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On the Problem of Periodic Evolution Inclusions of the Subdifferential Type

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Abstract. We examine nonlinear periodic evolution inclusions of the subdifferential type and prove two existence theorems: one for the "non-convex, lower semicontinuous" problem and the other for the "convex, h-upper semicontinuous" problem. Our method of proof is based on the theory of nonlinear operators of monotone type and on multi-valued analysis. We also present three examples from partial and ordinary differential inclusions, illustrating the applicability of our work.

Keywords: *Convex subdifferential, maximal monotone operators, pseudomonotone operators, operators of type* $(S)_+$, *resolvent, Yosida approximation, variational inequalities*

AMS subject classification: 34K30, 35K85

1. Introduction

The periodic problem for differential inclusions has been studied primarily under the assumption that the orientor field (multi-valued vector field) is convex-valued. We refer to the works of Macki, Nistri and Zecca [16], Haddad and Lasry [8], Pruszko [19] and the references therein. These papers deal with differential inclusions in $\mathbb{R}^N$. The non-convex periodic problem in $\mathbb{R}^N$ has been considered recently by De Blasi, Gorniewicz and Pianigiani [6], Hu, Kandilakis and Papageorgiou [10] and by Hu and Papageorgiou [11].

The study of the periodic problem for evolution inclusions is lagging behind. Only the "convex" problem has been investigated using a Nagumo-type tangential condition. Bader [2] considered semilinear problems and used semigroup theory and the Hausdorff measure of non-compactness. Hu and Papageorgiou [12] considered nonlinear problems driven by time-varying maximal monotone coercive operators defined in the context of an evolution triple and used Garlekin approximations. The work of Bader [2] extended to evolution inclusions the paper of Prüss [18], while the work of Hu and Papageorgiou [12] is related to the papers of Vrabie [21] and Hirano [9]. We should also mention the recent work of Avgerinos and Papageorgiou [1],

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who considered evolution equations defined in the framework of an evolution triple and driven by a time-varying pseudomonotone (in general not maximal monotone) operator.

In this paper we examine both the "convex" and the "non-convex" periodic problem for nonlinear evolution inclusions of the subdifferential type. Our work here appears to be the first on nonlinear, non-convex periodic evolution inclusions and also extends to a multi-valued setting the work of Hirano [9]. Our approach is based on techniques from the theory of nonlinear operators of monotone type and from multi-valued analysis.

2. Mathematical background

For easy reference, in this section we present some basic definitions and facts from nonlinear operator theory and multi-valued analysis, which we shall need in the sequel. Our main sources are the books [13, 14, 22].

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this paper we use the notations

$$P_{f(c)}(X) = \Big\{A \subseteq X : A \text{ is non-empty, closed (and convex)}\Big\}$$
$$P_{(w)k(c)}(X) = \Big\{A \subseteq X : A \text{ is non-empty, (weakly-) compact (and convex)}\Big\}.$$

A multifunction (set-valued function) $F : \Omega \to P_f(X)$ is said to be *measurable*, if for each $x \in X$ the function

$$\omega \mapsto d(x, F(\omega)) = \inf \big\{\|x - u\| : u \in F(\omega)\big\}$$

is Σ-measurable. Also, the multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ is said to be *graph measurable*, if

$$\mathrm{Gr}F = \big\{(\omega, x) \in \Omega \times X : x \in F(\omega)\big\} \in \Sigma \times B(X)$$

with $B(X)$ being the Borel σ-field of X. For a multifunction with values in $P_f(X)$, measurability implies graph measurability, while the converse is true if Σ is complete (i.e. $\Sigma = \hat{\Sigma} =$ the universal σ-field).

