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# Non-Compact $\lambda$-Hankel Operators

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**Abstract.** A $\lambda$-Hankel operator $X$ is a bounded operator on Hilbert space satisfying the operator equation $S^*X - XS = \lambda X$, where $S$ is the (unilateral) forward shift and $S^*$ is its adjoint. We prove that there are non-compact $\lambda$-Hankel operators for $\lambda$ a complex number of modulus less than 2, by first exhibiting a way to obtain bounded solutions to the above equation by associating to it a Carleson measure. We then show that an interpolating sequence can be given such that the $\lambda$-Hankel operator associated with the Carleson measure given by the interpolating sequence is non-compact.

**Keywords:** *Hankel operators, generalizations, Carleson measures, interpolating sequences*

**AMS subject classification:** Primary 47B35, secondary 47A62, 30E05

## 1. Introduction

In [3, 4] a class of operators, called $\lambda$-Hankel operators, is defined as those operators $X$ satisfying the operator equation

$$S^*X - XS = \lambda X \tag{1.1}$$

where $S$ is the unilateral forward shift on $\ell^2$, $S^*$ is its adjoint (usually known as the backward shift), and $\lambda$ is a complex number. Clearly, if $\lambda = 0$, the solutions to equation (1.1) are precisely the Hankel operators. Bounded $\lambda$-Hankel operators can be seen to have many of the properties of Hankel operators [3, 4], but many basic questions about them remain unanswered. For the basic facts about Hankel operators, the reader should consult [5, 6]. For a survey of recent advances on Hankel operators, the reader should see [7].

It is surprising that there are non-compact solutions to equation (1.1) (see below). Many similar-looking equations can be seen to have only solutions which are compact or solutions which are unitary multiples of Hankel operators, if they have non-zero solutions at all. The study of solutions of an equation of the above type is motivated by a (slightly different) question posed by Barría and Halmos ]1], answered by Sun [8]. It is hoped that the study of $\lambda$-Hankel operators will increase our understanding of Hankel operators. For example, as is shown in [4], some spectral properties of Hankel operators are shared by $\lambda$-Hankel operators, which suggests that a more general theory of generalizations of these types of operators may exist. Also, it would be

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interesting to see what type of function spaces one can obtain from studying the smoothness properties of $\lambda$-Hankel operators and how they generalize the function spaces related to Hankel operators.

If $\lambda$ has modulus less than 2, non-zero compact solutions to equation (1.1) are known to exist (see [3, 4]). For the cases where $\lambda$ has modulus equal to 1, or where $\lambda$ is purely imaginary and of modulus less than 2, it was also shown in [3, 4] that there are bounded, non-compact solutions to equation (1.1).

In this paper we will show that in fact, if $\lambda$ has modulus less than 2, there exist bounded non-compact solutions. This completely solves the question of existence of non-compact $\lambda$-Hankel operators, since it is also known that if $|\lambda| \geq 2$, there are no non-zero bounded solutions to the equation above. The case $|\lambda| > 2$ is an observation in [3, 4]. The case $|\lambda| = 2$ is due to L. Robert-Gonzalez and, independently, to A. Feintuch and A. Markus (personal communications).

First, we introduce some definitions. We will work on the Hardy space $\mathbb{H}^2$, the space of analytic functions on the unit disk $\mathbb{D}$ defined as

$$\mathbb{H}^2 = \Big\{ f(z) = \sum_{k=0}^{\infty} a_k z^k : \sum_{k=0}^{\infty} |a_k|^2 < \infty \Big\}.$$

The Hardy space is a Hilbert space with the inner product of two functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ defined by $(f,g) = \sum_{k=0}^{\infty} a_k \overline{b_k}$. The norm of $f \in \mathbb{H}^2$ is denoted by $\|f\|_2$. Clearly, there is a natural identification of this space with the classical space $\ell^2$ and we will think of $\mathbb{H}^2$ as $\ell^2$ occasionally. We denote the norm in $\ell^2$ as $\|\cdot\|$. The canonical basis in $\mathbb{H}^2$ consists of the functions $e_n(z) = z^n$ for $n \in \mathbb{N}_0$. The unilateral forward shift, simply called from now on the shift, is defined as $(Sf)(z) = zf(z)$ for every $f \in \mathbb{H}^2$. The only property of the shift that we will use is the fact that $Se_n = e_{n+1}$ for all $n$. The notation $h^* \in \mathbb{H}^2$ defined by $h^*(z) = \overline{h(\bar{z})}$ for a function $h \in \mathbb{H}^2$ will also be useful.

