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On the Minimal Displacement Problem of γ-Lipschitz Maps and γ-Lipschitz Retractions onto the Sphere

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Abstract. We give a general construction in arbitrary normed spaces to produce fixed-point free continuous maps with a large minimal displacement, contractions of the sphere, and retractions onto the sphere such that the corresponding maps have small measures of non-compactness.

Keywords: *Condensing operator, measure of non-compactness, fixed point, minimal displacement, retraction*

AMS subject classification: Primary 47H09, 47H10, secondary 46B20, 46B26, 47H12

1. Introduction

Let X be a normed space, and let

$$B_r(X) := \{x \in X : \|x\| \le r\}$$
$$S_r(X) := \{x \in X : \|x\| = r\}.$$

It is well-known that the following statements are equivalent in X:

1. There is a fixed-point free continuous map $F: B_1(X) \to B_1(X)$.

2. There is a homotopy $H: S_1(X) \times [0,1] \to S_1(X)$ which joins the identity with a constant map, i.e. $H(x,0) = x$ and $H(x,1) \equiv \text{const}$ for $x \in S_1(X)$.

3. There is a retraction of $B_1(X)$ onto $S_1(X)$, i.e. a continuous map $R: B_1(X) \to S_1(X)$ with $R(x) = x$ on $S_1(X)$.

Indeed, if F respectively H are given, then H respectively R can be obtained by well-known constructions (which we recall later). Conversely, $-R$ is a fixed-point free map. If X has finite dimensions, the above statements all fail in view of Brouwer's fixed point theorem. Conversely, if X has infinite dimensions, the existence of a retraction of $B_1(X)$ onto $S_1(X)$ was first proved in [5], using the axiom of choice.

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However, other constructions were found later, which require only the (countable) axiom of dependent choices: A very simple construction of a fixed-point free continuous map of $B_1(X)$ was given in [10]. This construction was later extended in many respects. Nowadays, it is known that (in each infinite-dimensional normed space) there exist Lipschitz maps with the above properties [3, 11, 12].

Almost nothing is known about the best possible Lipschitz constant; all known constructions of the above maps have enormous Lipschitz constants (see [9] for a summary of known results in this direction; see also [6, 7]). A well-known fixed point theorem on non-expanding maps implies that at least in uniformly convex spaces the Lipschitz constant of F (and thus of R) must be larger than 1 (see, e.g., [13, 15] or [9]). It is actually known that the Lipschitz constant of R must be at least 3.

It turns out that the Lipschitz constant for H and R by the above mentioned constructions not only depends on the Lipschitz constant of F but also on the minimal displacement

$$\operatorname{disp}(F,M) := \inf\big\{\|F(x)-x\| : x \in M\big\} \qquad (M \subseteq X)$$

of F on $M = B_1(X)$. Thus, one is interested not only in finding fixed-point free maps with a small Lipschitz constant but also with large $\operatorname{disp}(F, B_1(X))$. This problem has been discussed in [14].

We are interested in a related question: What are the best possible constants such that F, H, R are γ-Lipschitz where γ denotes a measure of non-compactness?

Definition 1. For a set M in a metric space, putting $\inf\emptyset = \infty$ and $\sup\emptyset = 0$, we define:

1. The *Kuratowski measure of non-compactness* $\alpha(M)$ is the infimum of all $\varepsilon > 0$ such that M has a finite covering of sets with diameter at most ε.

2. The *Hausdorff measure of non-compactness* $\chi_Y(M)$ (with respect to a set Y) is the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in Y.

3. The *lattice measure of non-compactness* $\beta(M)$ (in literature also called *separation measure of non-compactness*) is the supremum of all $\varepsilon > 0$ such that M contains a sequence x_n with $d(x_n, x_k) \geq \varepsilon \;\; (n \neq k)$.

The term "measure of non-compactness" is explained by the fact that the above measures vanish if and only if M is precompact, i.e. if the completion of M is compact (for the Hausdorff measure of non-compactness, assume here that $M \subseteq Y$). It is well-known and not hard to see that

$$\chi_Y(M) \leq \chi_{Y_0}(M) \leq \beta(M) \leq \alpha(M) \leq 2\chi_Y(M) \qquad (M \subseteq Y_0 \subseteq Y).$$

It is also known that for subsets of a normed space X the measures of non-compactness $\gamma \in \{\alpha, \beta, \chi_X\}$ are monotone, subadditive, homogeneous, and invariant under passage to the closed convex hull (see, e.g., [1, 2]). A map $F\colon M \subseteq X \to X$ is called *γ-Lipschitz*, if there is some constant $L < \infty$ with

$$\gamma(F(A)) \leq L\,\gamma(A) \qquad (A \subseteq M).$$

The name is explained by the fact that compact perturbations of Lipschitz maps are α-Lipschitz (with at most the same constant L). It follows from Darbo's fixed point theorem [4] (and its extension of Sadovskiĭ [16]) that any fixed-point free continuous map $F\colon B_1(X) \to B_1(X)$ must satisfy $L \geq 1$ for $\gamma \in \{\alpha, \beta, \chi_X\}$ (if X is a Banach space). In [17], we gave a construction of such a map which reaches the value $L = 2$ in any normed space and the best possible value $L = 1$ in a large class of spaces. The latter was formulated in [17] only for spaces with a "separable retraction property" as well as for Hilbert spaces and for "sufficiently large" subspaces of $l_p(S)$ $(1 \leq p \leq \infty)$, but an inspection of the proof shows that it works also for all normed spaces which contain an isometric copy of $c_{\text{fin},p}$ (the separable retraction property and containment of $c_{\text{fin},p}$ will also play a special role in the current paper).

