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Numerical Method of Lines for First Order Partial Differential-Functional Equations

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Abstract. We consider the Cauchy problem for a nonlinear equation on the Haar pyramid. By using a discretization with respect to spatial variables, the partial functional-differential equation is transformed into a system of ordinary functional-differential equations. We investigate the question of under what conditions the classical solutions of the original problem are approximated by solutions of associated systems of ordinary functional-differential equations. The proof of the convergence of the method of lines is based on the differential-inequalities technique. A numerical example is given. Differential equations with retarded variables and differential-integral equations are particular cases of a general model considered in the paper.

Keywords: *Cauchy problem, differential-difference inequalities, comparison technique, stability and convergence*

AMS subject classification: 65N40, 35A40

1. Introduction

For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let E be the Haar pyramid

$$E = \Big\{(t,x) \in \mathbb{R}^{1+n} : \, t \in [0,a] \;\text{ and }\; -b + Mt \le x \le b - Mt\Big\}$$

where $a > 0$, $M = (M_1, \ldots, M_n) \in \mathbb{R}^n$ with $M_i > 0$ for $1 \le i \le n$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$ and $b \ge Ma$. Write

$$E_0 = [-b_0, 0] \times [-b, b]$$
$$\Omega = E \times C(E_0 \cup E, \mathbb{R}) \times \mathbb{R}^n$$

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where $b_0 \in \mathbb{R}_+ = [0, +\infty)$. Suppose that functions $f : \Omega \to \mathbb{R}$ and $\varphi : E_0 \to \mathbb{R}$ are given. We consider the functional-differential initial-value problem

$$\left.\begin{aligned} \partial_t z(t,x) &= f\big(t, x, z, \partial_x z(t,x)\big) \\ z(t,x) &= \varphi(t,x) \quad \text{on } E_0 \end{aligned}\right\} \tag{1}$$

where $x = (x_1, \ldots, x_n)$ and $\partial_x z = (\partial_{x_1} z, \ldots, \partial_{x_n} z)$. A function $v : E_0 \cup E \to \mathbb{R}$ is called a classical solution of problem (1) if

(i) $v \in C(E_0 \cup E, \mathbb{R})$ and v is of class C^1 on E

(ii) v satisfies equation $(1)_1$ on E and condition $(1)_2$ holds.

The Haar pyramid is a natural set for the existence and uniqueness of solutions of initial problems for differential and functional-differential equations (see [3, 4, 10]).

Write

$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n) \qquad (0 \le t \le a).$$

We will denote by $\|\cdot\|_t$ the supremum norm in the space $C(E_t, \mathbb{R})$ $(0 \le t \le a)$. We will say that the function $f : \Omega \to \mathbb{R}$ satisfies the Volterra condition if for each $(t,x) \in E$ there is a subset $E[t,x] \subset E_t$ such that if $z, \bar{z} \in C(E_0 \cup E, \mathbb{R})$ and $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E[t,x]$, then $f(t,x,z,q) = f(t,x,\bar{z},q)$ where $q \in \mathbb{R}^n$. Note that the Volterra condition means that the value of f at the point $(t,x,z,q) \in \Omega$ depends on (t,x,q) and on the restriction of z to the set $E[t,x]$. In the paper we assume that f satisfies the Volterra condition and we consider classical solutions of problem (1).

We are interested in establishing a method of approximation of solutions of problem (1) by means of solutions of an associated system of ordinary functional-differential equations and in estimating the difference between the exact and approximate solutions. The systems of ordinary functional-differential equations mentioned above are obtained in the paper by using a discretization in the spatial variables of equation $(1)_1$ and therefore they are called differential-difference systems or method of lines.

There is an ample literature on the method of lines. The monographs [2, 13, 16] contain a large bibliography. The method is also treated as a tool for proving existence theorems for differential problems corresponding to parabolic equations [9, 11, 12, 14] or first order hyperbolic systems [8]. The papers [1, 6, 7, 17, 18] initiated the method of lines for functional-differential problems. Parabolic equations with initial boundary value conditions and Hamilton Jacobi equations with initial boundary conditions or initial conditions which are global with respect to spatial variables were considered. Error estimates implying the convergence of sequences of approximate solutions are obtained in these papers by using the method of differential inequalities.

The monograph [3] contains an exposition of the method of lines for hyperbolic functional-differential problems. Note that the results presented in [3: Chapter 6] and [17, 18] are not applicable to problem (1). The methods of lines for equations with generalized Hale operator are investigated in [3, 17, 18]. This model of functional dependence is not suitable for local Cauchy problems considered on the Haar pyramid.

Existence results for problem (1) are given in [4]. Differential equations with deviated variables and differential-integral equations can be obtained from (1) by specializing the operator f.

The paper is organized as follows. In Section 2 we transform problem (1) into a system of ordinary differential-functional equations. Theorems concerning the method of lines will be based on a comparison result where a function satisfying some differential-difference inequalities on the Haar pyramid is estimated by a solution of an adequate ordinary functional-differential problem. The comparison result is proved in Section 3. A convergence theorem and an error estimate for the method of lines are presented in Section 4. A numerical example is given in Section 5.

