Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 4, 915–929

L_p-L_q Estimates for the Bochner-Riesz Operator of Complex Order

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Abstract. We describe convex sets on the $(\frac{1}{p}, \frac{1}{q})$-plane for which the well-known Bochner-Riesz operator with the symbol $(1-|\xi|^2)_+^{-\alpha}$ $(0 < \operatorname{Re}\alpha < \frac{n+1}{2})$ is bounded from L_p into L_q.

Keywords: *Bochner-Riesz operator, potential-type operator, oscillating kernel, L_p-L_q estimates*

AMS subject classification: 46E35, 26A33

1. Introduction

The well known Bochner-Riesz operators B^α are widely used in various problems of analysis (see [13: Chapter IX, Section 2]). These operators are defined via Fourier transform by the equality

$$\widehat{(B^\alpha\varphi)}(\xi) = \frac{2^\alpha(2\pi)^{-\frac{n}{2}}}{\Gamma(1-\alpha)}\,(1-|\xi|^2)_+^{-\alpha}\widehat{\varphi}(\xi)$$

where

$$(1-|\xi|^2)_+^{-\alpha} = \begin{cases} (1-|\xi|^2)^{-\alpha} & \text{if } |\xi|<1 \\ 0 & \text{if } |\xi|>1 \end{cases}$$

and admit the integral representation

$$(B^\alpha\varphi)(x) = \int_{\mathbb{R}^n} |y|^{-\frac{n}{2}+\alpha} J_{\frac{n}{2}-\alpha}(|y|)\,\varphi(x-y)\,dy \tag{1.1}$$

for $\operatorname{Re}\alpha \le \frac{n+1}{2}$, J_ν being the Bessel function. One of the most interesting problems in the theory of operators (1.1) consists in establishing L_p-L_q estimates for them. The first results in this area, pertaining to the case $p = q$, are due to E. M. Stein, Ch. Fefferman, P. Sjölin and others (a comprehensive survey of these results and corresponding references are given in [13: Chapter IX]). The case $p \ne q$ is more complicated and also very interesting. Due to the oscillation of the Bessel function at infinity, it is possible to construct convex sets in the $(\frac{1}{p}, \frac{1}{q})$-plane for which B^α is

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ISSN 0232-2064 / $ 2.50

bounded from L_p into L_q. Such sets were constructed in [2], where the author dealt with the case $0 \le \alpha \le \frac{n+1}{2}$.

The goal of the present paper is to consider operator (1.1) for complex α with $0 < \operatorname{Re}\alpha < \frac{n+1}{2}$. At the same time, even in the case of real α we obtain stronger results in comparison with those presented in [2]: we fill in some gaps contained in the main theorem of [2] (see Remark 2.1). To establish the mentioned results, we develop a new approach to L_p-L_q estimates for potential-type operators with oscillating kernels. It is based on the investigation of mapping properties of some potentials with radial characteristics oscillating at infinity (see Theorem 4.2). This approach can also be applied to operators in a wide class containing, in particular, the acoustic potentials and Strichartz-type potentials over $\mathbb{R}^n$ with oscillating characteristics, which are known to be operators of essentially different nature. Such applications will be given in other papers.

At present time, there is great interest in potential-type operators (see the books [11, 12], the survey papers [5, 6, 10] and the bibliography therein). Nevertheless, the investigation of mapping properties of potentials with oscillating kernels is at the very beginning. Besides the mentioned papers dealing with the Bochner-Riesz operator, one can point out only the papers [7, 8] (see also [6]), which are devoted to L_p-L_q estimates for acoustic potentials and Riesz potentials with the characteristic $e^{i|t|}$, respectively.

The paper is organized as follows: In Section 2 we formulate our main result (Theorem 2.1) and give some comments. Section 3 contains necessary preliminaries. Section 4 can be regarded as background to the proof of Theorem 2.1. Here we prove some statements related to L_p-L_q estimates for the operator S_a^α given by

$$(S_a^\alpha \varphi)(x) = \int_{|y|\ge A} a(|y|)e^{i|y|}|y|^{\alpha-n}\varphi(x-y)\,dy \tag{1.2}$$

for $0 < \operatorname{Re}\alpha < n$ where the function a is sufficiently smooth on $(A,\infty]$ and such that $a(\infty) \ne 0$. We first consider the case when $a(r) \equiv 1$ in (1.2) (Theorem 4.1) and then pass to a more general situation (Theorem 4.2). We note that Theorems 4.1 and 4.2 are of special interest themselves because, as was mentioned above, they can be applied to obtain L_p-L_q estimates for a wide class of operators. Therefore we prove them for all α with $0 < \operatorname{Re}\alpha < n$, although to prove Theorem 2.1 we may restrict ourselves to the case $\frac{n-1}{2} < \operatorname{Re}\alpha < n$ only. Finally, Section 5 is devoted to the proof of Theorem 2.1.

2. The main result

Let $0 < \operatorname{Re}\alpha < n$. Everywhere below we use the following notation: $(A, B, \dots, K)$ is the open polygon in $\mathbb{R}^2$ with the vertices at the points $A, B, \dots, K$ and $[A, B, \dots, K]$ stands for its closure. By $\mathcal{L}(A)$ we denote the $\mathcal{L}$-characteristic of an operator A, that is the set of all the pairs $(\frac{1}{p}, \frac{1}{q})$ for which A is bounded from L_p into L_q. Let

$$A = \big(1, 1 - \tfrac{\operatorname{Re}\alpha}{n}\big) \qquad A' = \big(\tfrac{\operatorname{Re}\alpha}{n}, 0\big)$$
$$B = \big(1 - \tfrac{(n-1)(n-\operatorname{Re}\alpha)}{n(n+1)}, 1 - \tfrac{\operatorname{Re}\alpha}{n}\big) \qquad B' = \big(\tfrac{\operatorname{Re}\alpha}{n}, \tfrac{(n-1)(n-\operatorname{Re}\alpha)}{n(n+1)}\big)$$