As in the Lipschitz case, the γ-Lipschitz constants for H and R depend also on $\operatorname{displ}(F, B_1(X))$. Since $\operatorname{displ}(F, B_1(X)) = 0$ in the construction from [17], this example cannot be used to construct, e.g., γ-Lipschitz retractions onto the unit sphere. This difficulty is of principal nature as was observed in [8]. It is not accidental that $\operatorname{displ}(F, B_1(X)) = 0$ in the construction of [17]: If $L \geq 1$ is the γ-Lipschitz constant of F for some $\gamma \in \{\alpha, \beta, \chi_X\}$ (in a Banach space X), then

$$\operatorname{displ}(F, B_1(X)) \leq 1 - \tfrac{1}{L}. \tag{1}$$

This was proved in [8] for the case $\gamma = \alpha$, but the same proof works also for $\gamma = \beta$ and $\gamma = \chi_X$.

We are thus interested in the construction of a continuous map $F\colon B_1(X) \to B_1(X)$ with a small γ-Lipschitz constant such that $\operatorname{displ}(F, B_1(X))$ is large.

In the space $X = C([0,1])$, there exist χ_X-Lipschitz retractions onto $S_1(X)$ with any constant $L > 1$ [18]. It is unknown whether the constant $L = 1$ can be reached in some space. J. Wośko observed in [18] that this value cannot be reached for a Lipschitz map. Actually, a slightly stronger statement holds:

Proposition 1. *Let X be normed, and $F\colon B_1(X) \to B_1(X)$ be γ-Lipschitz for some $\gamma \in \{\alpha, \beta, \chi_X\}$ with constant $L \leq 1$. If F is uniformly continuous, then* $\operatorname{displ}(F, B_1(X)) = 0$ *and F is not a retraction onto $S_1(X)$.*

Proof. Since F is uniformly continuous, we can extend F to a continuous map on the completion $\overline{X}$ of X. Let $\varepsilon > 0$ be given. By Darbo's (or Sadovskiĭ's) fixed point theorem, the map $(1-\varepsilon)F$ has a fixed point in $B_1(\overline{X})$. By continuity, we find some $x \in B_1(X)$ in a neighborhood of this point with $\|x - (1-\varepsilon)F(x)\| \leq \varepsilon$; then $\|x - F(x)\| \leq 2\varepsilon$ and so $\operatorname{displ}(F) = 0$. Applying this result to $-F$, we find a sequence $x_n \in B_1(X)$ such that $y_n := -F(x_n)$ satisfies $\|x_n - y_n\| \to 0$. If F is a retraction onto $S_1(X)$, we would have $\|F(x_n) - F(y_n)\| = \|F(x_n) - y_n\| = 2$, contradicting the uniform continuity of F ∎

Our main interest is in finding good constants for a *large* class of spaces. To this end, we modify the construction of [17] carefully. Concerning the minimal displacement problem, we obtain the theoretically best possible constant from (1) (up to an arbitrary small error $\varepsilon > 0$) in a large class of spaces, and in the class of all normed spaces we obtain this constant up to a factor 2 (and some $\varepsilon > 0$). Concerning the constants for the homotopy and the retraction, we reach by our construction the value $6+\varepsilon$ for each γ. Moreover, in separable or reflexive spaces the value $4+\varepsilon$ is obtained, and in many spaces even $3+\varepsilon$ (the latter only for $\gamma=\chi_X$). Since the influence of the radius to our construction is not immediately evident, we do not restrict our attention to the unit sphere.

2. The abstract construction

Our main construction works in arbitrary metric spaces.

Definition 2. Given subsets M and Y of a metric space and $\delta_1,\delta_2\geq 0$, we say that M has a (δ_1,δ_2)-*path* with respect to Y, if there is a sequence of points $e_n\in M$ and of (continuous) paths $\Gamma_n\subseteq M$ joining e_n with e_{n+1} such that the following holds:

1. $\operatorname{dist}(\Gamma_n,\Gamma_j)>\delta_1$ for $n\geq j+2$.

2. For any $x\in Y$ and any $\varepsilon>0$ we have $\operatorname{dist}(x,\Gamma_n)\geq\delta_2-\varepsilon$ for all except at most finitely many numbers n.

In this case, we call $\Gamma:=\bigcup_n\Gamma_n$ a (δ_1,δ_2)-path.

One can always choose $\delta_2=\frac{\delta_1}{2}$:

Proposition 2. *If Γ is a $(\delta_1,0)$-path for M, then Γ is also a $(\delta_1,\frac{\delta_1}{2})$-path for M with respect to any Y.*

Proof. Let $y\in Y$ be given. If $\operatorname{dist}(y,\Gamma_n)>\frac{\delta_1}{2}$ for each n, we are done. Otherwise, there is some n and some $x\in\Gamma_n$ with $d(x,y)\leq\frac{\delta_1}{2}$. Then we have for all k with $|k-n|\geq 2$ that $\operatorname{dist}(y,\Gamma_k)\geq\operatorname{dist}(x,\Gamma_k)-d(x,y)\geq\delta_1-\frac{\delta_1}{2}\geq\frac{\delta_1}{2}$ ∎

Theorem 1. *Let a subset M of a metric space have a (δ_1,δ_2)-path Γ with respect to Y (without loss of generality $\delta_2\geq\frac{\delta_1}{2}$). Then for each $\delta\in(0,\frac{\delta_1}{2}]$ and each $\varepsilon\in(0,\delta)$ there is a continuous map $F\colon M\to\Gamma\subseteq M$ with the following properties:*

1. $\operatorname{displ}(F,M)>\delta-\varepsilon$.