A finite positive measure $\mu$ on the unit disk $\mathbb{D}$ is called a *Carleson measure* if for $0 < h < 1$ and for every set of the form

$$\Omega_h = \Big\{ z = re^{i\theta} \in \mathbb{D} : 1 - h \leq r < 1 \text{ and } \theta_0 \leq \theta \leq \theta_0 + h \Big\},$$

where $\theta_0 \in [0, 2\pi]$, there is a constant $A$ (independent of $h$ and of $\theta_0$) such that $\mu(\Omega_h) \leq Ah$. The sets $\Omega_h$ are called *Carleson sets* (centered at $\theta_0$).

A classical theorem of Carleson (a proof can be found in Duren's book [2: pp. 157 – 162]) is the following

**Theorem 1.1** (Carleson). *Let $\mu$ be a finite measure on $\mathbb{D}$. Then $\mu$ is a Carleson measure if and only if there exists a constant $C$ such that*

$$\Big( \int_{\mathbb{D}} |f(z)|^2 d\mu(z) \Big)^{1/2} \leq C\|f\|_2$$

*for all $f \in \mathbb{H}^2$.*

We will also need the concept of an interpolating sequence. A sequence $\{z_k\}_{k=0}^{\infty}$ in $\mathbb{D}$ is called an interpolating sequence if the operator $T_2 : \mathbb{H}^2 \longrightarrow \ell^2$ defined as

$$T_2(f) = \left\{(1-|z_k|^2)^{\frac{1}{2}} f(z_k)\right\}_{k=0}^{\infty}$$

is bounded and surjective. A sequence $\{z_k\}_{k=0}^{\infty}$ in $\mathbb{D}$ is called *uniformly separated* if there exists $\delta > 0$ such that

$$\prod_{j=0, j\neq k}^{\infty} \left|\frac{z_k - z_j}{1-\overline{z_j} z_k}\right| \geq \delta \qquad (k \in \mathbb{N}_0).$$

A theorem of Shapiro and Shields (see Duren's book [2: p. 149]) states that a sequence is uniformly separated if and only if it is an interpolating sequence. There are simple conditions which guarantee that interpolating sequences exist in abundance, for example, $\{z_k\}_{k=0}^{\infty}$ is a uniformly separated sequence if there exists a constant $c < 1$ with

$$1 - |z_{k+1}| \leq c\,(1 - |z_k|) \tag{1.2}$$

for $k \in \mathbb{N}_0$ (see Duren's book [2: p. 155] for references).

## 2. Construction of bounded λ-Hankel operators

In this section we will see how a Carleson measure defines a bounded $\lambda$-Hankel operator. From now on let us assume that $|\lambda| < 2$.

Let $R := \{z \in \mathbb{D} : |z+\lambda| < 1\}$ and let $\mu$ be a Carleson measure on $\mathbb{D}$ supported on $R \subset \mathbb{D}$, with the extra property that $\mu(\cdot - \lambda)$ is also a Carleson measure on $\mathbb{D}$ supported in $R + \lambda \subset \mathbb{D}$. That is, we can think of $\mu$ as a Carleson measure on the disk $\mathbb{D}$ and also on the disk $\mathbb{D} - \lambda$.

For integers $m \geq 0$ and $n \geq 0$ define an operator $X$ by

$$(Xe_m, e_n) = \int_R z^m (z+\lambda)^n d\mu(z). \tag{2.1}$$

Clearly, the right-hand side of this equation is well defined. Therefore, $X$ is a densely-defined operator (on polynomials).

