

On a Similarity Boundary Layer Equation

B. Brighi

Abstract. The purpose of this paper is to study the autonomous third order nonlinear differential equation $f''' + \frac{m+1}{2} f f'' - m f'^2 = 0$ on $(0, \infty)$, subject to the boundary conditions $f(0) = a \in \mathbb{R}$, $f'(0) = 1$ and $f'(t) \rightarrow 0$ as $t \rightarrow \infty$. This problem arises when looking for similarity solutions to problems of boundary-layer theory in some contexts of fluids mechanics, as free convection in porous medium or flow adjacent to a stretching wall. Our goal here is to investigate by a direct approach this boundary value problem as completely as possible, say studying existence or non-existence and uniqueness or non-uniqueness of solutions according to the values of the real parameter m . In particular, we will emphasize similarities and differences between the cases $a = 0$ and $a \neq 0$ in the boundary condition $f(0) = a$.

Keywords: *Third order differential equation, boundary layer, similarity solution*

AMS subject classification: 34B15, 34C11, 76D10

1. Introduction

The problem we consider consists in solving the autonomous third order nonlinear differential equation

$$f''' + \frac{m+1}{2} f f'' - m f'^2 = 0 \quad \text{on } (0, \infty) \quad (1.1)$$

subject to the boundary conditions

$$f(0) = a \quad (1.2)$$

$$f'(0) = 1 \quad (1.3)$$

$$f'(\infty) := \lim_{t \rightarrow \infty} f'(t) = 0. \quad (1.4)$$

The parameters m and a will be assumed to describe \mathbb{R} . But since the case $m = 0$, where (1.1) reduces to the so-called Blasius equation, is well-known (see [4, 10]), we will suppose $m \neq 0$. Moreover, for $a = 0$ we will refer to [5], where this case is partially investigated. As we will see, the results for $a \neq 0$ are sometimes the same as for $a = 0$, but as already noted in [8, 18], this is not always the rule. For instance, if $m = -\frac{1}{2}$, problem (1.1) - (1.4) has no solution for $a \leq 0$ and an infinite number of solutions for $a > 0$.

B. Brighi: Univ. de Haute Alsace, Lab. de math., F.S.T., 4 rue des frères Lumière, FR-68093 Mulhouse, France; b.brighi@uha.fr

Let us now look at the situations where the boundary-layer equation (1.1) arises. Essentially, two physically different contexts lead to problem (1.1) - (1.4) when looking for similarity solutions. The first one is concerned with free convection about a vertical flat surface embedded in a fluid-saturated porous medium, on which the temperature is prescribed as a power function with exponent equal to m , and through which fluid can be injected into the flow (in the case $a < 0$) or withdrawn from it (in the case $a > 0$); the case $a = 0$ corresponds to an impermeable surface (see [9, 12] for $a = 0$ and [8] for $a \neq 0$). Again in fluids mechanics, but in the study of the boundary-layer flow adjacent to a stretching wall with velocity involving a power-law exponent equal to m , one obtain a slightly different version of (1.1) given by

$$g''' + gg'' - \frac{2m}{m+1}g'^2 = 0 \quad (1.5)$$

subject to the boundary conditions $g(0) = b, g'(0) = 1, g'(\infty) = 0$, where $b = 0$ corresponds to an impermeable wall, $b > 0$ to suction and $b < 0$ to lateral injection of the fluid through a permeable wall (see [1, 2, 18]). In this second context, the set of relevant values of m is $(-1, \infty)$; this appears in the definition of similarity variables allowing to get (1.5). The way to pass from (1.5) to (1.1) is to set $\kappa = \sqrt{\frac{2}{m+1}}$ and $f(t) = \kappa g(\kappa^{-1}t)$, and for that we must have $m > -1$.

The particular values $m = -\frac{1}{3}$ and $m = 1$ have attracted special attention in the past and even recently. In both cases problem (1.1) - (1.4) is exactly solvable (see [1, 11, 15, 18, 20]). For $a \neq 0$ the value $m = 1$ corresponds to constant wall mass transfer rate in the first physical context (see [8]) and to lateral mass flux of constant velocity in the second one (see [18]). In the first framework, and if $m = -\frac{1}{3}$, the energy convected is constant and the local heat transfer rate along the surface of the flat plate is equal to 0; the value $m = -\frac{1}{3}$ can also be related to a horizontal line embedded in a porous medium (see [9]).

Recall also that by analogy with the Falkner-Skan equation (see [10, 16, 19, 21]), the constraint on the solution

$$0 \leq f'(t) \leq 1 \quad (t \geq 0) \quad (1.6)$$

could be considered. In the context of heated impermeable wall embedded in a porous medium, such a condition corresponds to assume that the temperature decreases away from the wall (see [3, 5]). But we will not consider this restriction here, as it is done in [1, 2, 8, 9, 12, 17, 18]. In most of these papers, after physical considerations, problem (1.1) - (1.4) is essentially studied from the numerical point of view, and only simple facts of mathematical analysis are investigated. Our goal here is to study deeper equation (1.1) and determine as precisely as possible, for given values of the parameter m and the initial condition a , if the boundary value problem (1.1) - (1.4) has solutions, and if uniqueness holds or not.

At this stage, we must note that due to the scaling invariance of (1.1) we could introduce the blow-up coordinates $u = \frac{f'}{f^2}$ and $v = \frac{f''}{f^3}$. After changing time, equation (1.1) reduces to an autonomous system on a plane. Its phase portrait can give some informations about f , that looks hard to get directly from (1.1). Especially, it is

possible to characterize, at least for $m \in [0, 1]$, the asymptotic behaviour of solutions and to improve the uniqueness results of Section 7. However, to use the coordinates u and v we have to assume that f does not vanish and so exclude the cases where $f(0) = a < 0$, and it seems difficult to get existence or non-existence results as Theorems 4.1, 6.1 or 7.1. Also, it is not so clear how to relate boundedness of f and properties of u and v . Both methods have arguments for and against, and our aim in this paper is to exploit as much as possible equation (1.1) and perhaps to show the limitation of this direct approach. We refer to [7] for further developments in the way involving the blow-up coordinates u and v .

Finally, as already said, some of the results below does not depend on the value of a , and the proofs are the same as those given in [5]. Nevertheless, most of the time we will give these proofs in order to be more convenient for the reader. Note also that we give answers to some questions shelved in [5].

2. Properties of the solutions

First let us remark that if f satisfies equation (1.1) on some interval I and if we denote by F any anti-derivative of f on I , we get the relation

$$(f'' e^{\frac{m+1}{2}F})' = m e^{\frac{m+1}{2}F} f'^2 \tag{2.1}$$

from which and the fact that f' and f'' cannot vanish at the same point, we easily deduce the following

Lemma 2.1. *Let $m \neq 0$ and let f be a solution of equation (1.1) on some interval I . Then, for $t_0 \in I$:*

- (i) *In the case $m < 0$, if $f''(t_0) \leq 0$, then $f''(t) < 0$ for $t > t_0$.*
- (ii) *In the case $m > 0$, if $f''(t_0) \geq 0$, then $f''(t) > 0$ for $t > t_0$.*

In other words, for $m < 0$ the function f is either convex, or convex-concave, or concave, and for $m > 0$ the function f is either convex, or concave, or concave-convex. Here, by convex and concave we mean strictly convex and strictly concave, respectively. Moreover, we say convex-concave for a function which is convex to a point t_0 and concave after this point.