2. Differential difference problem

Let $F(U, V)$ denote the set of all functions defined on U and taking values in V where U and V are arbitrary sets. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. For $x, y \in \mathbb{R}^n$ with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ we write

$$x \diamond y = (x_1 y_1, \dots, x_n y_n)$$
$$\|x\| = |x_1| + \dots + |x_n|.$$

We define a mesh with respect to spatial variables in the following way. Suppose that for $h = (h_1, \dots, h_n)$ with $h_i > 0$ for $1 \le i \le n$ there exists $N = (N_1, \dots, N_n) \in \mathbb{N}^n$ such that $N \diamond h = b$. We denote by Δ the set of all h having the above property. For $h \in \Delta$ and $m \in \mathbb{Z}^n$ with $m = (m_1, \dots, m_n)$ we put

$$x^{(m)} = m \diamond h = \big(x_1^{(m_1)}, \dots, x_n^{(m_n)}\big)$$
$$R^{1+n}_{t.h} = \big\{(t, x^{(m)}) : t \in \mathbb{R}, m \in \mathbb{Z}^n\big\}.$$

Further, we define the sets

$$E_{0.h} = E_0 \cap R^{1+n}_{t.h} \qquad \text{and} \qquad E_h = E \cap R^{1+n}_{t.h}.$$

Elements of $E_{0.h} \cup E_h$ will be denoted by $(t, x^{(m)})$ or (t, x). For a function $z : E_{0.h} \cup E_h \to \mathbb{R}$ we put $z^{(m)}(t) = z(t, x^{(m)})$. Let $x^{(m)} \in [-b, b]$ and write

$$I[m] = \Big\{t \in [-b_0, a] : (t, x^{(m)}) \in E_{0.h} \cup E_h\Big\}.$$

Denote by $F_c(E_{0.h} \cup E_h, R)$ the set of all $z : E_{0.h} \cup E_h \to \mathbb{R}$ such that $z^{(m)}(\cdot) : I[m] \to \mathbb{R}$ is a continuous function, for each fixed $x^{(m)} \in [-b, b]$. In a similar way we define the space $F_c(E_{0.h}, R)$. For a function $z \in F_c(E_{0.h} \cup E_h, \mathbb{R})$ and for a point $t \in [0, a]$ we write

$$\|z\|_{h.t} = \max\Big\{|z(\tau, x)| : (\tau, x) \in (E_{0.h} \cup E_h) \cap ([-b_0, t] \times \mathbb{R}^n)\Big\}.$$

In the sequel we will need the interpolating operator

$$T_h : F_c(E_{0.h} \cup E_h, R) \to C(E_0 \cup E, \mathbb{R}).$$

For its definition write

$$\tilde{E}_h = \Big\{(t, x^{(m)}) : t \in [-b_0, a] \text{ and } -N \le m \le N\Big\}$$

and suppose that $z \in F_c(E_{0.h} \cup E_h, \mathbb{R})$. First, if $-N \le m \le N$ and $I[m] = [-b_0, a_m]$, we define a function $\tilde{z} : \tilde{E}_h \to \mathbb{R}$ by

$$\tilde{z}(t, x^{(m)}) = \begin{cases} z(t, x^{(m)}) & \text{for } t \in I[m] \\ z(a_m, x^{(m)}) & \text{for } t \in [-b_0, a] \setminus I[m]. \end{cases} \tag{2}$$

Now, putting

$$S_+ = \Big\{s = (s_1, \dots, s_n) : s_i \in \{0, 1\} \ \ (1 \le i \le n)\Big\}$$

we define the function $T_h z : E_0 \cup E \to \mathbb{R}$ to define the announced operator T_h as follows: If $(t, x) \in E_0 \cup E$, then there exists $m \in \mathbb{Z}^n$ such that $x^{(m)} \le x \le x^{(m+1)}$ and $(t, x^{(m)}), (t, x^{(m+1)}) \in \tilde{E}_h$ where $m + 1 = (m_1 + 1, \dots, m_n + 1)$. Write

$$(T_h z)(t, x) = \sum_{s \in S_+} \tilde{z}^{(m+s)}(t) \Big(\frac{x - x^{(m)}}{h}\Big)^s \Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s}$$

where

$$\Big(\frac{x - x^{(m)}}{h}\Big)^s = \prod_{i=1}^{n} \Big(\frac{x_i - x_i^{(m_i)}}{h_i}\Big)^{s_i}$$

$$\Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s} = \prod_{i=1}^{n} \Big(1 - \frac{x_i - x_i^{(m_i)}}{h_i}\Big)^{1-s_i}$$

with putting $0^0 = 1$ in the above formulas. Then we set $T_h z : E_0 \cup E \to \mathbb{R}$ and T_h is a continuous function on $E_0 \cup E$.