We can now show the following

**Theorem 2.1.** *Let $|\lambda| < 2$ and let $X$ be the densely defined operator given by equation* (2.1) *where $\mu$ is a Carleson measure on $\mathbb{D}$ supported on $R$ and $\mu(\cdot - \lambda)$ is a Carleson measure on $\mathbb{D}$ supported on $R+\lambda$. Then $X$ is a bounded $\lambda$-Hankel operator.*

**Proof.** Assume for a moment that equation (2.1) defines a bounded operator. Then, to show that it is a $\lambda$-Hankel operator, it is enough to show that $(S^*X -$

$XS)e_m = \lambda X e_m$ for $m \geq 0$. But this holds since

$$\begin{aligned}\big((S^*X - XS)e_m, e_n\big) &= (Xe_m, e_{n+1}) - (Xe_{m+1}, e_n)\\ &= \int_R z^m (z+\lambda)^{n+1} d\mu(z) - \int_R z^{m+1}(z+\lambda)^n d\mu(z)\\ &= \int_R \big(z^m(z+\lambda)^{n+1} - z^{m+1}(z+\lambda)^n\big) d\mu(z)\\ &= \int_R \lambda z^m (z+\lambda)^n d\mu(z)\\ &= (\lambda X e_m, e_n)\end{aligned}$$

holds for every $n \geq 0$.

Let us show now that the operator defined by equation (2.1) is bounded. Let $f(z) = \sum_{k=0}^N a_k z^k$ and $g(z) = \sum_{j=0}^M b_j z^j$ be polynomials. Then

$$\begin{aligned}(Xf, g^*) &= \sum_{k=0}^N \sum_{j=0}^M a_k b_j (Xe_k, e_j)\\ &= \sum_{k=0}^N \sum_{j=0}^M a_k b_j \int_R z^k (z+\lambda)^j d\mu(z)\\ &= \int_R \left(\sum_{k=0}^N a_k z^k\right)\left(\sum_{j=0}^M b_j (z+\lambda)^j\right) d\mu(z)\\ &= \int_R f(z) g(z+\lambda)\, d\mu(z).\end{aligned}$$

But clearly, we also have

$$\begin{aligned}\left|\int_R f(z)g(z+\lambda)\, d\mu(z)\right| &\leq \int_R \big|f(z)g(z+\lambda)\big| d\mu(z)\\ &\leq \left(\int_R |f(z)|^2 d\mu(z)\right)^{1/2} \left(\int_R |g(z+\lambda)|^2 d\mu(z)\right)^{1/2}.\end{aligned}$$

Putting these two last equations together we get

$$|(Xf, g^*)| \leq \left(\int_R |f(z)|^2 d\mu(z)\right)^{1/2} \left(\int_R |g(z+\lambda)|^2 d\mu(z)\right)^{1/2}.$$

But since $\mu(\cdot)$ and $\mu(\cdot - \lambda)$ are both Carleson measures, we can apply Carleson's theorem (after a simple change of variables on the second integral) to get $|(Xf, g^*)| \leq C_1 C_2 \|f\|_2 \|g\|_2$ for some constants $C_1 > 0$ and $C_2 > 0$. Since $\|g^*\|_2 = \|g\|_2$, we have $|(Xf, g^*)| \leq C\|f\|_2 \|g^*\|_2$ for arbitrary polynomials $f$ and $g$ and a constant $C > 0$. Therefore $X$ is bounded ∎

## 3. Measures resulting in non-compact operators

In this section we will use an interpolating sequence to construct a measure whose associated $\lambda$-Hankel operator will not be compact. Assume from now on that $\lambda \neq 0$ (the case $\lambda = 0$ is well-known [6]).