2. *If $A\subseteq M$ is such that $\overline{F(A)}$ is not compact, then $\beta(A)\geq\delta_1-2\delta$ and $\chi_Y(A)\geq\delta_2-\delta$.*

Proof. Let e_n and Γ_n be as in Definition 2. Put

$$T_n=\{x\in M:\operatorname{dist}(x,\Gamma_n)\leq\delta-\varepsilon\}$$
$$S_n=\{x\in M:\operatorname{dist}(x,\Gamma_n)<\delta\}$$

and $S=\bigcup S_n$. Since $\delta\leq\frac{\delta_1}{2}$, the set T_n intersects T_k only if $|k-n|\leq 1$. Moreover, the intersections $I_{n,\pm}=T_n\cap T_{n\pm 1}$ contain $e_{n\pm 1}$, and $I_{n,+}\cap I_{n,-}=\emptyset$. By Urysohn's theorem, we find a continuous map $f\colon T_n\to[n,n+1]$ with $f|_{I_{n,-}}=n$ and $f|_{I_{n,+}}=n+1$; by the glueing lemma, f is continuous. For $x\notin S$, we put $f(x)=1$, and by Tietze-Urysohn, we extend f inductively to a continuous map $f\colon\bigcup_{k=1}^n\overline{S}_k\to[1,n+1]$. Hence, $f\colon M\to[1,\infty)$ is continuous.

Since Γ_n are paths, we find a map $g\colon [1,\infty) \to \Gamma$ with $g([n,n+1]) = \Gamma_n$, $g(n) = e_n$, $g(n+1) = e_{n+1}$. We claim that the map $F(x) = g(f(x)+2)$ has the required properties.

For any $x \in M \setminus \bigcup_n T_n$, the relation $F(x) \in \Gamma$ implies $d(F(x),x) > \delta - \varepsilon$. Moreover, if $x \in T_n$, then $f(x) \in [n, n+1]$, and so $F(x) \in \Gamma_{n+2}$ which implies $d(F(x),x) > \delta_1 - \delta \geq \frac{\delta_1}{2} \geq \delta$. Thus, $\operatorname{displ}(F,M) \geq \delta - \varepsilon$.

Now assume that $A \subseteq M$ is such that $\overline{F(A)}$ is not compact. Then $f(A)$ is unbounded, since otherwise $g(\overline{f(A)}+2)$ is compact (as a continuous image of a compact set) and contains $F(A)$. Hence, we find a sequence $x_k \in A$ with $f(x_k) = \alpha_k \to \infty$. By construction, we must have $x_k \in S_{n_k}$ where $n_k \geq \alpha_k - 1$, and by passing to a subsequence, we may assume that $|n_k - n_\ell| \geq 2$ for $k \neq \ell$. In particular, $|x_k - x_\ell| \geq \delta_1 - 2\delta$. Hence, $\beta(A) \geq \delta_1 - 2\delta$.

Finally, assume that $N \subseteq Y$ is a finite set. Given $\varepsilon_1 > 0$ and $y \in N$, the relation $\operatorname{dist}(y, S_k) \leq \delta_2 - \delta - \varepsilon_1$ holds at most for finitely many numbers k. In particular, $d(y,x_k) > \delta_2 - \delta - \varepsilon_1$ for infinitely many k. Since this holds for each $y \in N$, N cannot be a $(\delta_2 - \delta - \varepsilon_1)$-net, and so $\chi_Y(A) \geq \delta_2 - \delta - \varepsilon_1$ ∎

For the rest of this paper, let X be a normed space which does not have finite dimension.

Theorem 2. *Assume that there is a continuous map $F\colon B_r(X) \to B_r(X)$ with positive minimal displacement $\operatorname{displ}(F, B_r(X))$. Then:*

(i) *For each $\varepsilon > 0$ there is a homotopy $H\colon S_r(X) \times [0,1] \to S_r(X)$ joining the identity with a constant map such that, for each $\gamma \in \{\alpha, \beta, \chi_X\}$,*

$$\gamma\big(H(A \times [0,1])\big) \leq \frac{2r}{\operatorname{displ}(F, B_r(X))}\,\gamma(A) + \Big(\frac{2r}{\operatorname{displ}(F, B_r(X))} - 1 + \varepsilon\Big)\gamma\big(F(\operatorname{conv}(A \cup \{0\}))\big)$$

for all $A \subseteq S_r(X)$.

(ii) *For each $\varepsilon > 0$ there is a retraction R of $B_r(X)$ onto $S_r(X)$ such that, for each $\gamma \in \{\alpha, \beta, \chi_X\}$,*

$$\gamma(R(A)) \leq \frac{2r}{\operatorname{displ}(F, B_r(X))}(1+\varepsilon)\,\gamma(A) + \Big(\frac{2r}{\operatorname{displ}(F, B_r(X))} - 1 + \varepsilon\Big)\gamma\Big(F\big(B_r(X) \cap (1+\varepsilon)\operatorname{conv}(A \cup \{0\})\big)\Big)$$

for all $A \subseteq B_r(X)$.