Now let μ be a finite measure on Σ. Given a multifunction $F : \Omega \to 2^X \setminus \{\emptyset\}$ and $1 \leq p \leq \infty$, we define the set

$$S_F^p = \Big\{f \in L^p(\Omega, X) : f(\omega) \in F(\omega) \ \ \mu\text{-a.e. on } \Omega\Big\}$$

which may be empty. An easy application of a measurable selection theorem shows that, for a graph measurable function F, the set S_F^p is non-empty if and only if

$$\inf \big\{\|u\| : u \in F(\omega)\big\} \leq \varphi(\omega) \qquad \mu\text{-a.e. on } \Omega$$

with $\varphi \in L^p(\Omega)_+$. Moreover, the set S_F^p is closed or convex if and only if for μ-almost all $\omega \in \Omega$ the set $F(\omega)$ is closed or convex, respectively. Also, if $F : \Omega \to P_{wkc}(X)$ is measurable and

$$|F(\omega)| = \sup\{\|u\| : u \in F(\omega)\} \le \varphi_1(\omega) \ \mu\text{-a.e. on } \Omega$$

with $\varphi_1 \in L^p(\Omega)_+$ $(1 \le p < \infty)$, then $S_F^p \subset L^p(\Omega, X)$ is non-empty, weakly compact and convex. The set S_F^p is *decomposable* in the sense that if $(A, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then $\chi_A f_1 + \chi_{A^c} f_2 \in S_F^p$.

Let Y and Z be Hausdorff topological spaces. A multifunction $G : Y \to 2^Z$ is said to be *lower semicontinuous*, if for every $C \subset Z$ closed, the set

$$G^+(C) = \{y \in Y : G(y) \subset C\}$$

is closed. If Z is a metric space with metric d, then the multifunction G is lower semicontinuous if and only if for any $z \in Z$ the function $y \to d(z, G(y))$ is upper semicontinuous. Also, if Z is a metric space with metric d on $P_f(Z)$ we can define a generalized metric, known in the literature as *Hausdorff metric*, by setting

$$h(A,B) = \max\Big[\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big] \qquad \big(A, B \in P_f(Z)\big).$$

If Z is a complete metric space, then so is $(P_f(Z), h)$. A multifunction $G : Y \to P_f(Z)$ is said to be *h-continuous*, if it is continuous from Y into the metric space $(P_f(Z), h)$. Also, we set

$$h^*(A,B) = \sup_{a\in A} d(a,B),$$

and a multifunction $G : Y \to 2^Z \setminus \{\emptyset\}$ is said to be *h-upper semicontinuous* if for all $y \in Y$ the function $v \to h^*(G(v), G(y))$ is continuous at $y \in Y$.

Next, let X be a reflexive Banach space and X^* its (topological) dual. A map $A : D \subset X \to 2^{X^*}$ is said to be *monotone*, if $\langle x^* - y^*, x - y\rangle \ge 0$ for all $x, y \in D$ and all $x^* \in A(x), y^* \in A(y)$. Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). If in addition the equality $\langle x^* - y^*, x - y\rangle = 0$ implies $x = y$, then A is *strictly monotone*. The map A is said to be *maximal monotone* if $\langle x^* - y^*, x - y\rangle \ge 0$ for all $x \in D$ and all $x^* \in A(x)$ imply that $y \in D$ and $y^* \in A(y)$, i.e. the graph of A is maximal with respect to inclusion among the graphs of all monotone maps. It is easy to see that the graph of a maximal monotone map is closed in $X \times X_w^*$ and in $X_w \times X^*$. Here by X_w and X_w^* we denote the spaces X and X^*, respectively, furnished with the weak topology. If $X = H$ is a Hilbert space and $H^* = H$ (pivot space), for every maximal monotone operator $A : D \subset H \to 2^H$ and every $\lambda > 0$, we define the two well-known operators

$$J_\lambda = (I + \lambda A)^{-1} \qquad \text{(the } \textit{resolvent} \text{ of } A)$$
$$A_\lambda = \tfrac{1}{\lambda}(I - J_\lambda) \qquad \text{(the } \textit{Yosida approximation} \text{ of } A).$$

We have $D(J_\lambda) = D(A_\lambda) = H$ for all $\lambda > 0$. Both operators J_λ and A_λ are single-valued and have nice properties which are listed below:

(a) J_λ is non-expansive, i.e. $\|J_\lambda(x) - J_\lambda(y)\| \le \|x - y\|$ for all $x, y \in H$.
(b) A_λ is monotone and Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$ (hence A_λ is maximal monotone).
(c) $A_\lambda(x) \in A(J_\lambda(x))$ for all $x \in H$.
(d) $\|A_\lambda(x)\| \le \|A^0(x)\|$ for all $x \in D$, where $A^0(x)$ is the unique element of minimal norm in $A(x)$ and $A_\lambda(x) \to A^0(x)$ as $\lambda \downarrow 0$ for all $x \in D$.
(e) $\bar{D}$ is convex and $J_\lambda(x) \to \mathrm{proj}(x; \bar{D})$ for all $x \in H$ where $\mathrm{proj}(\cdot; \bar{D})$ denotes the metric projection on the convex set $\bar{D}$.