Lemma 2.2. *If a solution f of equation (1.1) is defined in a finite interval $[0, T)$ only, then $|f(t)|, |f'(t)|, |f''(t)| \rightarrow \infty$ as $t \rightarrow T$.*

Proof. We exploit an idea used in [10] for the Falkner-Skan equation. First, notice that

$$\lim_{t \rightarrow T} \{ |f(t)| + |f'(t)| + |f''(t)| \} = \infty. \tag{2.2}$$

If f'' were bounded for $t \rightarrow T$, then f' and f would also be bounded, which contradicts (2.2). Therefore, f'' is unbounded. Integrating equation (1.1) between 0 and $s < T$, we get

$$f''(s) - f''(0) + \frac{m+1}{2}(f(s)f'(s) - f(0)f'(0)) = \frac{3m+1}{2} \int_0^s f'(\eta)^2 d\eta \tag{2.3}$$

and we see that f' has to be unbounded. In view of Lemma 2.1, we deduce $|f'(t)| \rightarrow \infty$ as $t \rightarrow T$.

Finally, suppose f were bounded for $t \rightarrow T$. From (2.3) we get by integration

$$f'(t) + \frac{m+1}{4}f(t)^2 + \lambda t + \mu = \frac{3m+1}{2} \int_0^t \int_0^s f'(\eta)^2 d\eta ds$$

where λ and μ are constants. Since $|f'(t)| \rightarrow \infty$ as $t \rightarrow T$, the right-hand side does also and the proof is finished if $m = -\frac{1}{3}$. If $m \neq -\frac{1}{3}$ and if we put

$$w(t) = \int_0^t \int_0^s f'(\eta)^2 d\eta ds,$$

then $w''(t) = f'(t)^2 \sim \left(\frac{3m+1}{2}\right)^2 w(t)^2$ ($t \rightarrow T$). Multiplying by $w'(t)$, integrating and using the fact that $w(t) \rightarrow \infty$ as $t \rightarrow T$, we obtain

$$\begin{aligned} \frac{1}{2}w'(t)^2 &\sim \frac{(3m+1)^2}{12} w(t)^3 \\ w'(t) &\sim \frac{|3m+1|}{\sqrt{6}} w(t)^{\frac{3}{2}} \end{aligned} \quad (t \rightarrow T).$$

It follows that

$$\begin{aligned} w(t)^{-\frac{1}{2}} &\sim c_1(T-t) \\ f'(t) &\sim \frac{c_2}{(T-t)^2} \end{aligned} \quad (t \rightarrow T)$$

for some constants c_1 and c_2 . Integrating, we get $f(t) \sim \frac{c_2}{T-t}$ as $t \rightarrow T$ which contradicts the original assumption that f is bounded for $t \rightarrow T$. Therefore, f is unbounded and necessarily $|f(t)| \rightarrow \infty$ as $t \rightarrow T$.

It remains to show $|f''(t)| \rightarrow \infty$ as $t \rightarrow T$. To this end we use the equality

$$(f''' e^{\frac{m+1}{2}F})' = \frac{3m-1}{2} e^{\frac{m+1}{2}F} f' f'' \tag{2.4}$$

(where F is any anti-derivative of f) which can be obtained after differentiating (1.1). Thanks to Lemma 2.1, f' and f'' do not change of sign if we are sufficiently close to T . It follows that f''' does not also, which implies $|f''(t)| \rightarrow \infty$ as $t \rightarrow T$ ■

Clearly, the result of the Lemma 2.2 is still valid for a solution f of equation (1.1) defined only in a finite interval $(-T, 0]$.

For the rest of this section we will denote by f a solution, if it exists, of problem (1.1) - (1.4). The following propositions give properties of f . The proofs are similar to that of the case $a = 0$ (see [5]).

Proposition 2.1. *If $m < 0$, then $f > a$ and f is strictly increasing on $(0, \infty)$. Moreover,*

- *if $f''(0) \leq 0$, then f is strictly concave on $[0, \infty)$*
- *if $f''(0) > 0$, there exists $t_0 \in (0, \infty)$ such that f is strictly convex on $[0, t_0]$ and strictly concave on $[t_0, \infty)$.*

On the other hand, if $m > -1$ and $a < 0$, then $f(t)$ becomes positive for large t .

Proof. Assume first $f''(0) \leq 0$. It follows from Lemma 2.1 that $f'' < 0$ on $(0, \infty)$, so f is strictly concave on $[0, \infty)$ and f' is strictly decreasing on $[0, \infty)$. But $f'(\infty) = 0$, thus $f' > 0$ on $[0, \infty)$ and f is strictly increasing. Finally, since $f(0) = a$, we get $f > a$ on $(0, \infty)$.

Now, assume $f''(0) > 0$. There exists $t_0 \in (0, \infty)$ such that $f'' > 0$ on $[0, t_0)$ and $f''(t_0) = 0$. Indeed, if not, f' would be strictly increasing and thus, since $f'(0) = 1$, we could not have $f'(\infty) = 0$. Next, Lemma 2.1 implies $f'' < 0$ on (t_0, ∞) and as in the previous case, we deduce $f' > 0$ on $[t_0, \infty)$. Moreover, since $f' \geq 1$ on $[0, t_0]$, $f' > 0$ and $f > a$ on $[0, \infty)$.

If $m > -1$ and f is negative on $[0, \infty)$, then using the results above, we get

$$f'''(t) = mf'(t)^2 - \frac{m+1}{2}f(t)f''(t) < 0$$

for t large enough. So f' is concave at infinity, which contradicts $f' > 0$ and condition (1.4) ■

Proposition 2.2. *If $m > 0$, then $f''(0) < 0$. Moreover,*

- *either $f > a$, f is strictly increasing and strictly concave on $[0, \infty)$*
- *or there exists $t_0 \in (0, \infty)$ such that f is strictly concave on $[0, t_0]$ and f is positive, strictly decreasing and strictly convex on $[t_0, \infty)$.*

Proof. First, if $f''(0) \geq 0$, it follows from Lemma 2.1 that $f'' > 0$ on $(0, \infty)$. Hence f' is strictly increasing and we cannot have $f'(0) = 1$ and $f'(\infty) = 0$ together. So, $f''(0) < 0$.

Let us assume that f'' does not vanish on $(0, \infty)$. Then $f'' < 0$ and f is strictly concave on $[0, \infty)$. Hence, f' is strictly decreasing and, from (1.3), $f' > 0$ follows. Consequently, f is strictly increasing and $f > a$ on $(0, \infty)$.

Let us now assume that f'' vanishes somewhere and denote by t_0 the point of $(0, \infty)$ such that $f'' < 0$ on $[0, t_0)$ and $f''(t_0) = 0$. Thanks to Lemma 2.1, $f'' > 0$ on (t_0, ∞) . Therefore, f' is strictly increasing on $[t_0, \infty)$ and, due to (1.4), $f' < 0$ on $[t_0, \infty)$. So, f is strictly convex and strictly decreasing on $[t_0, \infty)$. It remains to prove $f > 0$. For that, suppose there exists $t_1 > t_0$ such that $f(t_1) \leq 0$. Since f is strictly decreasing on $[t_0, \infty)$, $f < 0$ on (t_1, ∞) . Therefore,

$$f'''(t) = mf'(t)^2 - \frac{m+1}{2}f(t)f''(t) > 0 \quad (t > t_1)$$

and f' is convex on $[t_1, \infty)$. But, this contradicts condition (1.4) and $f'(t_1) < 0$. So, $f > 0$ on $[t_0, \infty)$ ■

Remark 2.1. For $m > 0$ and $a < 0$, we do not know *a priori* if a concave solution of problem (1.1) - (1.4) becomes positive for large t . Nevertheless, this is true and will be deduced from Section 7 since for $m > 0$ we will construct a concave solution positive at infinity and prove that there is at most one concave solution.