We define the difference operator $\delta = (\delta_1, \dots, \delta_n)$ in the following way. For $1 \le i \le n$ we write $e_i = (1, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, with 1 standing on the i-th place. If $z : E_{0.h} \cup E_h \to \mathbb{R}$ and $(t, x^{(m)}) \in E_h$, then

$$\delta_i z^{(m)}(t) = \frac{1}{h_i} \times \begin{cases} \big(z^{(m+e_i)}(t) - z^{(m)}(t)\big) & \text{if } x_i^{(m_i)} < 0 \\ \big(z^{(m)}(t) - z^{(m-e_i)}(t)\big) & \text{if } x_i^{(m_i)} \ge 0. \end{cases} \tag{3}$$

We will approximate classical solutions of problem (1) by means of solutions of the system of the ordinary functional-differential initial-value problem

$$\left.\begin{aligned} \frac{d}{dt} z^{(m)}(t) &= f\big(t, x^{(m)}, T_h z, \delta z^{(m)}(t)\big) \\ z^{(m)}(t) &= \varphi_h^{(m)}(t) \ \text{ on } E_{0.h} \end{aligned}\right\} \tag{4}$$

where $\varphi_h : E_{0.h} \to \mathbb{R}$ is a given function and $\delta z = (\delta_1 z, \dots, \delta_n z)$. Let F_h be the Niemycki operator corresponding to $(4)_1$, i.e.

$$F_h[z]^{(m)}(t) = f\big(t, x^{(m)}, T_h z, \delta z^{(m)}(t)\big).$$

We prove that there exists a solution $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ of problem (4) and there is a function $\tilde{\alpha} : \Delta \to \mathbb{R}_+$ such that $|v^{(m)}(t) - u_h^{(m)}(t)| \le \tilde{\alpha}(h)$ on E_h where $v : E_0 \cup E \to \mathbb{R}$ is a solution of problem (1) and $\lim_{h \to 0} \tilde{\alpha}(h) = 0$.

3. Differential-difference inequalities

Different types of theorems on functional-differential inequalities are taken into considerations in literature. The first type allows to estimate a function of several variables by means of another function of several variables, while the second one, the so-called comparison theorems, give estimates for functions of several variables by means of functions of one variable [3, 5]. In this section we present a comparison theorem where a function satisfying differential-difference inequalities on the Haar pyramid is estimated by a solution of an ordinary differential equation with a retarded variable.

We will need the following assumption on comparison functions.

Assumption H$[\sigma, \alpha]$. Suppose that the functions $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha : [0,a] \to \mathbb{R}$ satisfy the following conditions:

1) σ is continuous on $[0,a] \times \mathbb{R}_+$ and for each $t \in [0,a]$ the function $\sigma(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing.

2) α is continuous on $[0,a]$, $-b_0 \le \alpha(t) \le t$ for $t \in [0,a]$, and for each function $\eta \in C([-b_0,0],\mathbb{R}_+)$ there exists on $[-b_0,a]$ the maximum solution of the Cauchy problem

$$\left.\begin{aligned} \omega'(t) &= \sigma(t,\omega(\alpha(t))) \\ \omega(t) &= \eta(t) \quad \text{on } [-b_0,0] \end{aligned}\right\}. \tag{5}$$

Assumption H$[\Lambda]$. Suppose that the function

$$\Lambda : E_h \times F_c(E_{0.h} \cup E_h, R) \to \mathbb{R}^n, \quad \Lambda = (\lambda_1, \ldots, \lambda_n)$$

satisfies the following conditions:

1) For $(t,x,z) \in E_h \times F_c(E_{0.h} \cup E_h, \mathbb{R})$, $x \diamond \Lambda(t,x,z) \le 0$.

2) Λ satisfies the following Volterra condition: for each $(t,x) \in E_h$ there is a subset $E_h[t,x] \subset (E_{0.h} \cup E_h) \cap ([-b_0,t] \times \mathbb{R}^n)$ such that if $z, \bar{z} \in F_c(E_{0.h} \cup E_h, \mathbb{R})$ and $z(\tau,y) = \bar{z}(\tau,y)$ for $(\tau,y) \in E_h[t,x]$, then $\Lambda(t,x,z) = \Lambda(t,x,\bar{z})$.

Lemma 3.1. *Suppose that Assumptions $H[\sigma,\alpha]$ and $H[\Lambda]$ are satisfied. Further, suppose the following:*

1) *The function $u \in F_c(E_{0.h} \cup E_h, \mathbb{R})$ satisfies the differential-difference inequality*

$$\left| D_- u^{(m)}(t) - \sum_{j=1}^{n} \lambda_j(t, x^{(m)}, u)\, \delta_j u^{(m)}(t) \right| \le \sigma(t, \|u\|_{h.\alpha(t)})$$

where $(t, x^{(m)}) \in E_h$ and $D_- u^{(m)}(t)$ is the left-hand lower Dini derivative of the function $u^{(m)}(\cdot)$ at the point t.

2) *For $(t, x^{(m)}) \in E_{0.h}$, $|u^{(m)}(t)| \le \eta(t)$ where $\eta \in C([-b_0,0],\mathbb{R}_+)$.*

Then

$$|u^{(m)}(t)| \le \omega(t,\eta) \qquad \text{on } E_h \tag{6}$$

where $\omega(\cdot,\eta)$ *is the maximum solution of problem* (4).