Define $L \subset R$ as $L = \{z \in \mathbb{D} : |z| = |z+\lambda|\}$. The set $L$ is just the straight line joining the intersection points of $\partial\mathbb{D}$ and $\partial\mathbb{D} - \lambda$, and clearly $L$ is contained in $R$. Let $z_\infty$ be one of the two intersection points of $\partial\mathbb{D}$ and $\partial\mathbb{D} - \lambda$. Clearly, $|z_\infty| = |z_\infty + \lambda| = 1$ and $z_\infty \in \partial L$. Let us choose a uniformly separated sequence $\{z_k\}_{k=0}^\infty$ in $L$ such that $z_k \to z_\infty$ as $k \to \infty$. This is clearly possible. In fact, let us choose the sequence such that it satisfies condition (1.2). Clearly, $|z_{k+1}| > |z_k|$ for all $k$.

The statement of the following proposition seems to be folklore and the type of argument in the proof is used frequently in theorems about Carleson measures coming from interpolating sequences. We include a proof here for completeness.

**Proposition 3.1.** *Let $z_\infty$ and $\{z_k\}_{k=0}^\infty$ defined as above. Define a sequence $\mu_k$ as $\mu_k = |z_\infty - z_k| = |(z_\infty + \lambda) - (z_k + \lambda)|$ for $k \in \mathbb{N}_0$, and define a measure $\mu$ on $\mathbb{D}$, supported on the set $\{z_k\}_{k=0}^\infty$, as $\mu(z_k) = \mu_k$. Then $\mu$ is a Carleson measure supported on the set $\{z_k\}_{k=0}^\infty$ and $\mu(\cdot - \lambda)$ is a Carleson measure supported on the set $\{z_k + \lambda\}_{k=0}^\infty$.*

**Proof.** A geometrical argument shows that, if $|z_k| < |z_{k+1}|$ and $z_k, z_{k+1} \in L$, then

$$\frac{|z_\infty - z_{k+1}|}{|z_\infty - z_k|} \leq \frac{1 - |z_{k+1}|}{1 - |z_k|}.$$

Alternatively, one can show by Calculus techniques that the real-valued function

$$f(t) = \frac{|z_\infty - ((1-t)z_k + tz_\infty)|}{1 - |(1-t)z_k + tz_\infty|}$$

is decreasing for $t \in [0,1]$; the result then follows by noticing that $z_{k+1} = (1-t_0)z_k + t_0 z_\infty$ for some $t_0 \in (0,1)$. Therefore, since $\{z_k\}_{k=0}^\infty$ satisfies condition (1.2),

$$|z_\infty - z_{k+1}| \leq c|z_\infty - z_k| \tag{3.1}$$

and thus, for all $k$, $|z_\infty - z_k| \leq c^k |z_\infty - z_0|$. Therefore

$$\sum_{k=0}^\infty \mu_k = \sum_{k=0}^\infty |z_\infty - z_k| \leq \frac{|z_\infty - z_0|}{1-c} < \infty.$$

This proves that the measure $\mu$ is finite.

To see that $\mu$ is a Carleson measure, let $0 < h < 1$ and let $\Omega_h$ be the corresponding Carleson set centered at $\theta_0$. It suffices to consider $h$ small enough, and we do so. If $\theta_0$ is such that $\Omega_h \cap \{z_k\}_{k\geq 0} = \emptyset$, then $\mu(\Omega_h) = 0$ and there is nothing to prove. If $\theta_0$ is such that $\Omega_h \cap \{z_k\}_{k\geq 0} \neq \emptyset$, then clearly $\Omega_h \cap L$ is a connected (non-empty) segment (recall we are assuming $h$ is small), and thus there must exist natural numbers $k_0$ and

$k_1$ (where we allow the case $k_1 = \infty$) such that $z_k \in \Omega_h$ for all $k \in [k_0, k_1)$ and $z_k \notin \Omega_h$ for $k \notin [k_0, k_1)$. From equation (3.1) we obtain that $|z_\infty - z_k| \le c^{k-k_0}|z_\infty - z_{k_0}|$ for $k \ge k_0$. Then

$$\mu(\Omega_h) = \sum_{k=k_0}^{k_1-1} \mu_k \le \sum_{k=k_0}^{\infty} \mu_k \le \sum_{k=k_0}^{\infty} c^{k-k_0}|z_\infty - z_{k_0}| = \frac{|z_\infty - z_{k_0}|}{1-c}.$$

If $k_1 = \infty$, since the length of any straight line inside $\Omega_h$ is at most $4h$, it then follows that $|z_\infty - z_{k_0}| \le 4h$. Therefore, $\mu(\Omega_h) \le \frac{4}{1-c}h$, i.e. $\mu$ is a Carleson measure.