Proof. For fixed $c_1 > 0$ and $c_2 \in (0,1)$ with $0 < 1 - c_1 c_2 < \frac{\operatorname{displ}(F, B_r(X))}{r}$ put

$$H(x,t) = \begin{cases} r\,\dfrac{x - c_1 t F(x)}{\|x - c_1 t F(x)\|} & \text{if } t \leq c_2 \\[2ex] r\,\dfrac{\frac{1-t}{1-c_2}x - c_1 c_2 F\big(\frac{1-t}{1-c_2}x\big)}{\big\|\frac{1-t}{1-c_2}x - c_1 c_2 F\big(\frac{1-t}{1-c_2}x\big)\big\|} & \text{if } t \geq c_2 \end{cases}$$

and note that

$$\|x - c_1 tF(x)\| \geq r - c_1 c_2 r \quad (\text{for } t \leq c_2)$$

and

$$\begin{aligned}\|x - c_1 c_2 F(x)\| &\geq \|x - F(x)\| - \|F(x) - c_1 c_2 F(x)\| \\ &\geq \operatorname{displ}(F, B_r(X)) - (1 - c_1 c_2) r\end{aligned}$$

are always positive. Clearly,

$$\begin{aligned}&\gamma\big(H(A \times [0,1])\big) \\ &\quad \leq r \max \left\{ \frac{\gamma(A) + c_1 \gamma\big(F(\operatorname{conv}(A \cup \{0\}))\big)}{r(1 - c_1 c_2)}, \frac{\gamma(A) + c_1 c_2 \gamma\big(F(\operatorname{conv}(A \cup \{0\}))\big)}{\operatorname{displ}(F, B_r(X)) - (1 - c_1 c_2) r} \right\}\end{aligned}$$

and the minimum of the denominators attains its optimal value

$$\frac{\operatorname{displ}(F, B_r(X))}{2} \qquad \text{if } \ (1 - c_1 c_2) r = \frac{\operatorname{displ}(F, B_r(X))}{2},$$

i.e. if

$$c_1 c_2 = 1 - \frac{\operatorname{displ}(F, B_r(X))}{2r}.$$

If we choose c_2 sufficiently close to 1, we obtain the claimed estimate. The required retraction is given by

$$R(x) = H\left(\frac{x}{\|x\|}, \min\big\{1, k(1 - \|x\|)\big\}\right)$$

with sufficiently large k. Indeed, since $R(x)$ is constant for $\|x\| \leq 1 - \frac{1}{k}$, we have

$$\begin{aligned}\gamma(R(A)) &= \gamma\Big(R\big(A \cap \{x : \|x\| > 1 - \tfrac{1}{k}\}\big)\Big) \\ &\leq \gamma\bigg(H\bigg(\Big(S_r(X) \cap \frac{\operatorname{conv}(A \cup \{0\})}{1 - \frac{1}{k}}\Big) \times [0,1]\bigg)\bigg).\end{aligned}$$

Thus the theorem is proved ∎

Remark 1. If $\operatorname{displ}(F, B_r(X)) = 0$ but F is fixed-point free, then the above constructions (with $c_1 > 1$ and $c_2 = \frac{1}{c_1}$) for H and R still work, but one cannot estimate the γ-Lipschitz constant. These are the constructions mentioned in the introduction.

3. Construction for a general normed space

The following lemma has been proved in [17]:

Lemma 1. *If $U \subseteq X$ is separable, complete, and convex, then for each $\varepsilon > 0$ there is a retraction $R\colon X \to U$ onto U which additionally satisfies $\|x - R(x)\| \le (1+\varepsilon)\mathrm{dist}(x, U)$.*

Using this lemma, we can show now:

Proposition 3. *Let $M \subseteq X$ contain a sphere with radius $r > 0$. Then, for each $\varepsilon > 0$, M has an $(r - \varepsilon, \frac{r-\varepsilon}{2})$-path in this sphere.*

Proof. There is no loss of generality to assume that the center of the sphere S is 0, and it suffices to prove that M has an $(r(1+\varepsilon)^{-1}, 0)$-path S. We choose $e_1 \in S$ arbitrarily and proceed inductively: Assuming that $e_1, \ldots, e_n \in S$ and $\Gamma_1, \ldots, \Gamma_{n-1} \subseteq S$ are already defined, we let U_n denote the linear hull of $e_1, \ldots, e_n$ and put $U_0 = \{0\}$. Let R_n be a retraction onto U_{n-1} with

$$\|x - R_n(x)\| \le (1+\varepsilon_n)\mathrm{dist}(x, U_{n-1})$$

where $\varepsilon_n > 0$ are chosen such that $r(1+\varepsilon_n)^{-1} \ge r - \varepsilon$. Choose some $f_n \notin U_n$. Putting

$$h_n(t) = e_n + t(f_n - e_n) \quad \text{and} \quad u_{n-1}(t) = R(h_n(t)) \qquad (0 \le t \le 1)$$

we define the path Γ_n by

$$g_n(t) = r\,\frac{h_n(t) - u_{n-1}(t)}{\|h_n(t) - u_{n-1}(t)\|}$$

and put $e_{n+1} = \Gamma_n(1)$. Then $\Gamma_n \subseteq U_{n+1} \cap S$, and for each $t \in [0,1]$ and each $u \in U_{n-1}$ we have

$$\begin{aligned}\|g_n(t) - u\| &= r\,\frac{\big\|h_n(t) - u_{n-1}(t) - r^{-1}\|h_n(t) - u_{n-1}(t)\|u\big\|}{\|h_n(t) - u_{n-1}(t)\|}\\ &\ge r\,\frac{\mathrm{dist}(h_n(t), U_{n-1})}{\|h_n(t) - R(h_n(t))\|}\\ &\ge r(1+\varepsilon_n)^{-1}.\end{aligned}$$

Hence, for $k \le n-2$ we have

$$\mathrm{dist}(\Gamma_n, \Gamma_k) \ge \mathrm{dist}(\Gamma_n, U_{n-1}) \ge r(1+\varepsilon_n)^{-1}$$

and the assertion is proved ∎

It is well-known that for a ball (and thus for a sphere) with radius $r > 0$ in X the Kuratowski and Hausdorff measures of non-compactness are always $2r$ and r, respectively (independent of the infinite-dimensional normed space X). Correspondingly, the value of the corresponding lattice measure of non-compactness belongs to $[r, 2r]$ (the precise value depends on X). Using this and recalling that the canonical retraction ρ of X onto a ball never increases any measure of non-compactness (because $\rho(A) \subseteq \overline{\mathrm{conv}}(A \cup \{x_0\})$ where x_0 denotes the center of the ball) we obtain from our previous results the following theorem.