Proposition 2.3. *For all $m \in \mathbb{R}$,*

$$\lim_{t \rightarrow \infty} f''(t) = 0 \tag{2.5}$$

and there exists a sequence $t_n \uparrow \infty$ such that

$$\lim_{n \rightarrow \infty} f'''(t_n) = \lim_{n \rightarrow \infty} f(t_n)f''(t_n) = 0 \tag{2.6}$$

Proof. Since $f'(\infty) = 0$, there exists a sequence $x_n \uparrow \infty$ satisfying $f''(x_n) \rightarrow 0$ (one can take x_n such that $f''(x_n) = f'(n+1) - f'(n)$). On the other hand, multiplying equation (1.1) by f'' and integrating by parts, we obtain

$$\frac{1}{2}f''(t)^2 - \frac{1}{2}f''(0)^2 - \frac{m}{3}f'(t)^3 + \frac{m}{3} = -\frac{m+1}{2} \int_0^t f(s)f''(s)^2 ds \tag{2.7}$$

for all t . But since f remains positive or negative for large t , the function $t \rightarrow \int_0^t f(s)f''(s)^2 ds$ has a limit as $t \rightarrow \infty$, and we deduce from (2.7) that $\lim_{t \rightarrow \infty} f''(t)^2$ exists. Then (2.5) holds. Furthermore, choosing (t_n) such that $f'''(t_n) = f''(n+1) - f''(n)$ and using (1.1) and (1.4) we get (2.6) ■

Remark 2.2. If $m > 0$, then f is bounded on $(0, \infty)$. In fact, if f is concave-convex, this is clear. Now, if f is concave and unbounded, then $f(\infty) = \infty$ and there exists t_1 such that

$$f'''(t) \geq -f''(t) \quad (t \geq t_1). \tag{2.8}$$

Since f'' is increasing on $[t_1, \infty)$, we deduce from (1.4) that $f''(\infty) = 0$, and by integrating (2.8) between $s \geq t_1$ and ∞ we obtain $-f''(s) \geq f'(s)$ for all $s \geq t_1$. Integrating again we get $-f'(t) + f'(t_1) \geq f(t) - f(t_1)$ for all $t \geq t_1$ and a contradiction with condition (1.4).

We will see that for some $m < 0$ there are unbounded solutions of problem (1.1) - (1.4).

3. Non-existence results for $m \leq -1$

Very often, the case $m \leq -1$ in equation (1.1) is not considered in physical papers (see Section 1). Nevertheless, it may be noted that in [22] one find a simple proof that problem (1.1) - (1.4) with $a = 0$ has no solutions for $m \leq -1$ (see also [5]). To be as exhaustive as possible, mathematically speaking, we propose the following (unfortunately, incomplete) generalization of that result from [22].

Theorem 3.1. *Let $m \leq -1$ in problem (1.1) - (1.4).*

- (i) *If $a \geq -\frac{2}{\sqrt{-m-1}}$, then the problem has no solution.*
- (ii) *If $a < -\frac{2}{\sqrt{-m-1}}$ and f is a solution of the problem, then necessarily $f < 0$.*

Proof. Let us assume that f is a solution of problem (1.1) - (1.4). Using Proposition 2.1, there exists $t_0 \geq 0$ such that $f''(t) < 0$ for $t > t_0$. On the other hand, if $f(t_1) \geq 0$ for some point t_1 , because f is increasing, $f'''(t) < 0$ for $t > \max(t_0, t_1)$ which contradicts (2.5) and the negativity of $f'''(t)$ for large t . Consequently, $f < 0$ and necessarily $a < 0$. Moreover, f is bounded, and integrating equation (1.1) between s and ∞ we obtain

$$-f''(s) - \frac{m+1}{2}f'(s)f(s) = \frac{3m+1}{2} \int_s^\infty f'(\eta)^2 d\eta < 0.$$

Integrating now on $[0, t]$ we get $1 - f'(t) - \frac{m+1}{4}(f(t)^2 - a^2) < 0$ ($t > 0$) which implies $0 \leq -\frac{m+1}{4}f(t)^2 < f'(t) - 1 - \frac{m+1}{4}a^2$ ($t > 0$) and $a < -\frac{2}{\sqrt{-m-1}}$. This completes the proof ■

Remark 3.1. Theorem 3.1 shows that, for $m = -1$ and for every $a \in \mathbb{R}$, there is no solution of problem (1.1) - (1.4). For $m < -1$ we do not know, if solutions of problem (1.1) - (1.4) exist for $-a$ large enough.

4. The case $m \in (-1, -1/3)$

For values $m \in (-1, -\frac{1}{3})$ existence and non-existence results for problem (1.1) - (1.4) again depend on a . In [8], a study of this case, using asymptotic expansions and numerical results, seems to indicate that solutions exist for $m > m_0(a)$ where $m_0(a) = -\frac{1}{2}$ if $a < 0$ and

$$m_0(a) = \begin{cases} -\frac{1}{2} & \text{as } a \rightarrow 0 \\ -1 & \text{as } a \rightarrow \infty \end{cases}$$

if $a > 0$. See also [18], and [5, 17] for the case $a = 0$. In what follows we will prove in a simple way non-existence when $a < 0$ and existence result for $m \in [-\frac{1}{2}, -\frac{1}{3})$ and $a > 0$. Obviously, this is very incomplete, because we have no existence results for $a < 0$, and no idea about the critical value $m_0(a)$.

Theorem 4.1. *For $m \in (-1, -\frac{1}{2}]$ and $a \leq 0$, problem (1.1) - (1.4) has no solution.*

Proof. Let us assume, contrary, that there is a solution f of problem (1.1) - (1.4). Thanks to Proposition 2.1, there exists some $s \geq 0$ such that $f(s) = 0$ and $f(t) > 0$ for $t > s$. Now, considering the sequence (t_n) defined by (2.6), multiplying equation (1.1) by f and integrating between s and t_n for n large enough, we get

$$\begin{aligned} 0 &\geq (2m + 1) \int_s^{t_n} f(\eta)f'(\eta)^2 d\eta \\ &= f(t_n)f''(t_n) - \frac{1}{2}f'(t_n)^2 + \frac{1}{2}f'(s)^2 + \frac{m+1}{2}f'(t_n)f(t_n)^2 \\ &\geq f(t_n)f''(t_n) - \frac{1}{2}f'(t_n)^2 + \frac{1}{2}f'(s)^2 \end{aligned}$$

which gives a contradiction as $n \rightarrow \infty$ (recall that $f' > 0$) ■

Theorem 4.2. *For $m \in [-\frac{1}{2}, -\frac{1}{3})$ and $a > 0$, problem (1.1) - (1.4) has an infinite number of solutions. Moreover, these solutions are unbounded.*