Proof. Consider the function

$$\tilde\omega(t) = \max\left\{|u^{(m)}(t)| : (t,x^{(m)}) \in E_{0.h} \cup E_h\right\} \qquad (-b_0 \le t \le a).$$

Then $\tilde\omega \in C([-b_0,a],\mathbb{R}_+)$ and estimate (6) is equivalent to

$$\tilde\omega(t) \le \omega(t,\eta) \qquad (t \in [0,a]). \tag{7}$$

It follows that the initial inequality

$$\tilde\omega(t) \le \eta(t) \qquad (t \in [-b_0,0]) \tag{8}$$

is satisfied. Let $J_+ = \{t \in (0,a] : \tilde\omega(t) > \omega(t,\eta)\}$. We prove that

$$D_-\tilde\omega(t) \le \sigma\big(t,\tilde\omega(\alpha(t))\big) \qquad (t \in J_+). \tag{9}$$

Indeed, let $t \in J_+$. Then $\tilde\omega(t) > 0$ and there is $m \in \mathbb{Z}^n$ such that $\tilde\omega(t) = |u^{(m)}(t)|$. Thus two possibilities can happen, namely either (i) $\tilde\omega(t) = u^{(m)}(t)$ or (ii) $\tilde\omega(t) = -u^{(m)}(t)$. Consider the case (i). We conclude from Assumption $H[\Lambda]$ and (3) that

$$\begin{aligned} D_-\tilde\omega(t) &\le D_-u^{(m)}(t) \\ &\le \sigma(t,\|u\|_{\alpha(t)}) + \sum_{j=1}^{n} \lambda_j(t,x^{(m)},u)\,\delta_j u^{(m)}(t) \\ &\le \sigma(t,\tilde\omega(\alpha(t)) \end{aligned}$$

and inequality (9) is proved. Analogously (9) can be proved if condition (ii) is satisfied. It follows from (8) - (9) and from a theorem on differential-functional inequalities (see [3: Lemma 5.12]) that estimate (7) is satisfied. This completes the proof of the lemma ∎

4. Convergence of the method of lines

Our basic assumptions on the function f in problem (1) are the following ones:

Assumption H[f]. Suppose that the function f of the variables (t,x,z,q) with $q = (q_1,\dots,q_n)$ satisfies the following conditions:

1) $f \in C(\Omega,\mathbb{R})$, the partial derivatives $(\partial_{q_1} f,\dots,\partial_{q_n} f) = \partial_q f$ exist on Ω, $\partial_q f(t,x,z,\cdot) \in C(\mathbb{R}^n,\mathbb{R}^n)$ for $(t,x,z) \in E \times C(E_0 \cup E,\mathbb{R})$ and $x \diamond \partial_q f(t,x,z,q) \le 0$ on Ω.

2) There are functions $\sigma : [0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha : [0,a] \to \mathbb{R}$ satisfying Assumption $H[\sigma,\alpha]$ and such that $\big|f(t,x,z,q) - f(t,x,\bar z,q)\big| \le \sigma(t,\|z-\bar z\|_{\alpha(t)})$ on Ω.

3) $\bar{\omega}(t) = 0 \ \ (t \in [0, a])$ is the maximum solution of problem (5) with $\eta(t) = 0$ for $t \in [-b_0, 0]$.

Assumption H$[z_0]$**.** Let $z_0 \in F_c(E_{0.h} \cup E_h, \mathbb{R})$ be a function such that the following conditions are satisfied:

1) z_0 satisfies initial condition $(4)_2$ and for each $x^{(m)} \in [-b, b]$ the derivative $\frac{d}{dt} z_0^{(m)}(\cdot)$ exists on $I[m] \cap [0, a]$.

(ii) There is a function $\gamma_0 \in C([0, a], \mathbb{R}_+)$ such that $\left|\frac{d}{dt} z_0^{(m)}(t) - F_h[z_0]^{(m)}(t)\right| \le \gamma_0(t)$ on E_h and the maximum solution ω_0 of the problem

$$\left.\begin{aligned} \omega'(t) &= \sigma(t, \omega(\alpha(t))) + \gamma_0(t) \\ \omega(t) &= 0 \ \ \text{for } t \in [-b_0, 0] \end{aligned}\right\} \tag{10}$$

is defined on $[-b_0, a]$.

Let us consider the sequence of functions $\omega_j : [-b_0, a] \to \mathbb{R}_+ \ \ (0 \le j < \infty)$ defined recursively in the following way:

(i) ω_0 is given in Assumption $H[z_0]$.

(ii) If ω_j is known, then

$$\omega_{j+1}(t) = \begin{cases} 0 & \text{for } t \in [-b_0, 0] \\ \int_0^t \sigma(\tau, \omega_j(\alpha(\tau)))\, d\tau & \text{for } t \in [0, a]. \end{cases}$$

Lemma 4.1. *If Assumptions* $H[\sigma, \alpha], H[z_0]$ *and condition* 3) *of Assumption* $H[f]$ *are satisfied, then* $\omega_{j+1}(t) \le \omega_j(t) \ \ (j \ge 0)$ *for* $t \in [0, a]$ *and* $\lim_{j\to\infty} \omega_j(t) = 0$ *uniformly on* $[0, a]$*.*

We omit the simple proof of this lemma.