Let $\varphi : H \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be a proper (i.e. the set $\{x \in H : \varphi(x) < +\infty\}$ is non-empty), convex and lower semicontinuous (i.e. $\varphi \in \Gamma_0(H)$) function. The *effective domain* of φ is the set

$$\mathrm{dom}\,\varphi = \big\{x \in H : \varphi(x) < +\infty\big\}.$$

The *subdifferential* of φ is the multi-valued operator $\partial\varphi : D(\partial\varphi) \subset H \to 2^H$ defined by

$$\partial\varphi(x) = \Big\{ \in H : \langle u, y - x\rangle \le \varphi(y) - \varphi(x) \text{ for all } y \in H\Big\}.$$

We have $D(\partial\varphi) \subset \mathrm{dom}\,\varphi$ and $\partial\varphi$ is a maximal monotone operator. For $\lambda > 0$ we define

$$\varphi_\lambda(x) = \inf\big[\varphi(y) + \tfrac{1}{2\lambda}\|x - y\|^2 : y \in Y\big]$$

and call φ_λ the *Moreau-Yosida approximation* of φ. We know that

- φ_λ is convex and Fréchet differentiable (hence continuous)
- $\varphi'_\lambda(x) = \partial\varphi_\lambda(x) = (\partial\varphi)_\lambda(x)$
- $\varphi(J_\lambda(x)) \le \varphi_\lambda(x) \le \varphi(x)$ for all $\lambda > 0$ and all $x \in H$
- $\varphi_\lambda(x) \to \varphi(x)$ as $\lambda \downarrow 0$ for all $x \in H$.

Now return to the more general situation where X is a reflexive Banach space. A single-valued and everywhere defined operator $A : X \to X^*$ is said to be *demicontinuous* if $x_n \to x$ in X implies $A(x_n) \xrightarrow{w} A(x)$ in X^*. A monotone demicontinuous operator is maximal monotone. A map $A : D \subset X \to 2^{X^*}$ is said to be *coercive* if $\inf\big[\|x^*\|_* : x^* \in A(x)\big] \to \infty$ as $\|x\| \to \infty$ where $\|\cdot\|$ denotes the norm of X and $\|\cdot\|_*$ the norm of X^*. A maximal monotone coercive operator is surjective.

An operator $A : X \to X^*$ is said to be *pseudomonotone*, if

$$\left.\begin{array}{l} x_n \xrightarrow{w} x \text{ in } X \\ A(x_n) \xrightarrow{w} u \text{ in } X^* \\ \limsup\langle A(x_n), x_n - x\rangle \le 0 \end{array}\right\} \implies \left\{\begin{array}{l} u = A(x) \\ \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle. \end{array}\right.$$

A monotone demicontinuous map is pseudomonotone. The sum of pseudomonotone maps is still pseudomonotone. Also, a pseudomonotone coercive map is surjective. Finally, a map $A : X \to X^*$ is said to be of *type* $(S)_+$ if

$$\left.\begin{array}{l} x_n \xrightarrow{w} x \\ \limsup\langle A(x_n), x_n - x\rangle \le 0 \end{array}\right\} \implies x_n \to x \text{ in } X.$$

The prototype map of type $(S)_+$ is a uniformly monotone map $A : X \to X^*$, i.e. A satisfies

$$\psi(\|x-y\|)\|x-y\| \le \langle A(x)-A(y), x-y\rangle \qquad (x,y \in X)$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly monotone, increasing and continuous with $\psi(0)=0$ and $\lim_{r\to\infty}\psi(r)=+\infty$. A demicontinuous map of type $(S)_+$ is pseudomonotone.