$$C = \left(\tfrac{3}{2} - \tfrac{2\operatorname{Re}\alpha}{n-1}, \tfrac{3}{2} - \tfrac{2\operatorname{Re}\alpha}{n-1}\right) \qquad C' = \left(\tfrac{2\operatorname{Re}\alpha}{n-1} - \tfrac{1}{2}, \tfrac{2\operatorname{Re}\alpha}{n-1} - \tfrac{1}{2}\right)$$
$$D = \left(\tfrac{\operatorname{Re}\alpha+1}{n+1}, \tfrac{n-\operatorname{Re}\alpha}{n+1}\right)$$
$$E = (1,0)$$
$$F = \left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$
$$G = \left(1 - \tfrac{(n-\operatorname{Re}\alpha)(n-1)}{n(n+3)}, 1 - \tfrac{\operatorname{Re}\alpha}{n}\right) \qquad G' = \left(\tfrac{\operatorname{Re}\alpha}{n}, \tfrac{(n-\operatorname{Re}\alpha)(n-1)}{n(n+3)}\right)$$
$$H = \left(1 - \tfrac{\operatorname{Re}\alpha}{n}, 1 - \tfrac{\operatorname{Re}\alpha}{n}\right) \qquad H' = \left(\tfrac{\operatorname{Re}\alpha}{n}, \tfrac{\operatorname{Re}\alpha}{n}\right)$$
$$K = \left(\tfrac{2(\operatorname{Re}\alpha+1)}{n+1} - \tfrac{1}{2}, \tfrac{1}{2}\right) \qquad K' = \left(\tfrac{1}{2}, \tfrac{3}{2} - \tfrac{2(\operatorname{Re}\alpha+1)}{n+1}\right).$$

To formulate our main result, we introduce the set

$$L(\alpha,n) = \begin{cases} (A',B',B,A,E) \cup (A,E] \cup (A',E) & \text{if } \frac{n}{2} \le \operatorname{Re}\alpha < n \\ (A',G',K',K,G,A,E) \cup (A,E] \cup (A',E) & \text{if } \frac{n-1}{2} < \operatorname{Re}\alpha < \frac{n}{2} \\ (A',G',F,G,A,E) \cup (A,E] \cup (A',E) & \text{if } \alpha = \frac{n-1}{2} \\ (A',G',F,G,A,E) \cup (A,E] \cup (A',E) \cup \{F\} & \text{if } \operatorname{Re}\alpha = \frac{n-1}{2}, \operatorname{Im}\alpha \ne 0 \\ (A',G',C',C,G,A,E) \cup (A,E] \cup (A',E) \cup (C',C) & \text{if } \frac{n(n-1)}{2(n+1)} < \operatorname{Re}\alpha < \frac{n-1}{2} \\ [A',H',H,A,E] \setminus ([A',H'] \cup [A,H]) & \text{if } 0 < \operatorname{Re}\alpha \le \frac{n(n-1)}{2(n+1)} \end{cases}$$

if $n \ge 3$ or if $n = 2$ and $\operatorname{Im}\alpha \ne 0$, and we put

$$L(\alpha,n) = \begin{cases} (A',B',B,A,E) \cup (A,E] \cup (A',E) & \text{if } \frac{1}{2} \le \alpha < 2, \alpha \ne 1 \\ (A',B',B,A,E) \cup (A,E] \cup (A',E) \cup (B',B) & \text{if } \alpha = 1 \\ [A',H',H,A,E] \setminus ([A',H'] \cup [A,H]) & \text{if } 0 < \alpha < \frac{1}{2} \end{cases}$$

for $n = 2$ and $\operatorname{Im}\alpha = 0$.

The main result of the paper is contained in the following theorem (see also Picture 1).

Theorem 2.1. *Let* $0 < \operatorname{Re}\alpha < \frac{n+1}{2}$. *Then the imbedding*

$$\mathcal{L}(B^\alpha) \supset \begin{cases} L(\alpha + \frac{n-1}{2}, n) & \text{if } \operatorname{Im}\alpha \ne 0 \\ L(\alpha + \frac{n-1}{2}, n) \cup \{D\} & \text{if } \operatorname{Im}\alpha = 0 \end{cases} \tag{2.1}$$

is valid.

Remark 2.1. We note that the main theorem from [2] (related to the case of real α with $0 \le \alpha \le \frac{n+1}{2}$) does not answer the question on the boundedness of the operator B^α from L_p into L_q if

$$\left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \begin{cases} [B',D,G'] \cup [B,D,G] \setminus ([B,G] \cup [B',G'] \cup \{D\}) & \text{for } n > 2 \text{ and } 0 < \alpha < \frac{n+1}{2} \\ (B',B) \setminus \{D\} & \text{for } n = 2 \text{ and } \alpha \ne 1. \end{cases}$$

Theorem 2.1 (which covers the case of complex α as well) gives a positive answer to this question (for $0 < \alpha < \frac{1}{2}$ only partially) if

$$\left(\tfrac{1}{p}, \tfrac{1}{q}\right) \in \begin{cases} (B,G,D) \cup (B',G',D) & \text{for } n \ge 3 \text{ and } \frac{1}{2} \le \alpha < \frac{n+1}{2} \\ (D,G,K) \cup (D,G',K') & \text{for } n \ge 3 \text{ and } 0 < \alpha < \frac{1}{2}. \end{cases}$$

Picture 1

Remark 2.2. As was proved in [2], in the case $\operatorname{Im}\alpha = 0$ the set $\mathcal{L}(B^\alpha)$ does not contain the points lying

1) on the straight line AB and above it
2) on the straight line $A'B'$ and to the left of it
3) above the straight line $B'B$.

In the case of complex α the question on the boundedness of the operator B^α in these regions still remains open. Nevertheless, in the case of operator (1.2), which is important for applications, we prove statements 1) and 2) for $0 < \operatorname{Re}\alpha < n$.