Theorem 4.2. *If Assumptions* $H[f], H[z_0]$ *are satisfied and* $\varphi_h \in F_c(E_{0.h}, \mathbb{R})$*, then there exists exactly one solution* $u_h : E_{0.h} \cup E_h \to \mathbb{R}$ *of problem* (4)*.*

Proof. Let the sequence $z_j : E_{0.h} \cup E_h \to \mathbb{R} \ \ (j \ge 0)$ be defined recursively in the following way:

(i) z_0 is given in Assumption $H[z_0]$.

(ii) If z_j is known, then z_{j+1} is the solution of the system

$$\frac{d}{dt} z^{(m)}(t) = f\big(t, x^{(m)}, T_h z_j, \delta z^{(m)}(t)\big) \tag{11}$$

with initial condition $(4)_2$. We prove that for $i, j \ge 0$

$$\left|z_{j+i}^{(m)}(t) - z_j^{(m)}(t)\right| \le \omega_j(t) \qquad \text{on } E_h. \tag{12}$$

First we prove (12) for $j = 0$ and $i \ge 0$. It follows that inequality (12) is satisfied for $j = i = 0$. If we assume that $\left|z_j^{(m)}(t) - z_0^{(m)}(t)\right| \le \omega_0(t)$ on E_h, then using the Hadamard mean value theorem we conclude that

$$\begin{aligned} &\left|\frac{d}{dt}(z_{i+1} - z_0)^{(m)}(t) - \sum_{p=0}^{n} \int_0^t \partial_{q_p} f(P_{i.0}^{(m)}(t, \tau))\, d\tau \ \delta_p (z_{i+1} - z_0)^{(m)}(t)\right| \\ &\qquad \le \sigma(t, \omega_0(\alpha(t))) + \gamma(t) \end{aligned}$$

on E_h where

$$P^{(m)}_{i.0}(t,\tau) = \Big(t, x^{(m)}, T_h z_i, \delta z^{(m)}_0(t) + \tau\delta(z_{i+1} - z_0)^{(m)}(t)\Big) .$$

It follows from Lemma 3.1 that $\big|(z_{i+1} - z_0)^{(m)}(t)\big| \le \omega_0(t)$ on E_h.

Suppose now that $\big|(z_{j+i} - z_j)^{(m)}(t)\big| \le \omega_j(t)$ on E_h with fixed $j \in \mathbb{N}$ and all $i \ge 0$. Then, using the Hadamard mean value theorem, we conclude that

$$\begin{aligned}\bigg|\frac{d}{dt}(z_{j+1+i} - z_{j+1})^{(m)}(t) - \sum_{p=0}^{n}\int_0^t \partial_{q_p} f(P^{(m)}_{i.j+1}(t,\tau))\,d\tau\ \delta_p(z_{j+1+i} - z_{j+1})^{(m)}(t)\bigg| \\ \le \sigma(t, \omega_j(\alpha(t)))\end{aligned}$$

on E_h where

$$P^{(m)}_{i.j+1}(t,\tau) = \Big(t, x^{(m)}, T_h z_{j+i}, \delta z^{(m)}_{j+1}(t) + \tau\delta(z_{j+1+i} - z_{j+1})^{(m)}(t)\Big) .$$

It follows from Lemma 3.1 that $\big|(z_{j+1+i} - z_{j+1})^{(m)}(t)\big| \le \tilde{\omega}(t)$ on E_h where $\tilde{\omega}$ is the maximum solution of the problem

$$\left.\begin{aligned}\omega'(t) &= \sigma(t, \omega_j(\alpha(t))) \\ \omega(t) &= 0 \quad \text{on } [-b_0, 0]\end{aligned}\right\} .$$

Since $\tilde{\omega} = \omega_{j+1}$, the proof of (12) can be completed by induction. It follows from Lemma 4.1 and (12) that the sequence $\{z_j\}_{j=0}^{\infty}$ is uniformly convergent on E_h. Write $u^{(m)}_h(t) = \lim_{j\to\infty} z^{(m)}_j(t)$ on E_h. According to (11) and $(4)_2$,

$$z^{(m)}_{j+1}(t) = \varphi^{(m)}_h(0) + \int_0^t f\big(\tau, x^{(m)}, T_h z_j, \delta z^{(m)}_{j+1}(\tau)\big)\,d\tau$$

where $j \ge 0$. By passing to the limit for $j \to \infty$ we obtain that u_h is the solution of problem (4). If both u_h and $\tilde{u}_h$ are solutions of problem (4), then we obtain the differential-difference inequality

$$\begin{aligned}\bigg|\frac{d}{dt}(u_h - \tilde{u}_h)^{(m)}(t) - \sum_{j=1}^{n}\int_0^1 \partial_{q_j} f(Q(t,\tau))\,d\tau\ \delta_j(u_h - \tilde{u}_h)^{(m)}(t)\bigg| \\ \le \sigma\big(t, \|u_h - \tilde{u}_h\|_{\alpha(t)}\big)\end{aligned}$$

on E_h where $Q(t,\tau)$ is a suitable intermediate point and $(u_h - \tilde{u}_h)^{(m)}(t) = 0$ on $E_{0.h}$. It follows from Lemma 3.1 and condition 3) of Assumption $H[f]$ that $u_h = \tilde{u}_h$. This completes the proof of the theorem ∎

The next theorem deals with convergence of the numerical method of lines.