Theorem 3. *Let $M \subseteq X$ contain a sphere S with radius $r > 0$. Let $c := \beta(B_1(X)) \in [1,2]$. Then, for each $\delta \in (0, \frac{r}{2})$ and each $\varepsilon > 0$, the following existence assertions on maps hold true:*

1. *There exists a fixed-point free continuous map $F\colon M \to S \subseteq M$ with* $\operatorname{displ}(F, M) \geq \delta - \varepsilon$ *such that*

$$\left.\begin{aligned} \chi_X(F(A)) &\leq \frac{1}{1-2\frac{\delta}{r}}\,\beta(A) \\ \beta(F(A)) &\leq \frac{c}{1-2\frac{\delta}{r}}\,\beta(A) \\ \alpha(F(A)) &\leq \frac{2}{1-2\frac{\delta}{r}}\,\beta(A) \end{aligned}\right\} \qquad (A \subseteq M).$$

2. *There exists a homotopy $H\colon S \times [0,1] \to S$ joining the identity with a constant map such that*

$$\chi_X\big(H(A \times [0,1])\big) \leq (4+1+\varepsilon)\,\beta(A) \qquad (A \subseteq S). \tag{2}$$

An analogous statement holds (*with a possibly different H*) *if one replaces* (2) *by one of the estimates*

$$\begin{aligned} \beta\big(H(A \times [0,1])\big) &\leq (4+c+\varepsilon)\,\beta(A) \\ \gamma\big(H(A \times [0,1])\big) &\leq (4+2+\varepsilon)\,\gamma(A) \end{aligned} \qquad (A \subseteq S)$$

where $\gamma \in \{\alpha, \beta, \chi_X\}$ is arbitrary but fixed.

3. *There exists a retraction R of X onto S such that*

$$\chi_X(R(A)) \leq (4+1+\varepsilon)\,\beta(A) \qquad (A \subseteq X). \tag{3}$$

An analogous statement holds (*with a possibly different R*) *if one replaces* (3) *by one of the estimates*

$$\begin{aligned} \beta(R(A)) &\leq (4+c+\varepsilon)\,\beta(A) \\ \gamma(R(A)) &\leq (4+2+\varepsilon)\,\gamma(A) \end{aligned} \qquad (A \subseteq X)$$

where $\gamma \in \{\alpha, \beta, \chi_X\}$ is arbitrary but fixed.

Proof. The statement for F is a straightforward calculation. Concerning H, note that our previous results show that for $0 < \varepsilon < \delta < \frac{r}{2}$ a homotopy $H\colon S \to S$ joining the identity with a constant map exists with

$$\gamma\big(H(A \times [0,1])\big) \leq \max\Big\{\frac{2r}{\delta - \varepsilon}\,\gamma(A), \frac{\gamma(S)}{r - 2\delta}\,\beta(A)\Big\} \qquad (A \subseteq S). \tag{4}$$

For $\gamma = \beta$ and $\varepsilon = 0$ (which is not admissible but which we assume by continuity arguments), the optimal choice for δ concerning the constant in (4) would be the value $\frac{r}{2+\frac{\gamma(S_1(X))}{2}}$ (which belongs to $(0, \frac{r}{2})$!) for which the constant becomes $\gamma(S_1(X)) + 4$; using the estimate $\beta(A) \leq \alpha(A)$ respectively $\chi_X(A) \leq \beta(A)$, an analogous calculation works in the cases $\gamma = \alpha$ and $\gamma = \chi_X$. For the case $\gamma = \chi_X$, we find for $\varepsilon = 0$ after the estimate $\beta(A) \leq 2\chi_X(A)$ that the optimal choice for δ would be $\frac{r}{2+\gamma(S_1(X))}$ with corresponding constant $2\gamma(S_1(X)) + 4$. The proof concerning R is similar ∎

The estimates for F imply in particular that for each $\gamma \in \{\alpha, \beta, \chi_X\}$ the estimate

$$\gamma(F(A)) \leq \Big(\frac{1}{2} - \frac{\delta}{r}\Big)^{-1} \gamma(A) \qquad (A \subseteq M) \tag{5}$$

holds.

Corollary 1. *For each infinite-dimensional normed space X, each $L > 1$ and each $\varepsilon > 0$, there is a continuous function $F\colon B_1(X) \to S_1(X)$ such that*

$$\operatorname{displ}\big(F, B_1(X)\big) \geq \frac{1}{2} - \frac{1}{L} - \varepsilon$$

holds where F is simultaneously α-, β-, and γ-Lipschitz with constant (at most) L.

If we could replace here (i.e. in (5)) the constant $\frac{1}{2}$ by 1 (and could put $\varepsilon = 0$), this estimate would complement (1). We prove now that one may actually do this in various classes of spaces (at least for $\gamma = \chi_X$).

4. Separable or reflexive spaces

If the underlying space is separable or has a geometry which allows a reduction to the separable case, we obtain sharper results.

Proposition 4. *Let $M \subseteq X$ contain a sphere S of radius $r > 0$. Then, for each $\varepsilon > 0$ and each separable subspace $Y \subseteq X$, the set M has an $(r - \varepsilon, r)$-path in $S \cap Y$ with respect to Y.*

Proof. The construction of the path is analogous to the construction in the proof of Proposition 3 with the difference that we assume in addition that $\varepsilon_n \to 0$ and that $\operatorname{span}\{e_1, e_2, \ldots\}$ is dense in Y. The latter can indeed be arranged: By the separability of Y there exist finite-dimensional subspaces $U_1 \subseteq U_2 \subseteq \ldots \subseteq Y$ with $Y \subseteq \overline{\bigcup_n U_n}$ (let, e.g., $U_n = \operatorname{span}\{y_1, \ldots, y_n\}$ where $\{y_1, y_2, \ldots\}$ is dense in Y). We may assume that $\dim U_n = n$, and in the construction of Proposition 3, we choose then $f_n \in U_{n+1} \setminus U_n$.