3. Preliminaries

3.1 Uniform asymptotic expansion for the Bessel function J_ν. Let $\Omega = \{z \in \mathbb{C} : |z| > \eta \text{ and } |\arg z| < \theta\}$, where $\eta > 0$ and $\theta \in (0, \frac{\pi}{2})$. Representing J_ν as a linear combination of the Hankel functions $H^{(1)}_{\pm\nu}$ and $H^{(2)}_{\pm\nu}$ (we take $+\nu$ if $\nu > -\frac{1}{2}$ and $-\nu$ otherwise) and applying the results of [15: p. 220] or [4: p. 167], we arrive at the equality

$$J_\nu(z) = \Big(\frac{\pi z}{2}\Big)^{-\frac{1}{2}} \Bigg[e^{-iz}\Bigg(\sum_{m=0}^{N} C^{(\nu)}_{m,-} z^{-m} + R^{(\nu)}_{N,-}(z)\Bigg) + e^{iz}\Bigg(\sum_{m=0}^{N} C^{(\nu)}_{m,+} z^{-m} + R^{(\nu)}_{N,+}(z)\Bigg)\Bigg] \tag{3.1}$$

where $C^{(\nu)}_{0,\pm} = \frac{1}{2} e^{\mp \frac{i\pi(2\nu+1)}{4}}$.

Remark 3.1. The remainder $R^{(\nu)}_{N,\pm}$ is analytic in Ω and $R^{(\nu)}_{N,\pm}(z) = O(|z|^{-N-1})$ as $|z| \to \infty$. Therefore $(\frac{d}{dz})^j R^{(\nu)}_{N,\pm}(z) = O(|z|^{-N-1-j})$ as $|z| \to \infty$ in any closed sector $\Omega_0 \subset \Omega$ (see [9: p. 21]).

3.2 On the L_p boundedness of a certain convolution operator with oscillating kernel. The following lemma is true:

Lemma 3.1 [13: p. 392]). *Let ψ be a smooth function in $\mathbb{R}^n$ with a compact support that vanishs in some neighborhood of the origin and let*

$$(G_\lambda \varphi)(x) = \int_{\mathbb{R}^n} e^{i\lambda|x-y|} \psi(|x-y|)\varphi(y)\, dy \qquad (\lambda > 0).$$

Then $\|G_\lambda \varphi\|_p \le A\lambda^{-\frac{n}{p'}} \|\varphi\|_p$ for $1 \le p \le \frac{2(n+1)}{n+3}$.

4. Some auxiliary statements

Here we study mapping properties of operator (1.2). We first dwell on the case $a(|y|) \equiv 1$; the corresponding operator is denoted by S^α and its kernel by $k_\alpha(|y|)$. The following theorem provides L_p-L_q estimates for this operator (see also Pictures 1 and 2 for the cases $\frac{n-1}{2} < \operatorname{Re}\alpha < n$ and $0 < \operatorname{Re}\alpha \le \frac{n-1}{2}$, respectively).

Theorem 4.1. *Let $0 < \operatorname{Re}\alpha < n$. Then:*

I. *The imbedding*

$$\mathcal{L}(S^\alpha) \supset L(\alpha, n) \tag{4.1}$$

is valid.

II. *The set $\mathcal{L}(S^\alpha)$ does not contain the points lying*

1) *on the segment $[A, H]$ and above it*
2) *on the segment $[A', H']$ and to the left of it*
3) *above the straight line $B'B$ in the case $\frac{n-1}{2} < \alpha < n$*
4) *on the segment $[O', O]$ if $\alpha = \frac{n-1}{2}$.*

Proof. To prove statement I, we first establish the estimate

$$\|S^\alpha \varphi\|_q \le C\|\varphi\|_p \qquad (\varphi \in \mathcal{S}), \tag{4.2}$$

the constant C not depending on φ, where

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in L(\alpha, n) \tag{4.3}$$

and $\mathcal{S}$ is the Schwartz class of rapidly decreasing smooth functions. We split $S^\alpha\varphi$ into

$$(S^\alpha \varphi)(x) = \sum_{\ell=0}^{\infty} (S^\alpha_\ell \varphi)(x) \qquad (x \in \mathbb{R}^n)$$

where

$$(S_\ell^\alpha \varphi)(x) = 2^{(\alpha-n)\ell} \int_{|y|\geq A} u_\alpha\Big(\frac{|y|}{2^\ell}\Big) e^{i|y|} \varphi(x-y)\, dy$$

for $0 < \operatorname{Re}\alpha < n$. The function u_α supported on $[\frac{A}{2}, 2A]$ is defined by

$$u_\alpha(r) = r^{\alpha-n}\big(\eta(r) - \eta(2r)\big)$$

where $\eta \in C^\infty(\mathbb{R}^1_+)$, $0 \leq \eta(r) \leq 1$, $\eta(r) = 1$ if $r \leq A$ and $\eta(r) = 0$ if $r \geq 2A$. We have

$$\|S^\alpha \varphi\|_q \leq \sum_{\ell=0}^{\infty} \|S_\ell^\alpha \varphi\|_q. \tag{4.4}$$

Picture 2

To estimate the series on the right-hand side herein, we need the following lemmas. (For the rest of the proof of Theorem 4.1 the same letter C will be used to denote various constants not depending on ℓ and not necessarily the same at each occurrence.)

Lemma 4.1. *Let $k_l^\alpha(|y|)$ be the kernel of S_ℓ^α. Then the estimates*

$$|\hat{k}_\ell^\alpha(|\xi|)| \leq C 2^{-M\ell} \begin{cases} 1 & \text{if } |\xi| \leq \frac{1}{2} \\ (1+|\xi|)^{-M} & \text{if } |\xi| \geq 2 \end{cases} \quad (\forall\, M > 0) \tag{4.5}$$

$$|\hat{k}_\ell^\alpha(|\xi|)| \leq C 2^{(\operatorname{Re}\alpha - \frac{n-1}{2})\ell} \quad \text{if } \tfrac{1}{2} < |\xi| < 2 \tag{4.6}$$

hold.