Proof. Let us introduce the initial value problem

$$(\mathcal{P}_{m,a,\mu}) \begin{cases} f''' + \frac{m+1}{2}ff'' - mf'^2 = 0 \\ f(0) = a \\ f'(0) = 1 \\ f''(0) = \mu \end{cases}$$

and let f_μ be its solution defined on $[0, T_\mu)$. Assume that f'_μ vanishes somewhere and denote by $t_1 > 0$ the point such that $f'_\mu(t_1) = 0$ and $f'_\mu > 0$ on $[0, t_1)$. Since f'_μ and f''_μ cannot vanish at the same point, $f''_\mu(t_1) < 0$. Moreover, $f_\mu > 0$ on $[0, t_1)$. Multiplying the differential equation in problem $(\mathcal{P}_{m,a,\mu})$ by f_μ and integrating it between 0 and t_1 , we derive

$$f_\mu(t_1)f''_\mu(t_1) - a\mu + \frac{1}{2} - \frac{m+1}{2}a^2 = (2m+1) \int_0^{t_1} f_\mu(\eta)f'_\mu(\eta)^2 d\eta \geq 0 \tag{4.1}$$

and therefore $\mu < \frac{1}{2a} - \frac{m+1}{2}a$. Consider now a $\mu \geq \frac{1}{2a} - \frac{m+1}{2}a$. Then f'_μ cannot vanish. On the other hand, integrating the differential equation in problem $(\mathcal{P}_{m,a,\mu})$ we get

$$f''_\mu + \frac{m+1}{2}f_\mu f'_\mu < \mu + \frac{m+1}{2}a. \tag{4.2}$$

Hence, in view of Lemma 2.1 and since $f'_\mu > 1$ as long as $f''_\mu > 0$, there is some t_0 such that $f''_\mu(t) < 0$ for $t > t_0$. It follows $T_\mu = \infty$ and $f'_\mu(t) \rightarrow l \geq 0$ as $t \rightarrow \infty$. If $l > 0$, then $f_\mu(t)f'_\mu(t) \rightarrow \infty$ as $t \rightarrow \infty$ and (4.2) implies that $f''_\mu(t) \rightarrow -\infty$ and f'_μ must become negative, which is a contradiction. Therefore, $l = 0$ and f_μ is a solution of problem (1.1) - (1.4). Coming back to (4.1) we easily get a contradiction if f_μ is assumed to be bounded ■

Remark 4.1. For $m \in [-\frac{1}{2}, -\frac{1}{3})$ and $a \geq \frac{1}{\sqrt{m+1}}$ we have $\frac{1}{2a} - \frac{m+1}{2}a \leq 0$, and if $\mu \in [\frac{1}{2a} - \frac{m+1}{2}a, 0]$, then the solution f_μ of problem $(\mathcal{P}_{m,a,\mu})$ is concave (i.e. it satisfies constraint (1.6)).

Remark 4.2. Let $m \in (-1, -\frac{1}{3})$ and let f be a solution of problem (1.1) - (1.4). Integrating equation (1.1) on $[0, t]$, we get

$$f''(t) - f''(0) + \frac{m+1}{2}(f'(t)f(t) - a) = \frac{3m+1}{2} \int_0^t f'(\eta)^2 d\eta.$$

Since $f' > 0$ and $f(t) > 0$ for large t , we obtain for such a t

$$0 < -\frac{3m+1}{2} \int_0^t f'(\eta)^2 d\eta \leq f''(0) - f''(t) + \frac{m+1}{2}a$$

for such a t . This implies $f' \in L^2(0, \infty)$ and, since $m \neq -1$, $f(t)f'(t)$ has a limit $c \geq 0$ as $t \rightarrow \infty$. But, if $c > 0$ we easily get a contradiction with the fact that $f' \in L^2(0, \infty)$ (see [5: Theorem 4.2]). Finally, $f'(t)f(t) \rightarrow 0$ as $t \rightarrow \infty$ and $f''(0) = -\frac{m+1}{2}a - \frac{3m+1}{2} \int_0^\infty f'(\eta)^2 d\eta$.

5. The special case $m = -1/3$

Let us denote by f_μ the solution of problem $(\mathcal{P}_{-\frac{1}{3},a,\mu})$ and by $[0, T_\mu)$ with $0 < T_\mu \leq \infty$ its right maximal interval of existence. Integrating twice the differential equation herein we get

$$f''_\mu + \frac{1}{3}f_\mu f'_\mu = \mu + \frac{1}{3}a \tag{5.1}$$

$$f'_\mu(t) + \frac{1}{6}f_\mu(t)^2 = (\mu + \frac{1}{3}a)t + 1 + \frac{1}{6}a^2. \tag{5.2}$$

The last equation is of Riccati type and give the explicit solution

$$f_{-\frac{a}{3}}(t) = \kappa \frac{a \cosh \frac{\kappa t}{6} + \kappa \sinh \frac{\kappa t}{6}}{a \sinh \frac{\kappa t}{6} + \kappa \cosh \frac{\kappa t}{6}}$$

of problem $(\mathcal{P}_{-\frac{1}{3},a,-\frac{a}{3}})$ where $\kappa = \sqrt{a^2 + 6}$. This solution is given for $a = 0$ in [5, 20] and for the general case in [18]. It is defined on $[0, \infty)$ and satisfies condition (1.4). But we have more:

Theorem 5.1. *For all $\mu \geq -\frac{a}{3}$ the solution f_μ of problem $(\mathcal{P}_{-\frac{1}{3},a,\mu})$ is defined on $[0, \infty)$ and satisfies $\lim_{t \rightarrow \infty} f'_\mu(t) = 0$, i.e. problem (1.1) – (1.4) has for $m = -\frac{1}{3}$ an infinite number of solutions.*

Proof. Let $\mu \geq -\frac{a}{3}$. Assume first $\mu \leq 0$. Then, from Lemma 2.1, $f''_\mu < 0$ and (5.1) shows that f'_μ cannot vanish. It follows $T_\mu = \infty$ and $f'_\mu(t)$ has a limit $l \geq 0$ as $t \rightarrow \infty$. Suppose $l > 0$. Then $f_\mu(t)f'_\mu(t) \rightarrow \infty$ as $t \rightarrow \infty$ and (5.1) implies $f''_\mu(t) \rightarrow -\infty$ and f'_μ must become negative, which is a contradiction. Therefore, $l = 0$.

Assume now $\mu > 0$. Since $f'_\mu > 1$ as long as $f''_\mu > 0$, we deduce from (5.1) that f''_μ has a zero at some point $t_0 > 0$. It then follows from Lemma 2.1 that $f''_\mu(t) < 0$ for $t > t_0$ and we can conclude as previously ■

Remark 5.1. If $\mu < -\frac{a}{3}$, then (5.2) shows that f_μ cannot satisfies condition (1.4).

Remark 5.2. It follows from Theorem 5.1 that for $m = -\frac{1}{3}$ one has the following possibilities:

- If $a > 0$, then problem (1.1) - (1.4) has an infinite number of concave solutions (i.e. such that (1.6) holds), which are f_μ for $\mu \in [-\frac{a}{3}, 0]$.
- If $a = 0$, then problem (1.1) - (1.4) has one and only one concave solution.
- If $a < 0$, then problem (1.1) - (1.4) has no concave solution.

Remark 5.3. It follows from (5.2) that, if $\mu > -\frac{a}{3}$, then f_μ is unbounded. Thus problem (1.1) - (1.4) has only one bounded solution, which is $f_{-\frac{a}{3}}$.

6. Existence, uniqueness and non-uniqueness results for $m \in (-1/3, 0)$

For $m \in (-\frac{1}{3}, 0)$ and $a = 0$, a proof of the existence of a concave and bounded solution is given in [5]. In this section, we obtain in Theorem 6.1 a similar result for $a \neq 0$, that we prove by using the solution corresponding to $a = 0$. A direct approach (in this case, as in the case $m > 0$) could be obtained, even if some difficulties should appear, especially when $a < 0$, by adapting the proof of [5], but the method we use in Theorems 6.1 and 7.1 is actually easier, and consists in to remark that, if g is a solution of equation (1.1), then it is so for the function $t \rightarrow kg(kt + t_0)$, for all $k > 0$ and all t_0 .

On the other hand, in [5] it was conjectured that there is one and only one concave solution of problem (1.1) - (1.4) (as for $m \geq 0$), but this is not true as shown in [14]. The concave solutions of problem (1.1) - (1.4) exhibited in [14] are unbounded, and in fact uniqueness holds for bounded solutions, at least when $a \geq 0$.

Theorem 6.1. *Let $a \in \mathbb{R}$. If $m \in (-\frac{1}{3}, 0)$, then problem (1.1) - (1.4) has a bounded solution f , which is positive at infinity, increasing and satisfying*

$$a \leq f(t) \leq \sqrt{a^2 + \frac{4}{m+1}} \quad (t \geq 0). \quad (6.1)$$

Moreover, if $a \geq 0$, such a solution is unique.