If $k_1 < \infty$, then there exists a constant $d$, depending on $\lambda$ only, such that $|z_\infty - z_{k_0}| \le dh$. Indeed, analogously to what we did at the beginning of this proof, notice that the real-valued function

$$f(t) = \frac{|z_\infty - ((1-t)z_0 + tz_\infty)|}{1 - |(1-t)z_0 + tz_\infty|}$$

is bounded by $d$ for $t \in [0,1]$; Therefore $\frac{|z_\infty - z_{k_0}|}{1-|z_{k_0}|} \le d$. Since clearly $1 - |z_{k_0}| \le h$, it follows that $|z_\infty - z_{k_0}| \le dh$. Therefore, $\mu(\Omega) \le \frac{d}{1-c}h$, i.e. $\mu$ is a Carleson measure.

To prove that $\mu(\cdot - \lambda)$ is a Carleson measure on $\mathbb{D}$ supported on $\{z_k + \lambda\}_{k=0}^{\infty}$, we just need to notice that the argument given above applies word by word to $\mu(\cdot - \lambda)$ and $\{z_k + \lambda\}_{k=0}^{\infty}$, since $|z_\infty + \lambda| = 1$, $\mu_k = |(z_\infty + \lambda) - (z_k + \lambda)|$ and $|z_k| = |z_k + \lambda|$ ∎

Let $X$ as in Theorem 2.1. If $f$ and $g$ are in $\mathbb{H}^2$, and since $X$ is bounded, one can use an argument similar to the one in the proof of Theorem 2.1 to prove that

$$(Xf, g^*) = \int_R f(z)g(z+\lambda)\, d\mu(z).$$

If $\mu$ is the measure given by Proposition 3.1, the integral becomes a sum and the above equation becomes

$$(Xf, g^*) = \sum_{k=0}^{\infty} f(z_k)g(z_k + \lambda)\mu_k. \tag{3.2}$$

We can now prove the main theorem.

**Theorem 3.2.** *Let $|\lambda| < 2$ and let $X$ be the bounded $\lambda$-Hankel operator given by the Carleson measure $\mu$, as in* (2.1), *where $\mu$ is the Carleson measure defined in Proposition* 3.1. *Then $X$ is a bounded non-compact $\lambda$-Hankel operator.*

**Proof.** That $X$ is bounded follows from Proposition 3.1. To check non-compactness requires the following construction. Since $\{z_k\}_{k=0}^{\infty}$ is uniformly separated, it follows by the Shapiro-Shields Theorem that it is an interpolating sequence. That is, the mapping $T_2 : \mathbb{H}^2 \longrightarrow \ell^2$ given by

$$T_2(f) = \left\{(1-|z_k|^2)^{1/2} f(z_k)\right\}$$

is bounded and surjective. This implies by the open mapping theorem that there exists a constant $c_1 > 0$ such that, for every sequence $\{x_k\} \in \ell^2$, there is a function $h \in \mathbb{H}^2$ with $T_2 h = \{x_k\}$ and $\|h\|_2 \leq c_1 \|\{x_k\}\|$. Therefore, for $n \geq 0$, given the $\ell^2$ sequence $\big\{(1-|z_k|^2)^{1/2} \frac{\delta_{n,k}}{\sqrt{\mu_n}}\big\}_{k=0}^{\infty}$ there exists a function $f_n \in \mathbb{H}^2$ with

$$f_n(z_k) = \begin{cases} 0 & \text{if } k \neq n \\ \frac{1}{\sqrt{\mu_n}} & \text{if } k = n \end{cases} \tag{3.3}$$

and

$$\|f_n\|_2 \leq c_1 \left\| \left\{ (1-|z_k|^2)^{1/2} \frac{\delta_{n,k}}{\sqrt{\mu_n}} \right\} \right\|.$$