Proof. By the Bochner formula we have

$$\hat{k}_\ell^\alpha(|\xi|) = \frac{2^{(\alpha-\frac{n}{2}+1)\ell}(2\pi)^{\frac{n}{2}}}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \rho^{\frac{n}{2}} u_\alpha(\rho) e^{i2^\ell\rho} J_{\frac{n-2}{2}}(\rho|\xi|2^\ell)\, d\rho.$$

Let first $|\xi| \le \frac{1}{2}$. Applying [1: p. 92/Formula (7)], interchanging the order of integration and integrating the inner integral by parts k times, $k = [\operatorname{Re}\alpha] + M + 1$, we arrive at the equality

$$\begin{aligned}\hat{k}_\ell^\alpha(|\xi|) = 2^{\ell(\alpha-k)} \frac{(2\pi)^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})} \int_{-1}^1 \frac{(1-t^2)^{\frac{n-3}{2}} dt}{(i(1+|\xi|t))^k} \\ \times \int_0^\infty e^{i2^\ell\rho(1+|\xi|t)} \Big(\frac{d}{d\rho}\Big)^k (\rho^{n-1} u_\alpha(\rho))\, d\rho\end{aligned}$$

which yields (4.5).

Let now $|\xi| > \frac{1}{2}$. Making use of formula (3.1) with $m = 0$, we get

$$\begin{aligned}\hat{k}_\ell^\alpha(|\xi|) = 2^{(\alpha-\frac{n-1}{2})\ell+\frac{n+1}{2}} \bigg(\frac{C_0^+}{|\xi|^{\frac{n-1}{2}}} I_{0,+}^{\alpha,\ell}(|\xi|) + C_0^- |\xi|^{\frac{n-1}{2}} I_{0,-}^{\alpha,\ell}(|\xi|) \\ + \frac{C_1^+}{|\xi|^{\frac{n-1}{2}}} I_{1,+}^{\alpha,\ell}(|\xi|) + \frac{C_1^-}{|\xi|^{\frac{n-1}{2}}} I_{1,-}^{\alpha,\ell}(|\xi|) \bigg)\end{aligned} \tag{4.7}$$

where

$$I_{0,\pm}^{\alpha,\ell}(|\xi|) = \int_0^\infty \rho^{\frac{n-1}{2}} u_\alpha(\rho)\, e^{i2^\ell\rho(1\pm|\xi|)} d\rho \tag{4.8}$$

$$I_{1,\pm}^{\alpha,\ell}(|\xi|) = \int_0^\infty \rho^{\frac{n-1}{2}} u_\alpha(\rho)\, e^{i2^\ell\rho(1\pm|\xi|)} R_{0,\pm}^{(\frac{n-2}{2})}(2^\ell\rho|\xi|)\, d\rho. \tag{4.9}$$

From here we derive (4.6). Integrating by parts $M + \big[|\operatorname{Re}\alpha - \frac{n-1}{2}|\big] + 1$ times in (4.8) - (4.9) and making some evident estimates, we obtain (4.5) for $|\xi| \ge 2$ (when evaluating the integral containing $R_{0,\pm}^{(\frac{n-2}{2})}$, we essentially use Remark 3.1) ∎

Lemma 4.2. *Let $\ell \ge 1$. Then*

$$\|S_\ell^\alpha \varphi\|_2 \le C 2^{\ell(\operatorname{Re}\alpha - \frac{n}{2})} \|\varphi\|_p \qquad (\varphi \in \mathcal{S}) \tag{4.10}$$

where $1 \le p \le \frac{2(n+1)}{n+3}$.

Proof. By the Parseval equality we have

$$\begin{aligned}\|S_\ell^\alpha \varphi\|_2^2 = \tfrac{1}{(2\pi)^n} \bigg(\int_{|\xi|\le\frac{1}{2}} + \int_{\frac{1}{2}<|\xi|<2} + \int_{|\xi|\ge 2} \bigg) |\hat{k}_\ell^\alpha(|\xi|)|^2 |\widehat{\varphi}(\xi)|^2 d\xi \\ =: J_1 + J_2 + J_3.\end{aligned}$$

With the aid of Lemma 4.1 we get

$$J_1 \le C2^{\ell(\operatorname{Re}\alpha-\frac{n}{2})}\int_{|\xi|\le\frac{1}{2}}|\widehat{\varphi}(\xi)|^2d\xi \le C2^{\ell(\operatorname{Re}\alpha-\frac{n}{2})}\|\varphi\|_p$$

and

$$J_3 \le C2^{\ell(\operatorname{Re}\alpha-\frac{n}{2})}\int_{|\xi|\ge 2}(1+|\xi|)^{-[\operatorname{Re}\alpha-\frac{n}{2}]}|\widehat{\varphi}(\xi)|^2d\xi \le C2^{\ell(\operatorname{Re}\alpha-\frac{n}{2})}\|\varphi\|_p.$$

To evaluate J_2, we invoke the following restriction theorem for the Fourier transform:

$$\int_{S^{n-1}}|\widehat{\varphi}(\sigma)|^2d\sigma \le C\|\varphi\|_p^2 \qquad \big(\varphi\in\mathcal{S}, 1\le p\le\tfrac{2(n+1)}{n+3}\big) \tag{4.11}$$

(see [14] for $1\le p<\frac{2(n+1)}{n+3}$ and [13: p. 386] for $p=\frac{2(n+1)}{n+3}$). From (4.6) and (4.11) we get

$$\begin{aligned} J_2 &\le C\int_{\frac{1}{2}}^2\rho^{n-1}|\hat{k}_\ell^\alpha(\rho)|^2d\rho\int_{S^{n-1}}|\widehat{\varphi}(\rho\sigma)|^2d\sigma \\ &\le C\|\varphi\|_p^2\int_{\frac{1}{2}}^2|\hat{k}_\ell^\alpha(\rho)|^2d\rho \\ &\le C2^{2\ell(\operatorname{Re}\alpha-\frac{n}{2})}\|\varphi\|_p^2. \end{aligned}$$

Gathering the above estimates, we arrive at (4.10) ∎

The next lemma deals with L_p estimates for the operator S^α. We will essentially use it to prove (4.2) in the case $\operatorname{Re}\alpha<\frac{n-1}{2}$.

Lemma 4.3. *Estimate* (4.2) *for $p=q$ is valid in the following cases:*

1) $0<\operatorname{Re}\alpha<\frac{n(n-1)}{2(n+1)}$ *and* $\frac{n}{n-\operatorname{Re}\alpha}<p<\frac{n}{\operatorname{Re}\alpha}$

2) $\frac{n(n-1)}{2(n+1)}<\operatorname{Re}\alpha<\frac{n(n-1)}{2}$ *and* $\frac{2(n-1)}{3(n-1)-4\operatorname{Re}\alpha}<p<\frac{2(n-1)}{4\operatorname{Re}\alpha-n+1}$.