Proof. *Existence:* Let us denote by g the solution of problem (1.1) - (1.4), constructed in [5] and corresponding to $a = 0$, i.e. such that $g(0) = 0$.

The case $a > 0$: Since for all $k > 0$ and all t_0 the function

$$f : t \mapsto kg(kt + t_0) \quad (6.2)$$

satisfies equation (1.1), let us try to choose k and t_0 in order to get a solution of problem (1.1) - (1.4) with $a \neq 0$. First, note that the function

$$h : t \mapsto \frac{g(t)^2}{g'(t)} \quad (6.3)$$

is well defined on $[0, \infty)$ and satisfies $h(0) = 0$ and $h(t) \rightarrow \infty$ as $t \rightarrow \infty$. Thus, there exists a point $t_0 > 0$ such that $h(t_0) = a^2$. Then choosing

$$k = \frac{a}{g(t_0)} \quad (6.4)$$

it is easy to see that, for $a > 0$, the function f defined by (6.2) with these values of k and t_0 is a solution of problem (1.1) - (1.4). Moreover, f as g is concave.

The case $a < 0$: Let us consider again the function h defined by (6.3). To apply the previous method, we have to look at $g(t)$ for negative values of t . Denote by $(-T, \infty)$ the maximal existence interval of g . It is easy to see that if g' does not vanish, then $T = \infty$. In fact, if $T < \infty$, then from Lemma 2.2, $g(t) \rightarrow -\infty$, $g'(t) \rightarrow \infty$ and $g''(t) \rightarrow -\infty$ as $t \rightarrow -T$. But in this case equation (1.1) gives $g'''(t) \rightarrow -\infty$ which is a contradiction.

- If g' vanishes somewhere, let $t_1 < 0$ be such that $g'(t_1) = 0$ and $g' > 0$ on $(t_1, 0)$. Then h is defined on $(t_1, 0]$ and $h(t) \rightarrow \infty$ as $t \rightarrow t_1$.

- Assume now that $g' > 0$. If h is bounded on $(-\infty, 0)$, then there is a $c > 0$ such that $\frac{g'(t)}{g(t)^2} > c$ for all $t < 0$. Integration gives $-\frac{1}{g(t)} + \frac{1}{g(s)} > c(t - s)$ for all $s < t < 0$ and, since $g(s) < 0$ for $s < 0$, $-\frac{1}{g(t)} > c(t - s)$ for all $s < t < 0$. This is a contradiction by passing to the limit as $s \rightarrow -\infty$. Hence, h is unbounded on $(-\infty, 0)$.

Therefore, in any case, h is unbounded and there exists $t_0 < 0$ such that $h(t_0) = a^2$. Now, if we choose k as in (6.4), the function f defined by (6.2) is a solution of problem (1.1) - (1.4).

Next, $f(t) > 0$ for $t > -\frac{t_0}{k}$, and since g is bounded, this is so for f . Moreover, to derive (6.1) we multiply equation (1.1) by t and integrate by parts to get

$$\begin{aligned}
 tf''(t) + \frac{m+1}{2}tf'(t)f(t) - f'(t) + 1 - \frac{m+1}{4}(f(t)^2 - a^2) \\
 = \frac{3m+1}{2} \int_0^t \eta f'(\eta)^2 d\eta.
 \end{aligned}
 \tag{6.5}$$

Now, using boundedness and concavity of f for large t , one deduce $\lim_{t \rightarrow \infty} tf'(t)f(t) = \lim_{t \rightarrow \infty} tf''(t) = 0$ and, passing to the limit as $t \rightarrow \infty$ in (6.5), we get

$$1 - \frac{m+1}{4}(f(\infty)^2 - a^2) = \frac{3m+1}{2} \int_0^\infty sf'(s)^2 ds > 0.
 \tag{6.6}$$

To conclude, it is enough to remark that f is increasing.

Uniqueness: We assume here $a \geq 0$. First of all, if f is a solution of problem (1.1) - (1.4), then f is increasing and we can define a function $v = v(y)$ such that $v(f(t)) = f'(t)$ for all $t \geq 0$. If f is bounded and $f(\infty) = \lambda$, then v is defined on $[a, \lambda)$, positive and

$$\begin{aligned}
 f''(t) &= f'(t)v'(f(t)) = v(f(t))v'(f(t)) \\
 f'''(t) &= f''(t)v'(f(t)) + f'(t)^2v''(f(t)) \\
 &= v(f(t))v'(f(t))^2 + v(f(t))^2v''(f(t))
 \end{aligned}$$

so much so that we get

$$v'' = -\frac{1}{v}(v' + \frac{m+1}{2}y)v' + m \quad (y \in [a, \lambda)).
 \tag{6.7}$$

In addition, $v(a) = f'(0) = 1$ and $v(\lambda) := \lim_{y \rightarrow \lambda} v(y) = \lim_{t \rightarrow \infty} f'(t) = 0$. Suppose there are two bounded solutions f_i of problem (1.1) - (1.4) and set $\lambda_i = f_i(\infty)$ ($i = 1, 2$). They give for equation (6.7) two solutions v_i defined on $[a, \lambda_i)$ such that $v_i(a) = 1$ and $v_i(\lambda_i) = 0$. Let us assume that $\lambda_1 \leq \lambda_2$ and prove that $v_1 \leq v_2$ on $[a, \lambda_1)$. Suppose $v_1(y) > v_2(y)$ for some $y \in (a, \lambda_1)$. Since $v_1(a) = v_2(a)$ and $v_1(\lambda_1) = 0 \leq v_2(\lambda_1)$, then $v_1 - v_2$ has a positive maximum at a point $x \in (a, \lambda_1)$. Hence $v_1(x) > v_2(x)$, $v_1'(x) = v_2'(x)$ and $v_1''(x) \leq v_2''(x)$. But, on the other hand,

$$v_1''(x) - v_2''(x) = \left(\frac{1}{v_2(x)} - \frac{1}{v_1(x)}\right)(v_1'(x) + \frac{m+1}{2}x)v_1'(x)
 \tag{6.8}$$

and

$$\begin{aligned} (v_1'(x) + \frac{m+1}{2}x)v_1'(x) &= (v_1(x)v_1'(x) + \frac{m+1}{2}xv_1(x))\frac{v_1'(x)}{v_1(x)} \\ &= (f_1''(s) + \frac{m+1}{2}f_1(s)f_1'(s))\frac{f_1''(s)}{f_1'(s)^2} \end{aligned} \tag{6.9}$$

for s such that $x = f_1(s)$. Since f_1 is bounded, by integrating equation (1.1) we get

$$f_1''(t) + \frac{m+1}{2}f_1(t)f_1'(t) = -\frac{3m+1}{2}\int_t^\infty f_1'(\eta)^2 d\eta < 0$$

for all $t \geq 0$. This together with $f_1 \geq 0$ imply $f_1'' < 0$ and, in view of (6.8) - (6.9), $v_1''(x) - v_2''(x) > 0$ which is a contradiction. Therefore, $v_1 \leq v_2$ on $[a, \lambda_1)$. It follows

$$\int_0^\infty f_1'(\eta)^2 d\eta = \int_a^{\lambda_1} v_1(y) dy \leq \int_a^{\lambda_1} v_2(y) dy \leq \int_a^{\lambda_2} v_2(y) dy = \int_0^\infty f_2'(\eta)^2 d\eta$$

and since $f_1''(0) + \frac{m+1}{2}a = -\frac{3m+1}{2}\int_0^\infty f_1'(\eta)^2 d\eta$ we get $f_1''(0) \geq f_2''(0)$. But, if $f_1''(0) > f_2''(0)$, we get $v_1'(a) > v_2'(a)$ which contradicts the fact that $v_1 \leq v_2$ on $[a, \lambda_1)$. So, $f_1''(0) = f_2''(0)$ and $f_1 = f_2$. This completes the proof ■

Remark 6.1. One can see in the previous proof that if there are two distinct solutions f_1 and f_2 of problem (1.1) - (1.4) when $a < 0$, then necessarily both are convex-concave and the point x where $v_1'(x) = v_2'(x)$ is negative and satisfies $v_1'(x) > 0$. This means that, if there is a bounded concave solution of problem (1.1) - (1.4), then there is no other bounded solution.