Theorem 4.3. *Suppose the following:*

1) *Assumptions* $H[f]$ *and* $H[z_0]$ *are satisfied and* $\varphi_h \in F_c(E_{0.h}, \mathbb{R})$.

2) $v : E_0 \cup E \to \mathbb{R}$ *is a solution of problem* (1) *and* v *is of class* C^1 *on* E.

3) *There is a function* $\alpha_0 : \Delta \to \mathbb{R}_+$ *such that*

$$|(\varphi - \varphi_h)^{(m)}(t)| \le \alpha_0(h) \quad \text{on } E_{0.h} \qquad \text{and} \qquad \lim_{h\to 0} \alpha_0(h) = 0. \tag{13}$$

Then there is a function $\tilde{\alpha} : \Delta \to \mathbb{R}_+$ *such that*

$$|v^{(m)}(t) - u_h^{(m)}(t)| \le \tilde{\alpha}(h) \quad \text{on } E_h \qquad \text{and} \qquad \lim_{h\to 0} \tilde{\alpha}(h) = 0. \tag{14}$$

Proof. We will denote by v_h the restriction of v to the set $E_{0.h} \cup E_h$. Let the function $\Gamma_h : E_h \to \mathbb{R}$ be defined by

$$\frac{d}{dt} v_h^{(m)}(t) = F_h[v_h]^{(m)}(t) + \Gamma_h^{(m)}(t). \tag{15}$$

It follows that there is a function $\gamma : \Delta \to \mathbb{R}_+$ such that

$$|\Gamma_h^{(m)}(t)| \le \gamma(h) \quad \text{on } E_h \qquad \text{and} \qquad \lim_{h\to 0} \gamma(h) = 0. \tag{16}$$

Write $w_h = v_h - u_h$. Then w_h satisfies the differential-difference inequality

$$\left| \frac{d}{dt} w_h^{(m)}(t) - \sum_{j=1}^{n} \int_0^1 \partial_{q_j} f(Q_m(t,\tau))\, d\tau \;\delta_j w_h^{(m)}(t) \right| \le \sigma(t, \|w_h\|_{h.\alpha(t)}) + \gamma(h)$$

for $(t, x^{(m)}) \in E_h$ where

$$Q_m(t,\tau) = \big(t, x^{(m)}, T_h v_h, \delta u_h^{(m)}(t) + \tau \delta w_h^{(m)}(t)\big)$$

and $|w_h^{(m)}(t)| \le \alpha_0(h)$ on $E_{0.h}$. Let $\omega(\cdot, h)$ be the maximum solution of the problem

$$\left.\begin{aligned} \omega'(t) &= \sigma(t, \omega(\alpha(t))) \\ \omega(t) &= \alpha_0(h) \quad \text{on } [-b_0, 0] \end{aligned}\right\}.$$

It follows from Lemma 3.1 that

$$|w_h^{(m)}(t)| \le \omega(t, h) \quad \text{on } E_h. \tag{17}$$

Then condition (15) is satisfied with $\tilde{\alpha}(h) = \omega(a, h)$. This proves the theorem ∎

Now we give a theorem on the error estimate for the numerical method of lines.

Theorem 4.4. *Suppose the following:*

1) *The function* $f : \Omega \to \mathbb{R}$ *satisfies condition* 1) *of Assumption* $H[f]$ *and* $|\partial_{q_i} f(t,x,z,q)| \le M_i$ *on* Ω *for* $1 \le i \le n$.

2 *There is an* $L \in \mathbb{R}_+$ *such that* $|f(t,x,z,q) - f(t,x,\bar{z},q)| \le L\|z - \bar{z}\|_t$ *on* Ω.

3) *The function* $v : E_0 \cup E \to \mathbb{R}$ *is a solution of problem* (1) *of class* C^1 *on* $E_0 \cup E$ *and there is a constant* $C \in \mathbb{R}_+$ *such that* $|\partial_t v(t,x)|, |\partial_{x_i} v(t,x)| \le C \quad (1 \le i \le n)$ *on* $E_0 \cup E$.

4) *There is a function* $\alpha_0 : \Delta \to \mathbb{R}_+$ *such that condition* (13) *holds.*

Then

$$\|v_h - u_h\|_{h.t} \le \alpha(h) \qquad (t \in [0,a])$$

where u_h *is the solution of problem* (4), v_h *is the restriction of* v *to the set* $E_{0.h} \cup E_h$,

$$\tilde{\alpha}(h) = \begin{cases} \alpha_0(h)e^{La} + \gamma(h)\frac{e^{La}-1}{L} & \text{if } L > 0 \\ \alpha_0(h) + a\gamma(h) & \text{if } L = 0 \end{cases}$$

and

$$\gamma(h) = LC(\xi(h) + \|h\|) + \frac{C}{2}\sum_{i=1}^{n} M_i h_i \tag{18}$$

with $\xi(h) = \max\left\{\frac{h_i}{M_i} : 1 \le i \le n\right\}$.