Proof. Since the kernel of the operator S_0^α is compactly supported and bounded, we have

$$\mathcal{L}(S_0^\alpha)=[O',O,E]. \tag{4.12}$$

Noting that

$$(S_\ell^\alpha\varphi)(x)=2^{\alpha\ell}\big(G_{2^\ell}\varphi(2^\ell y)\big)\big(\tfrac{x}{2^\ell}\big) \qquad (\ell\ge 1)$$

and applying Lemma 3.1, we get

$$\|S_\ell^\alpha\varphi\|_p\le C2^{\ell(\operatorname{Re}\alpha-\frac{n}{p'})}\|\varphi\|_p \qquad (\ell\ge 1) \tag{4.13}$$

where $1\le p\le\frac{2(n+1)}{n+3}$.

Applying (4.13) to each summand on the right-hand side of (4.4), except for the first one, in view of (4.12) we obtain the statement of Lemma 4.3 in the case 1).

Consider the case 2). Letting $p = \frac{2(n+1)}{n+3}$ in (4.13) we have

$$\|S_\ell^\alpha \varphi\|_{\frac{2(n+1)}{n+3}} \le C 2^{\ell(\operatorname{Re}\alpha - \frac{n(n-1)}{2(n+1)})} \|\varphi\|_{\frac{2(n+1)}{n+3}}. \tag{4.14}$$

Moreover,

$$\|S_\ell^\alpha \varphi\|_2 \le C 2^{\ell(\operatorname{Re}\alpha - \frac{n-1}{2})} \|\varphi\|_2 \tag{4.15}$$

by Lemma 3.1. Interpolating between (4.14) and (4.15), we obtain

$$\|S_\ell^\alpha \varphi\|_p \le C 2^{\ell(\operatorname{Re}\alpha + \frac{n-1}{2p} - 3\frac{n-1}{4})} \|\varphi\|_p \qquad \big(\tfrac{2(n+1)}{n+3} \le p \le 2\big). \tag{4.16}$$

From (4.4), (4.12) and (4.16) we derive the statement of the lemma in the case 2) ∎

Lemma 4.4. *In a neighbourhood of the unit sphere the symbol $\hat{k}_\alpha(|\xi|)$ of the operator S^α can be represented as*

$$\hat{k}_\alpha(|\xi|) =$$
$$\begin{cases} C_\alpha (1 - |\xi| + i0)^{\frac{n-1}{2} - \alpha} + u(|\xi|) & \text{if } \alpha - \frac{n+1}{2} \notin -\mathbb{N} \\ (1 - |\xi|)^{\frac{n-1}{2} - \alpha} (C'_\alpha + C''_\alpha \ln(1 - |\xi| + i0)) + v(|\xi|) & \text{if } \alpha - \frac{n+1}{2} \in -\mathbb{N} \end{cases}$$

where $u(|\xi|)(v(|\xi|)) = o\big(\big|1 - |\xi|\,\big|^{\frac{n-1}{2} - \alpha}\big)$ as $|\xi| \to 1$ and

$$C_\alpha = \frac{1}{(2\pi)^n} e^{i\frac{\pi}{2}(\alpha - \frac{n-1}{2})} \Gamma\big(\alpha - \tfrac{n-1}{2}\big)$$
$$C'_\alpha = \frac{\big(\psi(\frac{n+1}{2} - \alpha) + i\frac{\pi}{2}\big)\, e^{-i\frac{\pi}{2}(\alpha - \frac{n-1}{2})}}{(2\pi)^n (\frac{n-1}{2} - \alpha)!}$$
$$C''_\alpha = \frac{-e^{-i\frac{\pi}{2}(\alpha - \frac{n-1}{2})}}{(2\pi)^n (\frac{n-1}{2} - \alpha)!}\ .$$

Moreover, $\hat{k}_\alpha(|\xi|)$ is bounded outside of the mentioned neighborhood.

The statement of this lemma can be derived from Lemma 3.1 and [3: Lemma 2].

Returning in Theorem 4.1 to the proof of (4.2) under assumption (4.3), we note that $k_\alpha(|\cdot|) \in L_q$ if $0 \le \frac{1}{q} < 1 - \frac{\operatorname{Re}\alpha}{n}$, and that the statement of the Sobolev theorem is valid for the operator S^α. From this fact, by convexity and duality arguments we derive (4.2) for all points

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in [A', A, E] \setminus (\{A'\} \cup \{A\}). \tag{4.17}$$

Proceeding, we first consider the case $\frac{n}{2} \le \operatorname{Re}\alpha < n$ for $n \ge 2$. Taking into account estimate (4.10) and the evident estimate

$$\|S_\ell^\alpha \varphi\|_\infty \le C 2^{\ell(\operatorname{Re}\alpha - n)} \|\varphi\|_1,$$

by virtue of the Riesz-Thorin theorem we get

$$\|S_\ell^\alpha \varphi\|_q \le C 2^{\ell(t(\operatorname{Re}\alpha - \frac{n}{2}) + (1-t)(\operatorname{Re}\alpha - n))} \|\varphi\|_p \tag{4.18}$$

where $\frac{1}{p} = 1 - \frac{t}{p_0'}$ with $p_0 \in \left[1, \frac{2(n+1)}{n+3}\right]$ and $\frac{1}{q} = \frac{t}{2}$, $t \in [0,1]$. The exponent $\sigma = \frac{tn}{2} + \operatorname{Re}\alpha - n$ is negative if $t < \frac{2(n - \operatorname{Re}\alpha)}{n}$, which implies $\frac{1}{p} > 1 - \frac{(n-\operatorname{Re}\alpha)(n-1)}{n(n+1)}$ and $\frac{1}{q} < 1 - \frac{\operatorname{Re}\alpha}{n}$. Applying (4.18) to each summand on the right-hand side of (4.4), except for the first one, in view of (4.12) we obtain (4.2) if

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in (B, E]. \tag{4.19}$$

Interpolating between the points of the sets $(B, E]$ and $[A', A, E] \setminus (\{A'\} \cup \{A\})$, by duality we arrive at (4.2) under assumption (4.3).