To see that Γ is an $(r - \varepsilon, r)$-path with respect to Y, let $y \in Y$ and $\delta > 0$ be given. We find some $N > 1$ with $\operatorname{dist}(y, U_{N-1}) < \delta$ and $r(1 + \varepsilon_N)^{-1} > r - \delta$. For all $n \geq N$ our construction implies $\operatorname{dist}(\Gamma_n, U_{N-1}) \geq r(1 + \varepsilon_n)^{-1} > r - \delta$. Hence, $\operatorname{dist}(y, \Gamma_n) > r - 2\delta$ for all $n \geq N$ ∎

Definition 3. A normed space X has the *separable retraction property* if there is a separable subspace Y which does not have finite dimension such that for each $\varepsilon > 0$ we find a mapping $R\colon X \to Y$ which satisfies

$$\|R(x) - R(y)\| \leq (1 + \varepsilon)\|x - y\| + \varepsilon \qquad (x \in X,\, y \in B_1(Y)) \tag{6}$$

and

$$\|R(y) - y\| \leq \varepsilon \qquad (y \in B_1(Y)). \tag{7}$$

Of course, each (infinite-dimensional) separable space has the separable retraction property. More examples have been given in [17]. In particular, if we assume the axiom of choice, then each weakly compactly generated Banach space (and thus each reflexive space) has the separable retraction property.

Corollary 2. *Let X have the separable retraction property, and let $M \subseteq X$ contain a sphere S of radius $r > 0$. Then for each $\varepsilon > 0$ the set M has an $(r-\varepsilon, r)$-path in S with respect to X.*

Proof. Let Y be as above, and let Γ be an $(r-\varepsilon, r)$-path with respect to Y (Proposition 4). We claim that Γ is an $(r-\varepsilon, r)$-path with respect to X. Thus, let some $x \in X$ and $\delta > 0$ be given. We find a mapping $R\colon X \to Y$ which satisfies

$$\begin{aligned} \|R(x) - R(y)\| &\leq (1+\delta)\|x-y\| + \delta \\ \|R(y) - y)\| &\leq \delta \end{aligned}$$

for all $x \in X$ and all $y \in S \cap Y$. Since $\operatorname{dist}(R(x), \Gamma_n) \geq r - \delta$ for almost all n, we have in view of $\Gamma_n \subseteq S \cap Y$ that

$$\begin{aligned} \operatorname{dist}(x, \Gamma_n) &\geq (1+\delta)^{-1} \operatorname{dist}\big(R(x), R(\Gamma_n)\big) - \delta \\ &\geq (1+\delta)^{-1}\big(\operatorname{dist}(R(x), \Gamma_n) - \delta\big) - \delta \\ &\geq (1+\delta)^{-1}(r - 2\delta) - \delta \end{aligned}$$

for almost all n. Since the right-hand side is as close to r as we want (for sufficiently small $\delta > 0$), the claim follows ∎

Together with our previous results, we obtain:

Theorem 4. *Let X have the separable retraction property, and put $c = \beta(B_1(X)) \in [1, 2]$. Let $M \subseteq X$ contain a sphere S with radius $r > 0$. Then, for each $\delta \in (0, \frac{r}{2})$, each $\varepsilon > 0$, and each $\gamma \in \{\alpha, \beta, \chi_X\}$, the following existence assertions on maps hold true:*

1. *There exists a fixed-point free continuous map $F\colon M \to S \subseteq M$ with* $\operatorname{displ}(F, B_r(X)) \geq \delta - \varepsilon$ *such that*

$$\left.\begin{aligned} \chi_X(F(A)) &\leq \frac{1}{1-\frac{\delta}{r}}\,\chi_X(A) \\ \beta(F(A)) &\leq \frac{c}{1-\frac{\delta}{r}}\,\chi_X(A) \\ \alpha(F(A)) &\leq \frac{2}{1-\frac{\delta}{r}}\,\chi_X(A) \end{aligned}\right\} \qquad (A \subseteq M).$$

2. *There exists a homotopy $H\colon S \times [0,1] \to S$ joining the identity with a constant map such that*

$$\gamma\big(H(A \times [0,1])\big) \leq (4+\varepsilon)\,\gamma(A) \qquad (A \subseteq S).$$

3. *There exists a retraction R of X onto S such that*

$$\gamma(R(A)) \leq (4+\varepsilon)\,\gamma(A) \qquad (A \subseteq X).$$

Proof. We use essentially the same calculation as in the proof of Theorem 3. The most remarkable difference is that (4) can be replaced by

$$\gamma\big(H(A \times [0,1])\big) \leq \max\Big\{\frac{2r}{\delta - \varepsilon}\,\gamma(A), \frac{\gamma(S)}{r-\delta}\,\chi_X(A)\Big\} \qquad (A \subseteq S). \tag{8}$$

We estimate $\chi_X(A) \le \gamma(A)$ in (8). For $\varepsilon = 0$ the best possible choice for δ would be $(1 + \frac{\gamma(S_1(X))}{2})^{-1} r$ which unfortunately does not belong to $(0, \frac{r}{2})$. However, this calculation shows that δ should be chosen as large as possible. For δ close to $\frac{r}{2}$ the estimate in the claim is obtained ∎

Corollary 3. *Let the normed space X have the separable retraction property. Then, for each $L > 1$ and each $\varepsilon > 0$, there exists a continuous function $F\colon B_1(X) \to S_1(X)$ with*

$$\operatorname{displ}\bigl(F, B_1(X)\bigr) \ge 1 - \frac{1}{L} - \varepsilon$$

where F is χ_X-Lipschitz with constant (at most) L.