Remark 6.2. For $a \geq 0$, the solution f of problem (1.1) - (1.4) constructed in Theorem 6.1 is strictly concave on $[0, \infty)$, and thus $sf'(s) < f(s) - a$ for $s > 0$ which implies $\int_0^\infty sf'(s)^2 ds < \frac{1}{2}(f(\infty) - a)^2$. Therefore (6.6) gives the relation

$$(2m + 1)f(\infty)^2 - (3m + 1)af(\infty) + ma^2 - 2 > 0$$

from which and (6.1) we derive the estimate

$$\frac{(3m+1)a + \sqrt{(m+1)^2 a^2 + 4(4m+2)}}{4m+2} \leq f(\infty) \leq \sqrt{a^2 + \frac{4}{m+1}}. \tag{6.10}$$

Note that these bounds have some optimality, in the sense that both are equal to $\sqrt{a^2 + 6}$ for $m = -\frac{1}{3}$.

Theorem 6.2. *If $m \in (-\frac{1}{3}, 0)$, then problem (1.1) - (1.4) has an infinite number of unbounded solutions.*

Proof (following an idea of [14]). Let us consider problem $(\mathcal{P}_{m,a,\mu})$ and again let f_μ be its solution defined on $[0, T_\mu)$. Integrating the differential equation in $(\mathcal{P}_{m,a,\mu})$ between 0 and $t < T_\mu$ we get

$$f_\mu''(t) + \frac{m+1}{2}f_\mu'(t)f_\mu(t) = \mu + \frac{m+1}{2}a + \frac{3m+1}{2}\int_0^t f_\mu'(\eta)^2 d\eta. \tag{6.11}$$

For the rest of this proof we will assume $\mu \geq -\frac{m+1}{2}a$. Then

$$f''_{\mu}(t) + \frac{m+1}{2}f'_{\mu}(t)f_{\mu}(t) > 0 \quad (t \in [0, T_{\mu})) \tag{6.12}$$

from which $f'_{\mu} > 0$ follows. Indeed, since $f'_{\mu}(0) = 1$, $f''_{\mu}(t_1) \leq 0$ for t_1 the first point where f'_{μ} vanishes, which contradicts (6.12). In view of Lemma 2.1, $f'_{\mu}(t) \rightarrow l \in [0, \infty]$ as $t \rightarrow T_{\mu}$. Suppose $l \neq 0$. Then there exists s such that $f_{\mu} > 0$ on $[s, T_{\mu})$. Indeed, either $T_{\mu} = \infty$ and it is clear, or $T_{\mu} < \infty$ and $f_{\mu}(t) \rightarrow \infty$ as $t \rightarrow T_{\mu}$. Next, multiplying the differential equation in $(\mathcal{P}_{m,a,\mu})$ by f''_{μ} and integrating by parts, we obtain

$$\frac{1}{2}f''_{\mu}(t)^2 - \frac{1}{2}f''_{\mu}(s)^2 - \frac{m}{3}f'_{\mu}(t)^3 + \frac{m}{3}f'_{\mu}(s)^3 = -\frac{m+1}{2} \int_s^t f_{\mu}(\eta)f''_{\mu}(\eta)^2 d\eta$$

for all $t \in (s, T_{\mu})$. Since $m < 0$, l is finite and $T_{\mu} = \infty$ by virtue of Lemma 2.2. Moreover, we assumed $l \neq 0$ and thus $f_{\mu}(t) \sim lt$ for $t \rightarrow \infty$. Coming back to (6.11), we obtain $f''_{\mu}(t) \sim -\frac{m+1}{2}l^2t + \frac{3m+1}{2}l^2t = ml^2t$ ($t \rightarrow \infty$) which contradicts $f'_{\mu}(t) \sim l$ for $t \rightarrow \infty$. Finally, $l = 0$ and f_{μ} is a solution of problem (1.1) - (1.4) which is necessarily unbounded in accordance with (6.11) ■

Remark 6.3. For $a \geq 0$ and $\mu \in [-\frac{m+1}{2}a, 0]$, the solution f_{μ} of problem $(\mathcal{P}_{m,a,\mu})$ is concave (i.e. it satisfies constraint (1.6)).

Remark 6.4. One can show (see [14]), for unbounded solutions f of problem (1.1) - (1.4) exhibited in the proof of Theorem 6.2, $f'(t)f(t) \rightarrow \infty$ as $t \rightarrow \infty$ (compare with Remark 4.2).

7. Existence and uniqueness results for $m \geq 0$

The first result of this section says that, for $m \geq 0$, there is one and only concave solution of problem (1.1) - (1.4).

Theorem 7.1. *Let $a \neq 0$. If $m \geq 0$, then problem (1.1) – (1.4) has one and only one concave solution f , which is positive at infinity and such that*

$$a \leq f(t) \leq \sqrt{a^2 + \frac{4}{m+1}} \tag{7.1}$$

for all $t \geq 0$.

Proof. *Existence:* Let us denote again by g the solution of problem (1.1) - (1.4), constructed in [5] and corresponding to $a = 0$.

The case $a > 0$: The proof is exactly the same as that of Theorem 6.1.

The case $a < 0$: As in Theorem 6.1, denote by $(-T, \infty)$ the maximal existence interval of the function g and consider the function h defined by (6.2). In view of Lemma 2.1, g is concave, $g' > 0$ and h is defined on $(-T, \infty)$. Our goal is to prove that h is unbounded on $(-T, 0)$.

If $T = \infty$, then we can reproduce exactly the reasoning in the proof of Theorem 6.1. Suppose now $T < \infty$. Thanks to Lemma 2.2, $g(t), g''(t) \rightarrow -\infty$ and $g'(t) \rightarrow \infty$ as $t \rightarrow -T$. Then, using (2.4) and the fact that $g'''(0) = m \geq 0$ we see that $g''' > 0$ on $(-T, 0)$. Hence, if we set $\beta = \frac{2m}{m+1}$, we get $-gg'' + \beta g'^2 > 0$ from where we deduce that the function $\varphi = g'(-g)^{-\beta}$ is positive and increasing on $(-T, 0)$. Thus, φ is bounded as $t \rightarrow -T$ and $h(t)^{-1} = \varphi(t)|g(t)|^{\beta-2} \rightarrow 0$ as $t \rightarrow -T$, since $g(t) \rightarrow -\infty$ as $t \rightarrow -T$ and $\beta < 2$. Hence, h is unbounded on $(-T, 0)$.

Therefore, in any case, h is unbounded and we get, as in Theorem 6.1, that problem (1.1) - (1.4) has a solution f positive at infinity and satisfying estimate (7.1). Moreover, f as g is concave.