Proof. We first prove that

$$\|T_h v_h - v\|_t \le C(\xi(h) + \|h\|) \qquad (0 \le t \le a). \tag{19}$$

Let $\tilde{v}_h : \tilde{E}_h \to \mathbb{R}$ be the function defined by (2) with v_h instead of z. Suppose that $(t,x) \in E_0 \cup E$. Then there is an $m \in \mathbb{Z}^n$ such that $x^{(m)} \le x \le x^{(m+1)}$ and $(t,x^{(m)}), (t,x^{(m+1)}) \in \tilde{E}_h$ and

$$(T_h v_h)(t,x) = \sum_{s \in S_+} \tilde{v}_h^{(m+s)}(t)\Big(\frac{x - x^{(m)}}{h}\Big)^s\Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s}. \tag{20}$$

It is easy to prove by induction with respect to n that

$$\sum_{s \in S_+}\Big(\frac{x - x^{(m)}}{h}\Big)^s\Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s} = 1 \qquad \big(x^{(m)} \le x \le x^{(m+1)}\big). \tag{21}$$

There are two cases to be distinguished:

1) If $(t,x^{(m)}), (t,x^{(m+1)}) \in E_{0.h} \cup E_h$, then using (20) - (21) we get

$$\begin{aligned}&(T_h v_h)(t,x) - v(t,x) \\ &\quad = \sum_{s \in S_+}\big[v_h^{(m+s)}(t) - v(t,x)\big]\Big(\frac{x - x^{(m)}}{h}\Big)^s\Big(1 - \frac{x - x^{(m)}}{h}\Big)^{1-s}\end{aligned}$$

and

$$\big|(T_h v_h)(t,x) - v(t,x)\big| \le C\|h\|. \tag{22}$$

2) Suppose that $(t, x^{(m)}) \in E_{0.h} \cup E_h$ and $(t, x^{(m+1)}) \notin E_h$. Write

$$J_{m.0}[t] = \big\{ s \in S_+ : (t, x^{(m+s)}) \in E_h \big\}$$
$$J_{m.+}[t] = \big\{ s \in S_+ : (t, x^{(m+s)}) \notin E_h \big\}.$$

Then

$$\begin{aligned}(T_h v_h)(tx) &- v(t,x)\\ &= \sum_{s \in J_{m.0}[t]} \big[v_h^{(m+s)}(t) - v(t,x) \big] \Big(\frac{x - x^{(m)}}{h} \Big)^s \Big(1 - \frac{x - x^{(m)}}{h} \Big)^{1-s} \\ &\quad + \sum_{s \in J_{m.+}[t]} \big[v_h^{(m+s)}(a_{m+s}) - v(t,x) \big] \Big(\frac{x - x^{(m)}}{h} \Big)^s \Big(1 - \frac{x - x^{(m)}}{h} \Big)^{1-s}\end{aligned}$$

which results in

$$\big|(T_h v_h)(t,x) - v(t,x)\big| \le C(\xi(h) + \|h\|). \tag{23}$$

In a similar way we prove this estimate if $(t, x^{(m+1)}) \in E_{0.h} \cup E_h$ and $(t, x^{(m)}) \notin E_h$.

Estimates (22) - (23) imply (19). Let $\Gamma_h : E_h \to \mathbb{R}$ be defined by (15). It follows from assumptions 1) - 2) and from (19) that condition (16) is satisfied with γ given by (18). Then we obtain the assertion of the theorem from (17) ∎

Remark 4.5. By using a discretization with respect to t of problem (4) we obtain difference methods for problem (1).

5. Numerical examples

The classical difference methods for nonlinear partial functional-differential equations consist in replacing partial derivatives by difference expressions. Then solutions of difference equations approximate, under suitable assumptions on given functions and on the mesh, solutions of the original problem. We formulate a new class of difference problems corresponding to problem (1). Namely, we transform the nonlinear differential equation into a system of ordinary functional-differential equations. The system thus obained is solved numerically by the Runge-Kutta method. We illustrate this approach on the following example.

Put $n = 1$ and

$$E = \Big\{ (t,x) \in R^2 : t \in [1,3] \text{ and } -3 + t \le x \le 3 - t \Big\}.$$

Consider the differential-integral initial-value problem

$$\left.\begin{aligned} \partial_t z(t,x) &= -\frac{x}{4} \partial_x z(t,x) + \int_{-x}^{x} z(t,s)\,ds + f(t,x) \\ z(1,x) &= 3x^2 \quad \text{on } [-2,2] \end{aligned}\right\} \tag{24}$$

where

$$f(t,x) = tx^2\big(6 - \tfrac{3}{2}t - 2tx\big).$$

The solution of this problem is known – it is $v(t,x) = 3(tx)^2$. We define a mesh on E in the following way. Suppose that (h_0, h) stand for steps of the mesh and

$$\begin{aligned} t^{(r)} &= 1 + rh_0 \qquad (0 \le r \le N_0)\\ x^{(m)} &= mh \qquad\quad (-N \le m \le N) \end{aligned}$$

where $N_0 h_0 \le 2 < (N_0+1)h_0$ and $Nh = 2$. Write

$$E_{(h_0,h)} = \Big\{(t^{(r)}, x^{(m)}) :\ 1 \le t^{(r)} \le 3 \ \text{ and } \ -3 + t^{(r)} \le x^{(m)} \le 3 - t^{(r)}\Big\}.$$