That is, $\|f_n\|_2 \leq c_1 \big(\frac{1-|z_n|^2}{\mu_n}\big)^{1/2}$, and since $\frac{1-|z_n|^2}{\mu_n} = \frac{1-|z_n|}{|z_\infty - z_n|}(1+|z_n|) \leq 1 \cdot 2 = 2$, it follows that

$$\|f_n\|_2 \leq \sqrt{2} c_1 \tag{3.4}$$

for all $n$. Analogously, since $\{z_k + \lambda\}_{k=0}^{\infty}$ is uniformly separated, there exists a constant $c_2 > 0$ such that, for each $n, m \in \mathbb{N}_0$ with $n \neq m$, given the $\ell^2$ sequence

$$\left\{ (1-|z_k|^2)^{1/2} \left( \frac{\delta_{n,k}}{\sqrt{\mu_n}} - \frac{\delta_{m,k}}{\sqrt{\mu_m}} \right) \right\}_{k=0}^{\infty},$$

there exists a function $g_{n,m} \in \mathbb{H}^2$ with

$$g_{n,m}(z_k + \lambda) = \begin{cases} 0 & \text{if } k \neq n, m \\ \frac{1}{\sqrt{\mu_n}} & \text{if } k = n \\ -\frac{1}{\sqrt{\mu_m}} & \text{if } k = m \end{cases} \tag{3.5}$$

and with

$$\|g_{n,m}\|_2 \leq c_2 \left\| \left\{ (1-|z_k|^2)^{1/2} \left( \frac{\delta_{n,k}}{\sqrt{\mu_n}} - \frac{\delta_{m,k}}{\sqrt{\mu_m}} \right) \right\} \right\|.$$

That is, $\|g_{n,m}\|_2 \leq c_2 \big(\frac{1-|z_n|^2}{\mu_n} + \frac{1-|z_m|^2}{\mu_m}\big)^{1/2}$, and since $\frac{1-|z_n|^2}{\mu_n} \leq 2$ and $\frac{1-|z_m|^2}{\mu_m} \leq 2$, it follows that

$$\|g_{n,m}\|_2 \leq 2 c_2 \tag{3.6}$$

for all $n$ and $m$ with $n \neq m$. Now, we also have that for $n \neq m$

$$\begin{aligned} \big(X f_n - X f_m, g_{n,m}^*\big) &= (X f_n, g_{n,m}^*) - (X f_m, g_{n,m}^*) \\ &= \sum_{k=0}^{\infty} f_n(z_k) g_{n,m}(z_k + \lambda) \mu_k - \sum_{k=0}^{\infty} f_m(z_k) g_{n,m}(z_k + \lambda) \mu_k \\ &= \frac{1}{\sqrt{\mu_n}} \frac{1}{\sqrt{\mu_n}} \mu_n - \frac{1}{\sqrt{\mu_m}} \Big( - \frac{1}{\sqrt{\mu_m}} \Big) \mu_m \\ &= 2 \end{aligned}$$

where the second-to-last equality follows from (3.3) and (3.5). This implies

$$2 = \left|\left(Xf_n - Xf_m, g^*_{n,m}\right)\right| \le \|Xf_n - Xf_m\|_2 \|g^*_{n,m}\|_2$$

and, since $\|g_{n,m}\|_2 = \|g^*_{n,m}\|_2$, inequality (3.6) implies $2 \le \|Xf_n - Xf_m\|_2 2c_2$, i.e.

$$\frac{1}{c_2} \le \|Xf_n - Xf_m\|_2. \tag{3.7}$$

Since the sequence $\{f_j\}_{j=0}^{\infty}$ is bounded (by (3.4)), there must be a subsequence $\{f_{j_n}\}$ of $\{f_j\}$ which converges weakly. If $X$ was compact, it would follow that $\{Xf_{j_n}\}$ converges in norm. In particular, $\{Xf_{j_n}\}$ is a Cauchy sequence. But this is impossible by (3.7). Thus the bounded $\lambda$-Hankel operator $X$ cannot be compact ∎

**Remark on the proof.** In fact, there is an exact formula for $f_n$:

$$f_n(z) = \frac{1}{\sqrt{\mu_n}} \frac{B_1(z)}{B_1'(z_n)(z - z_n)} \tag{3.8}$$

where $B_1(z)$ is the Blaschke product associated with the uniformly separated sequence $\{z_k\}$. In this case $f_n$ satisfies (3.3), and inequality (3.4) holds with $c_1 = \frac{1}{\delta_1}$ (where $\delta_1$ is the uniform lower bound in the definition of uniform separation).