Uniqueness: Let f_1 and f_2 be two concave solutions of problem (1.1) - (1.4). Let us assume $f_1''(0) > f_2''(0)$ and introduce the function $k = f_1 - f_2$. One has $k(0) = 0$, $k'(0) = 0$ and $k''(0) > 0$. Since $k'(\infty) = 0$, there is a point $t_0 > 0$ such that $k' > 0$ on $(0, t_0]$, $k''(t_0) = 0$ and

$$k'''(t_0) \leq 0. \tag{7.2}$$

Moreover, $k(t_0) > 0$. Now, using the equality $f_1''(t_0) = f_2''(t_0)$, we can write

$$k'''(t_0) = f_1'''(t_0) - f_2'''(t_0) = mk'(t_0)(f_1'(t_0) + f_2'(t_0)) - \frac{m+1}{2} f_1''(t_0)k(t_0)$$

which gives $k'''(t_0) > 0$ and a contradiction with (7.2) ■

Remark 7.1. For $m = 1$, the function g is given by $g(t) = 1 - e^{-t}$ (see [5, 11, 20]) and the function f defined by (6.2) can be computed; we get

$$f(t) = a + (c - a)(1 - e^{-ct}) \tag{7.3}$$

with $c = \frac{1}{2}(a + \sqrt{a^2 + 4})$. This explicit form was first given in [15] (see also [18]). Note that we can recover (7.3) directly by the method used in [5] to get the function g . On the other hand, f is the unique concave solution of problem (1.1) - (1.4).

Remark 7.2. The function f constructed in Theorem 7.1 is strictly concave on $[0, \infty)$, and thus estimate (6.10) still holds. Since f is positive at infinity, the lower bound in (6.10) can be replaced by 0 when this one is negative. In fact, this is the case for $m > 0$ and $a < -\sqrt{\frac{2}{m}}$. Note also that for $a \geq 0$ the lower and upper bounds tend to a as $m \rightarrow \infty$.

Remark 7.3. The upper bound in (7.1) is still valid for a concave-convex solution f of problem (1.1) - (1.4) when $m > 0$. To see that, it is enough to write relation (6.5) for $t = t_1$, where $t_1 > 0$ is the point such that $f'(t_1) = 0$, i.e. where f achieves its maximum. The lower bound has to be replaced by $\min(a, 0)$.

We would like to finish this section by dealing with uniqueness for problem (1.1) - (1.4). We saw in the previous parts that in several cases this problem has more than one solution. In [5], uniqueness is obtained for $m \in [0, \frac{1}{3}]$. The proof consists to remark that if a concave-convex solution f of problem (1.1) - (1.4) exists, then f''' has to vanish, and to get a contradiction by considering f'''' and using the fact that f' and f'' do not vanish after the point where f''' vanishes.

In order to improve this result, say to obtain that uniqueness holds for $m \in [0, 1)$ (as numerical attempts seem to indicate, see [6]), the idea is to look at higher derivatives of solutions. The method consists again to exclude concave-convex solutions. But to conclude, we *a priori* need to know that for all k the derivative $f^{(k)}$ of a possible concave-convex solution f does not vanish in some interval (t_k, ∞) . As we will see, this is true if $f(t)$ tends to some $\lambda > 0$ as $t \rightarrow \infty$. Unfortunately, we are unable to prove this property when $f(t) \rightarrow 0$ as $t \rightarrow \infty$ (recall that $f(\infty) \geq 0$; see Proposition 2.2).

In fact, it seems that, at least for $m \in [0, 1]$, any solution f of equation (1.1) tending to 0 at infinity satisfies

$$f(t) \sim \frac{6}{t}, \quad f'(t) \sim -\frac{6}{t^2}, \quad f''(t) \sim \frac{12}{t^3} \quad (t \rightarrow \infty). \tag{7.4}$$

If these relations were true, by using (1.1) we could assert $f^{(k)}(t) \sim (-1)^k \frac{c_k}{t^{k+1}}$ as $t \rightarrow \infty$ for some constant $c_k > 0$, and conclude.

In order to have partial results, let us say that a solution of equation (1.1) is *oscillating*, if there is an integer k (necessarily $k > 2$) such that, for all $n > 0$, there is a point $s_n > n$ such that $f^{(k)}(s_n) = 0$. If f is a non-oscillating concave-convex solution of problem (1.1) - (1.4), then it is easy to show by induction that there exists a sequence $0 < t_k \uparrow$ such that

$$f^{(k)}(t_k) = 0 \quad (k \geq 1) \quad \text{and} \quad (-1)^i f^{(i)} > 0 \quad (1 \leq i \leq k) \quad \text{on} \quad (t_k, \infty). \tag{7.5}$$

The following results show that, for $m \in (\frac{1}{3}, 1)$, problem (1.1) - (1.4) may have as concave-convex solutions only oscillating solutions tending to 0 at infinity, and thus such that (7.4) does not hold. In fact, we would deduce uniqueness for $m \in [0, 1)$ if we were able to proof relations (7.4) for concave-convex solutions tending to 0 at infinity.

Proposition 7.1 (see [5]). *For $m \in [0, \frac{1}{3}]$, problem (1.1) – (1.4) has one and only one solution.*

Proposition 7.2. *Let $m > \frac{1}{3}$ and assume f is a concave-convex solution of problem (1.1) – (1.4) tending to $\lambda > 0$ at infinity. Then f is non-oscillating.*

Proof. We will use asymptotic relations at infinity of f , f' and f'' , and we relegate to the Appendix for the proof of these facts.

Consider f , a concave-convex solution of problem (1.1) - (1.4) such that $f(t) \rightarrow \lambda > 0$ as $t \rightarrow \infty$. Then

$$f(t) \sim \lambda, \quad f'(t) \sim -c_1 e^{-\frac{m+1}{2}\lambda t}, \quad f''(t) \sim c_2 e^{-\frac{m+1}{2}\lambda t} \quad (t \rightarrow \infty) \tag{7.6}$$

where $c_1, c_2 > 0$. Hence from equation (1.1) we get $f'''(t) \sim -c_3 e^{-\frac{m+1}{2}\lambda t}$ as $t \rightarrow \infty$ and easily, by induction, we see that for all $k \geq 1$ there is a constant $c_k > 0$ such that $f^{(k)}(t) \sim (-1)^k c_k e^{-\frac{m+1}{2}\lambda t}$ as $t \rightarrow \infty$. Therefore, f is non-oscillating ■

Proposition 7.3 *Let $m \in (\frac{1}{3}, 1)$. If problem (1.1) – (1.4) has a concave-convex solution f , then f is oscillating and $f(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Proof. Suppose there is a solution f of problem (1.1) - (1.4) which is concave-convex. Suppose, moreover, f is non-oscillating. Successive differentiations of equation (1.1) lead to the formula

$$f^{(2n+2)} = \sum_{k=0}^n (\mu_{n,k}m - \nu_{n,k}) f^{(2n+1-k)} f^{(k)} \quad (n \geq 1)$$

where

$$\begin{aligned} \nu_{n,0} &= \frac{1}{2} & \mu_{n,0} &= -\frac{1}{2} \\ \nu_{n,1} &= n - \frac{1}{2} & \mu_{n,1} &= -n + \frac{3}{2} \\ \nu_{n,k} &= \frac{1}{2} \left[\binom{2n-1}{k-2} + \binom{2n-1}{k} \right], & \mu_{n,k} &= 2 \binom{2n-1}{k-1} - \nu_{n,k} \quad (2 \leq k \leq n). \end{aligned}$$

Moreover, one has

$$\frac{\mu_{n,k}}{\nu_{n,k}} = 2 \left(\frac{n(2n+1)}{k(2n+1-k)} - 1 \right)^{-1} - 1 \quad (2 \leq k \leq n)$$

in such a way that

$$\frac{\mu_{n,2}}{\nu_{n,2}} < \dots < \frac{\mu_{n,k}}{\nu_{n,k}} < \dots < \frac{\mu_{n,n}}{\nu_{n,n}}. \tag{7.7}$$

Now, since $\frac{\mu_{n,n}}{\nu_{n,n}} = \frac{n+2}{n} \rightarrow 1^+$ as $n \rightarrow \infty$, there exists p such that $\frac{\mu_{p,p}}{\nu_{p,p}} < \frac{1}{m}$. Then, using (7.7) we get $\mu_{p,k}m - \nu_{p,k} < \mu_{p,p}m - \nu_{p,p} < 0$ ($0 \leq k \leq p$) from where and (7.5) we deduce $f^{(2p+2)}(t) > 0$ for all $t > t_{2p+1}$. But this contradicts $f^{(2p+2)}(t_{2p+2}) = 0$. Therefore, f is oscillating and, in view of Propositions 2.2 and 7.2, $f(t) \rightarrow 0$ as $t \rightarrow \infty$. This completes the proof ■

8. Appendix

Here we will prove relations (7.6). These results will be based on asymptotic integrations of second order and linear differential equations (see [13, 16]).