Consider the case $\frac{n}{2} > \operatorname{Re}\alpha \ge \frac{n-1}{2}$. Interpolation between (4.15) and (4.13) with $p = \frac{2(n+1)}{n+3}$ yields

$$\|S_\ell^\alpha \varphi\|_q \le C 2^{\ell(t(n(\frac{1}{p_0} - \frac{1}{2}) + \operatorname{Re}\alpha - \frac{n+1}{2}) + (1-t)(\operatorname{Re}\alpha - n))} \|\varphi\|_p \qquad (\ell \ge 1) \tag{4.20}$$

where $\frac{1}{p} = \frac{t}{p_0} + 1 - t$ and $\frac{1}{q} = \frac{t}{p_0}$ with $p_0 = \frac{2(n+1)}{n+3}$. Now applying (4.20) and (4.12) to the right-hand side of (4.4), we obtain (4.2) if the point $(\frac{1}{p}, \frac{1}{q})$ satisfies the relation

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in (G, E]. \tag{4.21}$$

Making use of (4.18) with $t = 1$ and (4.15), after interpolation we get

$$\|S_\ell^\alpha \varphi\|_2 \le C 2^{\ell(t(\operatorname{Re}\alpha - \frac{n-1}{2}) + (1-t)(\operatorname{Re}\alpha - \frac{n}{2}))} \|\varphi\|_p \tag{4.22}$$

where $\frac{1}{p} = \frac{t}{2} + \frac{(1-t)(n+3)}{2(n+1)}$. Further application of (4.22) to the right-hand side of (4.4) yields (4.2) if

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in (K, E]. \tag{4.23}$$

By virtue of (4.17), (4.21) and (4.23) and convexity and duality arguments we obtain (4.2) in the case $\frac{n}{2} > \operatorname{Re}\alpha > \frac{n-1}{2}$ if (4.3) is fulfilled.

In the case $\operatorname{Re}\alpha = \frac{n-1}{2}$ and $\operatorname{Im}\alpha \ne 0$, in accordance with Lemma 4.4, the symbol $\hat{k}_\alpha(|\xi|)$ of S^α is a 2-multiplier, which implies that S^α is bounded in L_2.

Let us consider the situation in which $0 < \operatorname{Re}\alpha < \frac{n-1}{2}$. In the case $\frac{n(n-1)}{2(n+1)} < \operatorname{Re}\alpha < \frac{n-1}{2}$ the application of Lemma 4.3 (statement 3)) yields (4.2) if

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in (C', C). \tag{4.24}$$

Moreover, as in the case $\frac{n-1}{2} < \operatorname{Re}\alpha < \frac{n}{2}$, we also have (4.2) if (4.21) is fulfilled. From (4.17), (4.21) and (4.24), by convexity and duality we derive (4.2) under assumption (4.3) in the case under consideration.

Let finally $0 < \operatorname{Re}\alpha \le \frac{n(n-1)}{2(n+1)}$. Then (4.2) holds if

$$(\tfrac{1}{p}, \tfrac{1}{q}) \in (H', H) \tag{4.25}$$

in view of Lemma 4.3 (statement 2)). By virtue of (4.17) and (4.25) and by usual convexity and duality arguments we also arrive at (4.2) for $(\frac{1}{p}, \frac{1}{q})$ satisfying (4.3).

Direct analysis of the proof of the main result from [2] shows the validity of (4.2) in the case $n = 2$ and $\frac{1}{3} < \alpha \le 1$.

Thus we have proved (4.2) under assumption (4.3). Therefore the operator S^α can be extended by continuity to a bounded operator acting from L_p into L_q under assumption (4.3). As is easily verified, this extension coincides with the right-hand side of (1.2) (with $a(r) \equiv 1$). Statement I of Theorem 4.1 has been proved completely.

Let us prove statement II of Theorem 4.1, item 1). Consider the characteristic function $\chi_{\frac{1}{10}}$ of the ball $|y| < \frac{1}{10}$. Clearly, $\chi_{\frac{1}{10}} \in L_p$ for $1 \le p \le \infty$. It suffices to prove that $S^\alpha \chi_{\frac{1}{10}} \notin L_{\frac{n}{n-\operatorname{Re}\alpha}}$. We have

$$(S^\alpha \chi_{\frac{1}{10}})(x) = \int_{|t|<\frac{1}{10}} \frac{e^{i|x-t|}}{|x-t|^{n-\alpha}}\, dt - \int_{|t|<\frac{1}{10}} \frac{(1-\chi(|x-t|))e^{i|x-t|}}{|x-t|^{n-\alpha}}\, dt, \tag{4.26}$$

χ being the characteristic function of the unit ball. Evidently, the second summand on the right-hand side herein belongs to $L_{\frac{n}{n-\operatorname{Re}\alpha}}$. We will prove that for the function J defined by

$$J(x) = \int_{|t|<\frac{1}{10}} \frac{e^{i|x-t|}}{|x-t|^{n-\alpha}}\, dt$$

we have $J \notin L_{\frac{n}{n-\operatorname{Re}\alpha}}$ and thus come to the desired conclusion. For $|x| \to \infty$ we have

$$|J(x)| \ge \left| \int_{|t|<\frac{1}{10}} \frac{dt}{|x-t|^{n-\alpha}} \right| - \int_{|t|<\frac{1}{10}} \frac{|t|\, dt}{|x-t|^{n-\operatorname{Re}\alpha}}.$$