5. Spaces which contain a copy of l_p or c_0

By $c_{\mathrm{fin},p}$ we denote the (incomplete) space of eventually zero sequences, endowed with the p-norm. Many spaces contain an isometric copy of some $c_{\mathrm{fin},p}$, for example Hilbert spaces (put $p = 2$), $L_p(\mu)$ (if it is infinite-dimensional), $C([0,1])$, or c_0 (put $p = \infty$). For such spaces our approach works best.

Lemma 2. *Let X contain an isometric copy of $c_{\mathrm{fin},p}$ for some $p \in [1,\infty]$. Let $M \subseteq X$ contain a sphere S of radius $r > 0$. Then M has a $(2^{\frac{1}{p}} r - \varepsilon, r)$-path $\Gamma \subseteq S$ with respect to $Y := X$ (simultaneously for each $\varepsilon > 0$) such that* $\operatorname{diam}(\Gamma) = 2^{\frac{1}{p}} r$. *In particular, $\beta(\Gamma) = \alpha(\Gamma) = 2^{\frac{1}{p}} r$.*

Proof. We may assume that $X = c_{\mathrm{fin},p}$ and that S has center 0. Let $e_n \in S$ be the canonical base vectors (with norm r) of $c_{\mathrm{fin},p}$. We let Γ_n be determined by the paths $g_n(t) = \frac{r h_n(t)}{\|h_n(t)\|}$ in S where $h_n(t) = e_n + t(e_{n+1} - e_n)$ $(0 \le t \le 1)$. For $|k - n| \ge 2$ the vectors $h_n(t)$ and $h_k(s)$ have disjoint "support" (if we think of $c_{\mathrm{fin},p}$ as a space of functions $\mathbb{N} \to \mathbb{R}$). Since they have norm r, it follows that $\|h_n(t) - h_k(s)\| = 2^{\frac{1}{p}} r$. Moreover, since the vectors $h_n(t)$ and $h_k(s)$ correspond to non-negative functions $\mathbb{N} \to \mathbb{R}$, we have also in the case $|k - n| \le 1$ the estimate $\|h_n(t) - h_k(s)\| \le 2^{\frac{1}{p}} r$. Hence, $\operatorname{dist}(\Gamma_n, \Gamma_k) \ge 2^{\frac{1}{p}} r$ for $|k - n\| \ge 2$, and the path $\Gamma = \bigcup_n \Gamma_n$ satisfies $\operatorname{diam}(\Gamma) = 2^{\frac{1}{p}} r$.

Given some $x \in X$ and some $\varepsilon > 0$, since x is a null sequence and $g_n(t)$ (considered as a function $\mathbb{N} \to \mathbb{R}$) has its "support" at $\{n, n+1\}$ and norm r, we have $\operatorname{dist}(x, g_n(t)) \ge r - \varepsilon$ if only n is sufficiently large. Hence, $\operatorname{dist}(x, \Gamma_n) \ge r - \varepsilon$ for almost all n ∎

Theorem 5. *Let X contain an isometric copy of $c_{\mathrm{fin},p}$ for some $p \in [1,\infty]$, and put $c = \beta(S_1(X))$. Let $M \subseteq X$ contain a sphere S of radius $r > 0$. Then, for each $\delta \in (0, 2^{\frac{1}{p}-1} r]$ and each $\varepsilon > 0$, the following existence assertions on maps hold true:*

1. *There exists a fixed-point free continuous map $F\colon M \to S \subseteq M$ with* displ$(F,$

$B_r(X)) > \delta - \varepsilon$ *such that*

$$\left.\begin{aligned} \chi_X(F(A)) &\le \min\left\{\frac{1}{1-\frac{\delta}{r}}\,\chi_X(A), \frac{1+\varepsilon}{2^{\frac{1}{p}}-2\frac{\delta}{r}}\,\beta(A)\right\} \\ \alpha(F(A)) &\le \min\left\{\frac{2^{\frac{1}{p}}}{1-\frac{\delta}{r}}\,\chi_X(A), \frac{1+\varepsilon}{1-2^{1-\frac{1}{p}}\frac{\delta}{r}}\,\beta(A)\right\} \end{aligned}\right\} \qquad (A \subseteq M).$$

2. *There exists a homotopy* $H\colon S \times [0,1] \to S$ *joining the identity with a constant map such that*

$$\chi_X\big(H(A\times[0,1])\big) \le \big(\max\{2+1, 2^{2-\frac{1}{p}}\}+\varepsilon\big)\,\chi_X(A) \qquad (A \subseteq S). \tag{9}$$

An analogous statement holds (*with a possibly different* H) *if one replaces* (9) *by one of the estimates*

$$\left.\begin{aligned} &\beta\big(H(A\times[0,1])\big) \le \big(\max\{2+c, 2^{2-\frac{1}{p}}\}+\varepsilon\big)\,\beta(A) \\ &\alpha\big(H(A\times[0,1])\big) \le (4+\varepsilon)\,\alpha(A) \end{aligned}\right. \qquad (A \subseteq S).$$

3. *There exists a retraction* R *of* X *onto* S *such that*

$$\chi_X(R(A)) \le \big(\max\{2+1, 2^{2-\frac{1}{p}}\}+\varepsilon\big)\,\chi_X(A) \qquad (A \subseteq X). \tag{10}$$