Analogously, there is an exact formula for $g_{n,m}$:

$$g_{n,m}(z) = \frac{1}{\sqrt{\mu_n}} \frac{B_2(z)}{B_2'(z_n + \lambda)(z - z_n - \lambda)} - \frac{1}{\sqrt{\mu_m}} \frac{B_2(z)}{B_2'(z_m + \lambda)(z - z_m - \lambda)} \tag{3.9}$$

where $B_2(z)$ is the Blaschke product associated with the uniformly separated sequence $\{z_k + \lambda\}$. In this case $g_{n,m}$ satisfies (3.5), and inequality (3.6) holds with $c_2 = \frac{1}{\delta_2}$ (where $\delta_2$ is the uniform lower bound in the definition of uniform separation).

Thus we do not need to use the Shapiro-Shields theorem to justify the existence of the sequences of functions $f_n$ and $g_{n,m}$. Nevertheless, we prefer to present the whole construction, rather than just put $f_n$ and $g_{n,m}$ as in (3.8) and (3.9) and then check (3.7) without any explanation of why this choice works (after all, such functions exist thanks to the Shapiro-Shields theorem anyway!).

**Final comments.** A deep fact about Hankel operators (see [6: p. 10] is that every bounded Hankel operator $H$ is given by a (possible complex) Carleson measure $\mu$ on $\mathbb{D}$ with

$$(He_m, e_n) = \int_{\mathbb{D}} z^m z^n d\mu(z).$$

It would be interesting to know if a fact like this could be proven for $\lambda$-Hankel operators. At present, we do not seem to have the tools to obtain such a result.

It is also known that the operator $H$ above is compact if and only if the measure $\mu$ is a vanishing Carleson measure. It would be interesting to see if $X$, the $\lambda$-Hankel operator defined by (2.1), is compact if and only if the measure $\mu$ supported in $R \subset \mathbb{D}$ is a vanishing Carleson measure (with respect to $\mathbb{D}$ and $\lambda+\mathbb{D}$). It would be even more interesting if indeed it was the case that all bounded $\lambda$-Hankel operators are given by a measure as in (2.1). Since at the moment we are only interested in showing that there are non-compact bounded $\lambda$-Hankel operators, we leave the questions above open for future research.

## References


[1] Barría, J. and P. Halmos: *Asymptotic Toeplitz operators.* Trans. Amer. Math. Soc. 273 (1982), 621 – 630.

[2] Duren, P.: *Theory of $H^p$ Spaces*, 2nd ed. New York: Dover Publ. 2000.

[3] Martínez-Avendaño, R.: *Hankel operators and generalizations.* Ph.D. Dissertation. University of Toronto 2000.

[4] Martínez-Avendaño, R.: *A generalization of Hankel operators.* J. Funct. Anal. 190 (2002), 418 – 446.

[5] Power, S.: *Hankel operators on Hilbert space.* Bull. London Math. Soc. 12 (1980), 422 – 442.

[6] Power, S.: *Hankel Operators on Hilbert Space.* Boston: Pitman 1982.

[7] Sarason, D.: *Holomorphic spaces: a brief and selective survey.* In: Holomorphic spaces, Berkeley, CA, 1995 (eds.: S. Axler et al.; Math. Sci. Res. Inst. Publ.: Vol. 33). Cambridge: Cambridge Univ. Press 1998, pp. 1 – 34.

[8] Sun, S.: *On the operator equation $U^*TU = \lambda T$.* Kexue Tongbao (English Ed.) 29 (1984), 298 – 299.


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