Let us evaluate (from below) the first summand on the right-hand side herein:

$$\left| \int_{|t|<\frac{1}{10}} \frac{dt}{|x-t|^{n-\alpha}} \right| = |x|^{\operatorname{Re}\alpha} \left| \int_{|y|<\frac{1}{10|x|}} \frac{e^{i\operatorname{Im}\alpha(\ln|x|+\ln|x'-y|)}}{|x'-y|^{n-\operatorname{Re}\alpha}}\, dy \right|$$

where $x' = \frac{x}{|x|}$. After the change of variable $y = \omega_x(\tau)$, where $\omega_x(\tau)$ is the rotation in $\mathbb{R}^n$ such that $x' = \omega_x(e_1)$ with $e_1 = (1, 0, \dots, 0)$, we get

$$\begin{aligned} \left| \int_{|t|<\frac{1}{10}} \frac{dt}{|x-t|^{n-\alpha}} \right| &\ge |x|^{\operatorname{Re}\alpha} \int_{|\tau|<\frac{1}{10|x|}} \frac{\cos(\operatorname{Im}\alpha \ln|\tau - e_1|)\, d\tau}{|\tau - e_1|^{n-\operatorname{Re}\alpha}} \\ &\ge A|x|^{\operatorname{Re}\alpha} \int_{|\tau|<\frac{1}{10|x|}} \frac{d\tau}{|\tau - e_1|^{n-\operatorname{Re}\alpha}} \end{aligned} \tag{4.27}$$

where A is close to one. Besides this,

$$\int_{|t|<\frac{1}{10}} \frac{|t|}{|x-t|^{n-\operatorname{Re}\alpha}}\,dt \le \frac{|x|^{\operatorname{Re}\alpha}}{10}\int_{|\tau|<\frac{1}{10|x|}} \frac{d\tau}{|\tau-e_1|^{n-\operatorname{Re}\alpha}}. \tag{4.28}$$

From both estimates we derive that

$$|J(x)| \ge |x|^{\operatorname{Re}\alpha}\big(A-\tfrac{1}{10}\big)\int_{|\tau|<\frac{1}{10|x|}} \frac{d\tau}{|\tau-e_1|^{n-\operatorname{Re}\alpha}} \ge \frac{C}{|x|^{n-\operatorname{Re}\alpha}} \notin L_{\frac{n}{n-\operatorname{Re}\alpha}}.$$

Statement II/2) follows from statement II/1) by duality. To prove statement II/3) we have to use the counter-example constructed in [2: pp. 231 - 232]. Finally, statement II/4) follows from Lemma 4.4. This ends the proof of Theorem 4.1 ∎

Remark 4.1. We observe that for every α with $1 < \operatorname{Re}\alpha < n$ there exists a neighbourhood $\Omega(\alpha, n)$ of the set $\mathcal{L}(S^\alpha)$ such that

$$\Omega(\alpha, n) \cap [O', O, E] \subset L(\alpha - 1, n).$$

This is a consequence of the definition of $L(\alpha, n)$ and statements I and II/items 1) - 2) of Theorem 4.1.

Remark 4.2. We note that in the case of real α with $0 \le \alpha < \frac{n-1}{2}$ or $\frac{n}{2} \le \alpha < n$ the statement of Theorem 4.1 was partially proved by the authors jointly with E. E. Urnysheva.

Passing to the case of operator (1.2), we first describe the class of characteristics a. Let a be such that the function a^* defined by $a^*(r) = a(r^{-1})$ $(r > 0)$ and $a^*(0) = \lim_{r\to\infty} a(r^{-1})$ is continuously differentiable up to the order $[\operatorname{Re}\alpha] + 2$ on the interval $[0, A^{-1})$ and $a(\infty) = a^*(0) \ne 0$.

The next theorem describes mapping properties of the operator S_a^α. It will play a crucial role in the proof of Theorem 2.1.

Theorem 4.2. *Let $0 < \operatorname{Re}\alpha < n$. The statements of Theorem* 4.1 *are also valid for the operator S_a^α. Moreover, $\mathcal{L}(S_a^\alpha) = \mathcal{L}(S^\alpha)$.*

Proof. Let first $\operatorname{Re}\alpha \notin \mathbb{N}$. We make use of the following equality, which is obtained by application of the Taylor formula to $a^*(r)$ $(r \in [0, A^{-1}))$:

$$a(r) = \sum_{k=0}^{m-1} \frac{(a^*)^{(k)}(0)}{k!\,r^k} + R_m(r^{-1}) \qquad \big(r > A, m = [\operatorname{Re}\alpha] + 1\big)$$

where

$$R_m(r^{-1}) = \frac{r^{-m}}{(m-1)!}\int_0^1 (1-u)^{m-1}(a^*)^{(m)}(ur^{-1})\,du.$$

Correspondingly,

$$(S_a^\alpha\varphi)(x) = \sum_{k=0}^{m-1} \frac{(a^*)^{(k)}(0)}{k!}(S^{\alpha-k}\varphi)(x) + (T_m^\alpha\varphi)(x) \tag{4.29}$$

where

$$(T_m^\alpha \varphi)(x) = \int_{|y|\geq A} \frac{e^{i|y|} R_m(|y|^{-1})}{|y|^{n-\alpha}} \varphi(x-y)\, dy.$$

Since the kernel of T_m^α belongs to $L_1 \cap L_\infty$, we have

$$\mathcal{L}(T_m^\alpha) = [O', O, E]. \tag{4.30}$$

Besides this, the imbedding

$$L(\alpha, n) \subset L(\alpha - 1, n) \qquad (1 < \operatorname{Re}\alpha < n) \tag{4.31}$$

holds which follows directly from the definition of the set $L(\alpha, n)$ (see Section 2). Now (4.1) and (4.31) imply

$$\mathcal{L}(S^{\alpha-k}) \supset L(\alpha, n) \qquad (k = 0, 1, \ldots, m-1). \tag{4.32}$$

Now the statement of Theorem 4.2 follows from (4.29), (4.30), (4.32) and Remark 4.1.