An analogous statement holds (*with a possibly different* R) *if one replaces* (10) *by one of the estimates*

$$\left.\begin{aligned} &\beta(R(A)) \le \big(\max\{2+c, 2^{2-\frac{1}{p}}\}+\varepsilon\big)\,\beta(A) \\ &\alpha(R(A)) \le (4+\varepsilon)\,\alpha(A) \end{aligned}\right. \qquad (A \subseteq X).$$

Proof. Since for F as in Theorem 1 corresponding to the previously constructed path Γ the inclusion $F(A) \subseteq \Gamma$ holds, we have

$$\gamma(F(A)) \le \gamma(\Gamma)\,\min\left\{\frac{\chi_X(A)}{r-\delta}, \frac{\beta(A)}{2^{\frac{1}{p}}r-2\delta-\varepsilon}\right\}.$$

For the corresponding homotopy H of Theorem 2 we have (since $\gamma(F(A)) = 0$ for $\chi_X(A) \le r-\delta$ or $\beta(A) < 2^{\frac{1}{p}}r - 2\delta - \varepsilon$):

$$\begin{aligned} &\gamma\big(H(A\times[0,1])\big) \\ &\quad\le \min\Bigg\{\max\left\{\frac{2r}{\delta-\varepsilon}\,\gamma(A), \min\left\{\frac{\gamma(S)\chi_X(A)}{r-\delta}, \frac{\gamma(S)\beta(A)}{2^{\frac{1}{p}}r-2\delta-\varepsilon}\right\}\right\}, \\ &\qquad\frac{2r}{\delta-\varepsilon}\,\gamma(A) + \Big(\frac{2r}{\delta-\varepsilon}-1\Big)\,\gamma(\Gamma)\,\min\left\{\frac{\chi_X(A)}{r-\delta}, \frac{\beta(A)}{2^{\frac{1}{p}}r-2\delta-\varepsilon}\right\}\Bigg\}. \end{aligned}$$

For $\gamma = \chi_X$, we estimate throughout $\beta(A) \le 2\chi_X(A)$ and find the bound $C\chi_X(A)$ where

$$C = \min\big\{\max\{C_0, C_1\}, C_2\big\}$$

with

$$C_0 = \frac{2r}{\delta - \varepsilon}, \quad C_1 = \frac{1}{1 - \frac{\delta}{r}}, \quad C_2 = C_0 + (C_0 - 1)C_1.$$

In the case $2^{\frac{1}{p}-1} \geq \frac{2}{3}$ the optimal choice concerning $\max\{C_0, C_1\}$ would be $\delta = \frac{2}{3}r$ and $\varepsilon = 0$ in which case $C_0 = C_1 = 3$. In the case $2^{\frac{1}{p}-1} < \frac{2}{3}$ the optimal choice would be $\delta = 2^{\frac{1}{p}-1}r$ and $\varepsilon = 0$ for which $C_1 \leq C_0 = 2^{2-\frac{1}{p}} \geq 3$, and we obtain (9). Note that we have in view of $C_0 - 1 > 1$ always $C_2 \geq C_0 + C_1$, and so the quantity C_2 does never improve this estimate.

For $\gamma = \beta$ and $\gamma = \alpha$ we introduce the shortcut $\gamma_0 = \gamma(S_1(X))$ and estimate throughout $\chi_X(A), \beta(A) \leq \gamma(A)$ to find the bound $D\gamma(A)$ where

$$D = \min\big\{ \max\{C_0, D_1\}, \max\{C_0, D_2\}, D_3\big\}$$

with

$$D_1 = \frac{\gamma_0}{1 - \frac{\delta}{r}}, \quad D_2 = \frac{\gamma_0}{2^{\frac{1}{p}} - 2\frac{\delta}{r} - \varepsilon}, \quad D_3 = C_0 + (C_0 - 1)\, \min\{D_1, D_2\}\, \frac{2^{\frac{1}{p}}}{\gamma_0}.$$

Similarly, a straightforward calculation shows that the quantity $\max\{C_0, D_1\}$ attains its minimal value for $\delta \in (0, 2^{\frac{1}{p}-1}]$ for the choice $\delta = \min\big\{(1 + \frac{\gamma_0}{2})^{-1}r, 2^{\frac{1}{p}-1}r\big\}$ in which case $D_1 \leq C_0 = \max\{2 + \gamma_0, 2^{2-\frac{1}{p}}\}$. Moreover, $\max\{C_0, D_2\}$ attains for the choice $\delta = 2^{\frac{1}{p}}(\gamma_0 + 2)r$ (in $(0, 2^{\frac{1}{p}-1}r)$) and $\varepsilon = 0$ its minimal value $(4+\gamma_0)2^{-\frac{1}{p}}$ which is a worse estimate than $2^{2-\frac{1}{p}}$. The proof concerning R is similar.

Theorems 4 and 5 contain the following complement of (1) for a large class of spaces.

Corollary 4. *Let the normed space X have the separable retraction property or let it contain an isometric copy of $c_{\mathrm{fin},p}$ $(1 \leq p \leq \infty)$. Then, for each $L \in (1,2)$ (respectively even for each $L \in (1, (1 - 2^{\frac{1}{p}-1})^{-1}]$) and each $\varepsilon > 0$, there exists a continuous function $F\colon B_1(X) \to S_1(X)$ with*

$$\mathrm{displ}(F, B_1(X)) \geq 1 - \frac{1}{L} - \varepsilon$$

where F is χ_X-Lipschitz with constant (at most) L. Moreover, if X contains an isometric copy of $c_{\mathrm{fin},\infty}$, it may be arranged that F is also simultaneously α-Lipschitz and β-Lipschitz with constant (at most) L.

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