For a function $z :\ E_{h_0,h)} \to \mathbb{R}$ we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$. The classical difference method for problem (24) has the form

$$\begin{aligned} z^{(r+1,m)} &= \tfrac{1}{2} z^{(r,m+1)}\Big[1 - \frac{x^{(m)}h_0}{4h}\Big]\\ &\quad + \tfrac{1}{2} z^{(r,m-1)}\Big[1 + \frac{x^{(m)}h_0}{4h}\Big] + h_0 I^{(r,m)}[z] + h_0 f^{(r,m)}\\ z(1,x^{(m)}) &= 3(x^{(m)})^2 \quad (-N \le m \le N) \end{aligned} \tag{25}$$

where $I^{(r,0)}[z] = 0$ and

$$I^{(r,m)}[z] = \begin{cases} \frac{h}{2}\big(z^{(r,-m)} + z^{(r,m)}\big) + h\sum_{j=-m+1}^{m-1} z^{(r,j)} & \text{for } m > 0\\ -\frac{h}{2}\big(z^{(r,m)} + z^{(r,-m)}\big) - h\sum_{j=m+1}^{-m-1} z^{(r,j)} & \text{for } m < 0. \end{cases} \tag{26}$$

Let us denote by $z_h :\ E_{(h_0.h)} \to \mathbb{R}$ the solution of the above problem. The numerical method of lines for problem (24) has the form

$$\begin{aligned} \frac{d}{dt} z^{(m)}(t) &= \begin{cases} -\frac{x^{(m)}\left(z^{(m)}(t) - z^{(m-1)}(t)\right)}{4h} + I^{(m)}(t) + f^{(m)}(t) & \text{for } m \ge 0\\ -\frac{x^{(m)}\left(z^{(m+1)}(t) - z^{(m)}(t)\right)}{4h} + I^{(m)}(t) + f^{(m)}(t) & \text{for } m < 0 \end{cases}\\ z^{(m)}(1) &= 3(x^{(m)})^2 \quad (-N \le m \le N) \end{aligned} \tag{27}$$

where

$$I^{(m)}(t) = \int_{-x^{(m)}}^{x^{(m)}} T_h(t,\tau)\, d\tau$$

and T_h is the interpolating operator for $n = 1$.

We apply the Runge-Kutta method of the second order to problem (27). This leads to the difference equations problem

$$\begin{aligned} z^{(r+1)} &= z^{(r,m)} - \frac{x^{(m)}h_0}{4h}\\ &\qquad \times \begin{cases} \big(y^{(r,m)} - y^{(r,m-1)}\big) + h_0 I^{(r,m)}[y] + h_0 f^{(r+1.m)} & \text{for } m \ge 0\\ \big(y^{(r,m+1)} - y^{(r,m)}\big) + h_0 I^{(r,m)}[y] + h_0 f^{(r+1.m)} & \text{for } m < 0 \end{cases}\\ z^{(0,m)} &= 3(x^{(m)})^2 \quad (-N \le m \le N) \end{aligned} \tag{28}$$

where

$$y^{(r,m)} = z^{(r,m)} - \frac{x^{(m)}h_0}{8h} \times \begin{cases} \left(z^{(r,m)} - z^{(r,m-1)}\right) + \frac{h_0}{2} I^{(r,m)}[z] + \frac{h_0}{2} f^{(r,m)} & \text{for } m \geq 0 \\ \left(z^{(r,m+1)} - z^{(r,m)}\right) + \frac{h_0}{2} I^{(r,m)}[z] + \frac{h_0}{2} f^{(r,m)} & \text{for } m < 0 \end{cases}$$

with $I^{(r,m)}[z]$ and $I^{(r,m)}[y]$ given by (26). Denote by $u_h : E_{(h_0,h)} \to \mathbb{R}$ the solution of problem (28).

We give the following information on the errors of methods (25) and (28). Write

$$\begin{aligned} \varepsilon^{(r,m)} &= \left|v^{(r,m)} - z_h^{(r,m)}\right| \\ \tilde{\varepsilon}^{(r,m)} &= \left|v^{(r,m)} - u_h^{(r,m)}\right| \end{aligned} \quad \text{and} \quad \begin{aligned} \eta^{(r)} &= \max\left\{|\varepsilon^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h_0,h}\right\} \\ \tilde{\eta}^{(r)} &= \max\left\{|\tilde{\varepsilon}^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h_0,h}\right\} \end{aligned}$$

where $0 \leq r \leq N_0$. In Table A and B experimental values of the functions η and $\tilde{\eta}$ are given.

Table A: Values of η and $\tilde{\eta}$ for $h_0 = 0,01$ and $h = 0.01$

Table B: Values of η and $\tilde{\eta}$ for $h_0 = 0,001$ and $h = 0.001$

The errors for the classical method (25) are larger than the errors for (28). This is due to the fact that problem (27) is solved by the Runge-Kutta method.

Methods described in Theorem 4.3 have potential for applications in the numerical solving of initial problems for functional-differential equations on the Haar pyramid. Difference problems obtained by a discretization of problem (4) with respect to t have the following property: a large number of previous values $z^{(r,m)}$ must be preserved, because they are needed to compute an approximate solution with $t = t^{(r+1)}$.

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