The case $\operatorname{Re}\alpha = \ell$, $1, 2, \ldots, n-1$ can be considered similarly on the basis of the equality

$$(S_a^\alpha \varphi)(x) = \sum_{k=0}^{\ell-1} \frac{(a^*)^{(k)}(0)}{k!} (S^{\alpha-k}\varphi)(x) + \frac{(a^*)^{(\ell)}(0)}{\ell!} (S_0^\alpha \varphi)(x) + (T_{\ell+1}^\alpha \varphi)(x)$$

where

$$(S_0^\alpha \varphi)(x) = \int_{|y|\geq A} \frac{\chi(|y|) e^{i|y|}}{|y|^{n-i\operatorname{Im}\alpha}} \varphi(x-y)\, dy.$$

We only have to note that

$$\mathcal{L}(S_0^\alpha) = [O', O, E] \setminus (\{O'\} \cap \{O\}). \tag{4.33}$$

Indeed, since $[O', O, E] \setminus [O', O] \subset \mathcal{L}(S_0^\alpha)$, equality (4.33) will follow from the evident relation $\{O\}, \{O'\} \notin \mathcal{L}(S_0^\alpha)$ and the estimate

$$\|S_0^\alpha \varphi\|_p \leq C \|\varphi\|_p \qquad (\varphi \in \mathcal{S}, 1 < p < \infty) \tag{4.34}$$

with constant C not depending on φ. To prove (4.34), we split $S_0^\alpha \varphi$ into

$$(S_0^\alpha \varphi)(x) = \sum_{j=0}^{\infty} (S_{0,j}^\alpha \varphi)(x) \tag{4.35}$$

where

$$(S_{0,j}^\alpha \varphi)(x) = 2^{j(i\operatorname{Im}\alpha - n)} \int_{|y|\geq A} u_\alpha\Big(\frac{|y|}{2^j}\Big) e^{i|y|} \varphi(x-y)\, dy.$$

Evidently, $\mathcal{L}(S_{0,0}^\alpha) = [O', O, E]$. Besides this, the estimate

$$\|S_{0,j}^\alpha \varphi\|_p \leq C 2^{-\frac{jn}{p'}} \|\varphi\|_p \qquad (j \geq 1) \tag{4.36}$$

is valid, which can be proved in just the same way as (4.13). Now (4.36) yields (4.34) by (4.35) ∎

Remark 4.3. Together with the operator S_a^α we consider the operator

$$(S_a^{\alpha,-}\varphi)(x) = \int_{|y|\geq A} a(|y|)|y|^{\alpha-n}e^{-i|y|}\varphi(x-y)\,dy$$

for $0 < \operatorname{Re}\alpha < n$ where a is assumed to possess the same properties as the characteristic of S_a^α. As is easily seen,

$$\mathcal{L}(S_a^{\alpha,-}) \supset L(\alpha, n). \tag{4.37}$$

5. Proof of the main Theorem 2.1

Representing $J_{\frac{n}{2}-\alpha}$ as linear combination of Hankel functions

$$J_{\frac{n}{2}-\alpha} = \tfrac{1}{2}\big(H^{(1)}_{\frac{n}{2}-\alpha} + H^{(2)}_{\frac{n}{2}-\alpha}\big)$$

we get

$$(B^\alpha\varphi)(x) = (M_+^\alpha\varphi)(x) + (M_-^\alpha\varphi)(x) + (N^\alpha\varphi)(x) \tag{5.1}$$

where

$$(M_+^\alpha\varphi)(x) = \tfrac{1}{2}\int_{|y|>A} |y|^{-\frac{n}{2}+\alpha}H^{(1)}_{\frac{n}{2}-\alpha}(|y|)\varphi(x-y)\,dy$$

$$(M_-^\alpha\varphi)(x) = \tfrac{1}{2}\int_{|y|>A} |y|^{-\frac{n}{2}+\alpha}H^{(2)}_{\frac{n}{2}-\alpha}(|y|)\varphi(x-y)\,dy$$

(the sense of the lower indices $''+''$ and $''-''$ will be seen from (5.4)) and

$$(N^\alpha\varphi)(x) = \int_{|y|<A} |y|^{-\frac{n}{2}+\alpha}J_{\frac{n}{2}-\alpha}(|y|)\varphi(x-y)\,dy.$$

We make use of the integral representations

$$H^{(1)}_\nu(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{i(z-\frac{\pi\nu}{2}-\frac{\pi}{4})}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}\Big(1+\frac{it}{2z}\Big)^{\nu-\frac{1}{2}}dt \tag{5.2}$$

$$H^{(2)}_\nu(z) = \sqrt{\frac{2}{\pi z}}\,\frac{e^{-i(z-\frac{\pi\nu}{2}-\frac{\pi}{4})}}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-t}t^{\nu-\frac{1}{2}}\Big(1-\frac{it}{2z}\Big)^{\nu-\frac{1}{2}}dt \tag{5.3}$$

(see [4: p. 165]). In view of these representations we can rewrite $M_\pm^\alpha\varphi$ as

$$(M_\pm^\alpha\varphi)(x) = C^\pm\int_{|y|\geq A} |y|^{-\frac{n+1}{2}+\alpha}e^{\pm i|y|}m_\pm(|y|)\varphi(x-y)\,dy \tag{5.4}$$

where

$$C^\pm = (2\pi)^{-\frac{1}{2}}e^{\pm\frac{i\pi(n+1-2\alpha)}{4}}\Gamma^{-1}(\tfrac{n+1}{2}-\alpha)$$

$$m_\pm(|y|) = \int_0^\infty e^{-t}t^{\frac{n-1}{2}-\alpha}\big(1\pm\tfrac{it}{2|y|}\big)^{\frac{n-1}{2}-\alpha}dt.$$

It is evident that $m_\pm$ satisfy the assumptions of Theorem 4.2 and Remark 4.3. Applying (4.1) and (4.37), we obtain

$$\mathcal{L}(M_\pm^\alpha) \supset \mathcal{L}(\alpha+\tfrac{n-1}{2}, n). \tag{5.5}$$

Moreover,

$$\mathcal{L}(N^\alpha) = [O', O, E]. \tag{5.6}$$

Now (5.5) and (5.6) yield (2.1) by (5.1) in the case $\operatorname{Im}\alpha \neq 0$. Applying the corresponding result from [13: p.238] we also arrive at (2.1) in the case $\operatorname{Im}\alpha = 0$ ∎

Acknowledgements. The authors are thankful to Prof. S. G. Samko, Prof. B. Silbermann and Prof. A. Böttcher for useful discussions of the results.

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Received 25.02.2002