If f is a solution of equation (1.1), for τ large enough put

$$y(t) = f'(t) e^{\frac{m+1}{4} \int_{\tau}^t f(s) ds}. \tag{8.1}$$

Then y satisfies the differential equation

$$y'' - qy = 0 \tag{8.2}$$

where $q = \frac{5m+1}{4} f' + \left(\frac{m+1}{4}\right)^2 f^2$. Assume now that $\lambda = f(\infty) > 0$. Hence

$$q(t) \sim \left(\frac{m+1}{4}\right)^2 \lambda^2 \quad (t \rightarrow \infty) \tag{8.3}$$

and it is easy to verify that the integrals

$$\int^{\infty} q''(s)q(s)^{-\frac{3}{2}} ds \quad \text{and} \quad \int^{\infty} q'(s)^2 q(s)^{-\frac{5}{2}} ds$$

converge. Therefore (see [13] or [16]) equation (8.2) has a fundamental system of solutions $\{y_1, y_2\}$ such that

$$\begin{aligned} y_1(t) &\sim q(t)^{-\frac{1}{4}} e^{-\int_{\tau}^t \sqrt{q(s)} ds} \\ y_2(t) &\sim q(t)^{-\frac{1}{4}} e^{\int_{\tau}^t \sqrt{q(s)} ds} \end{aligned} \quad (t \rightarrow \infty).$$

Then, in view of (8.3) we get

$$\begin{aligned} y_1(t) &\sim \frac{2}{\sqrt{m+1}} e^{-\frac{m+1}{4} \lambda t} \\ y_2(t) &\sim \frac{2}{\sqrt{m+1}} e^{\frac{m+1}{4} \lambda t} \end{aligned} \quad (t \rightarrow \infty).$$

But, on the other hand, definition (8.1) gives $f'(t) \sim y(t)e^{-\frac{m+1}{4} \lambda t}$ as $t \rightarrow \infty$, and since $f'(t) \rightarrow 0$ as $t \rightarrow \infty$, then necessarily y is proportional to y_1 and

$$f'(t) \sim c_1 e^{-\frac{m+1}{2} \lambda t} \quad (t \rightarrow \infty) \tag{8.4}$$

for some $c_1 \in \mathbb{R}^*$. In addition, integrating equation (1.1) between t and ∞ we get $f''(t) + \frac{m+1}{2} f(t)f'(t) = -\frac{3m+1}{2} \int_t^{\infty} f'(s)^2 ds$ which implies

$$f''(t) \sim -\frac{m+1}{2} \lambda c_1 e^{-\frac{m+1}{2} \lambda t} \quad (t \rightarrow \infty). \tag{8.5}$$

This completes the proof of relations (7.6). Note also that asymptotic relations (8.4) and (8.5) hold, with $c_1 > 0$, for all bounded solutions concave at infinity and, with $c_1 < 0$, for all concave-convex solutions of problem (1.1) - (1.4) tending to $\lambda > 0$ at infinity; recall that we do not know whether such concave-convex solutions exist or not.

References

- [1] Banks, W. H. H.: *Similarity solutions of the boundary layer equations for a stretching wall.* J. de Mécan. théor. et appl. 2 (1983), 375 – 392.
- [2] Banks, W. H. H. and M. B. Zaturka: *Eigensolutions in boundary layer flow adjacent to a stretching wall.* IMA J. Appl. Math. 36 (1986), 263 – 273.
- [3] Belhachmi, Z., Brighi, B. and K. Taous: *Solutions similaires pour un problème de couches limites en milieux poreux.* C.R. Mécanique 328 (2000), 407 – 410.
- [4] Belhachmi, Z., Brighi, B. and K. Taous: *On the concave solutions of the Blasius equation.* Acta Math. Univ. Comenianae 49 (2000), 199 – 214.
- [5] Belhachmi, Z., Brighi, B. and K. Taous: *On a family of differential equations for boundary layer approximations in porous media.* European J. Appl. Math. 12 (2001), 513 – 528.

- [6] Belhachmi, Z., Brighi, B., Sac Epée, J. M. and K. Taous: *Numerical simulations of free convection about vertical flat plate embedded in porous media.* (to appear).
- [7] Brighi, B. and T. Sari (in preparation).
- [8] Chaudary, M. A., Merkin, J. H. and I. Pop: *Similarity solutions in free convection boundary layer flows adjacent to vertical permeable surfaces in porous media.* Part I: *Prescribed surface temperature.* European J. Mech. B - Fluids 14 (1995), 217 – 237.
- [9] Cheng, P. and W. J. Minkowycz: *Free convection about a vertical flat plate embedded in a porous medium with application to heat transfer from a dike.* J. Geophys. Res. 82 (1977), 2040 – 2044.
- [10] Coppel, W. A.: *On a differential equation of boundary layer theory.* Phil. Trans. Roy. Soc. London (Ser. A) 253 (1960), 101 – 136.
- [11] Crane, L. E.: *Flow past a stretching plane.* Z. Ang. Math. Phys. 21 (1970), 645 – 647.
- [12] Ene, H. I. and D. Poliševski: *Thermal Flow in Porous Media.* Dordrecht: D. Reidel Publ. Comp. 1987.
- [13] Fedoryuk, M. V.: *Asymptotic Analysis.* Berlin - Heidelberg: Springer-Verlag 1993.
- [14] Guedda, M.: *Nonuniqueness of solutions to differential equations for boundary layer approximations in porous media.* C.R. Mecanique 330 (2002), 279 – 283.
- [15] Gupta, P. S. and A. S. Gupta: *Heat and mass transfer on a stretching sheet with suction or blowing.* Can. J. Chem. Eng. 55 (1977), 744 – 746.
- [16] Hartmann, P.: *Ordinary Differential Equations.* New York: Wiley 1964.
- [17] Ingham, D. B. and S. N. Brown: *Flow past a suddenly heated vertical plate in a porous medium.* Proc. R. Soc. Lond. A 403 (1986), 51 – 80.
- [18] Magyari, E. and B. Keller: *Exact solutions for self-similar boundary layer flows induced by permeable stretching walls.* European J. Mech. B - Fluids 19 (2000), 109 – 122.
- [19] Oleinik, O. A. and V. N. Samokhin: *Mathematical Models in Boundary Layer Theory.* Boca Raton: Chapman and Hall/CRC 1999.
- [20] Stuart, J. T.: *Double boundary layers in oscillatory viscous flow.* J. Fluid Mech. 24 (1966), 673 – 687.
- [21] Rosenhead, L.: *Laminar Boundary Layers.* Oxford: Oxford Univ. Press 1963.
- [22] Watson, J.: *On the existence of solutions for a class of rotating-disc flows and the convergence of a successive approximation scheme.* NPL Aero Rep. 1134 (1965).