \frac{3\varepsilon}{4}$. Set $N = \max(n_2, k_1 q)$. Then for any $n \geq N$ and for any $x \in \mathbb{R}^s$ one obtains

$$P_n(x) \leq \left(A + \frac{\varepsilon}{2}\right)^{\frac{n-q}{n}} C^{\frac{q}{n}} < \left(A + \frac{3\varepsilon}{4}\right)\left(1 + \frac{\varepsilon}{2A}\right) < A + \varepsilon.$$

This completes the proof of assertion 1. Since the norm of M is now greater than one, the second assertion is true for $E'_n = \left\{x \in \mathbb{R}^s : 0 < \|x\| < \frac{e^{-k_0}}{\|M\|^{n+1}}\right\}$ ∎

The conditions of Theorem 13 imposed on the eigenvalues of the dilation matrix M look very restrictive. Though main formula (24) for spectral radius does not require limit relation (2), in fact, it was used while obtaining (22). It would be interesting to find the spectral radius of $W_M(a)$ without requiring limit relation (2). In connection with this we consider real-valued symmetric dilations M with positive eigenvalues. However, if we assume that the dilation matrix possesses such a property, then the requirements for the eigenvalues of the dilation matrix can be weakened. Thus let M be a real-valued symmetric matrix and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s > 0$ be the eigenvalues of M. Then there exists a real unitary matrix U such that

$$M^T = M = UDU^{-1} \tag{27}$$

where $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_s)$.

Let V_j $(j = 1, 2, \ldots, 2^s)$ denote the hyper octants of $\mathbb{R}^s$, and let the function v be defined by

$$v(x) = |a(Ux)| \qquad (x \in \mathbb{R}^s).$$

Theorem 14. *Assume that the eigenvalues of the dilation matrix M are positive and different from one. If at least for one $j_0 \in \{1, 2, \ldots, 2^s\}$ and for one $r_0 \in \{0, \infty\}$ there exists the limit*

$$A_{j_0}(r_0) = \lim_{V_{j_0} \ni x \to r_0} v(x)$$

such that

$$A_{j_0}(r_0) = \max\left\{\limsup_{x\to 0} v(x), \limsup_{x\to\infty} v(x)\right\},$$

then for the spectral radius of the operator $W_M(a)$ the formula

$$\rho(W_M(a)) = \frac{A_{j_0}(r_0)}{|\det M|}$$

holds.

Proof. Representation (27) implies $(M^T)^j = UD^jU^{-1}$, and applying general formula for the spectral radius one obtains

$$\begin{aligned}\rho(W_M(a)) &= \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \left(\prod_{j=0}^{n-1} |a((M^T)^j x)|\right)^{1/n} \\ &= \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \left(\prod_{j=0}^{n-1} |a(UD^jU^{-1}x)|\right)^{1/n} \\ &= \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \left(\prod_{j=0}^{n-1} |v(D^j x)|\right)^{1/n}.\end{aligned}$$

Now we can proceed as in the proof of Theorem 13. However, it is more convenient to replace spherical segments (26) by other subsets of $\mathbb{R}^s$. More precisely, let $k_0, k_1 > 0$ be numbers and let Q_{k_r} $(r = 0, 1)$ refer to the cube $\{x = (x_l) \in \mathbb{R}^s : |x_l| \leq k_r\}$. Assume that $k_0 < k_1$ and consider the subset $S_{k_0,k_1} = Q_{k_1} \setminus Q_{k_0}$. Let $x = (x_1, \ldots, x_s) \in S_{k_0,k_1}$. Then at least one coordinate x_l of x is not equal to zero. Taking into account that all eigenvalues of the dilation matrix M are different from one we obtain that the number of the points $D^j x$ $(j \in \mathbb{N})$ located in the subset S_{k_0,k_1} is bounded by a number q_0 independent of x. Further steps in the proof are analogous to those in the proof of Theorem 13 and will be omitted ∎

The later assertion allows us to calculate the spectral radius of the operator T_c of (1) in the case of symmetric dilation matrices which do not necessarily satisfy condition (2).

Corollary 15. *Let $c \in L_1(\mathbb{R}^s)$. If M is a symmetric matrix of real numbers the eigenvalues of which are positive and not equal to one, then the spectral radius of the operator $T_c : L_2(\mathbb{R}^s) \to L_2(\mathbb{R}^s)$ is*

$$\rho(T_c) = \frac{1}{\sqrt{|\det M|}} \left| \int_{\mathbb{R}^s} c(t)\, dt \right|.$$

For the proof one has to mention that under the above conditions the limit $\lim_{x\to\infty} a(x)$ exists and equal to 0. Moreover, the function a is continuous at the origin. Applying Theorem 14 one obtains the result.

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