3. Non-convex problem

Let $T=[0,b]$ and H a separable Hilbert space with inner product $(\cdot,\cdot)$. We study the multi-valued periodic problem

$$\left.\begin{aligned} -\dot{x}(t) &\in \partial\varphi(x(t)) + F(t,x(t)) \ \text{ a.e. on } T \\ x(0) &= x(b) \end{aligned}\right\} \tag{1}$$

where $\varphi \in \Gamma_0(H)$ and $F : T\times H \to 2^H \setminus \{\emptyset\}$. The precise hypotheses on the data of this problem are the following ones:

H(φ) $\varphi \in \Gamma_0(H)$ is of compact-type, i.e. the set $\{x \in H : \varphi(x) + \|x\|^2 \le \theta\}$ is compact for all $\theta \ge 0$ and $0 \in \partial\varphi(0)$.

Remark. We know that φ is of compact type if and only if for every $\lambda > 0$ the resolvent J_λ of $\partial\varphi$ is compact (see [13: p. 412]). Also, the condition $0 \in \partial\varphi(0)$ implies that $\varphi(0) = \inf_H \varphi$, i.e. φ attains its infimum at $x=0$.

H(F)$_1$ $F : T\times H \to P_f(H)$ is a multifunction such that the following conditions are satisfied:

- **(i)** $(t,x) \to F(t,x)$ is graph measurable.
- **(ii)** For a.a. $t\in T$, $x \to F(t,x)$ is lower semicontinuous.
- **(iii)** For a.a. $t \in T$, all $x \in H$ and all $v \in F(t,x)$, $\|v\| \le c_1(t) + c_2(t)\|x\|$ with $c_1, c_2 \in L^2(T)_+$.
- **(iv)** For a.a. $t\in T$, all $x \in D(\partial\varphi)$, all $w \in \partial\varphi(x)$ and all $v \in F(t,x)$, $(w+v,x) \ge c_3\|x\|^2 - c_4(t)$ with $c_3 > 0$ and $c_4 \in L^1(T)_+$.

Definition. A function $x \in W^{1,2}(T,H)$ is said to be a *strong solution* of problem (1) if $x(t) \in D(\partial\varphi)$ for all $t \in T$, $x(0)=x(b)$ and there exist $u \in S^2_{\partial\varphi(x(\cdot))}$ and $f \in S^2_{F(\cdot,x(\cdot))}$ such that $-\dot{x}(t) = u(t) + f(t)$ a.e. on T.

Remark. We know (see, for example, [14: p. 6]) that a function $x \in W^{1,2}(T,H)$ is absolutely continuous, hence strongly differentiable almost everywhere on T.

Consider the vectorial Sobolev space $W^{1,2}_{per}(T,H)$ defined by

$$W^{1,2}_{per} = \big\{x \in W^{1,2}(T,H) : x(0) = x(b)\big\}.$$

Since $W^{1,2}(T,H) \subset C(T,H)$, the pointwise evaluations at $t=0$ and $t=b$ make sense. Let $W^{1,2}_{per}(T,H)^*$ be the dual of $W^{1,2}_{per}(T,H)$. Then the triple

$$\big(W^{1,2}_{per}(T,H), L^2(T,H), W^{1,2}_{per}(T,H)^*\big)$$

is an evolution triple (see [14: p. 3]) or [22: p. 416]) and by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $\big(W^{1,2}_{per}(T,H), W^{1,2}_{per}(T,H)^*\big)$. Note that $\langle\cdot,\cdot\rangle|_{W^{1,2}_{per}(T,H)\times L^2(T,H)}$ is the inner product in the Hilbert space $L^2(T,H)$. Also, let $\hat{J}_{\frac{1}{n}} : L^2(T,H) \to L^2(T,H)$ be the Nemitsky operator corresponding to the resolvent operator $J_{\frac{1}{n}}$ of the maximal monotone map $\partial\varphi$, i.e. $\hat{J}_{\frac{1}{n}}(x)(\cdot) = J_{\frac{1}{n}}(x(\cdot))$ for all $x \in L^2(T,H)$.

Let

$$R : C(T,H) \to P_f(L^2(T,H))$$

be the multi-valued Nemitsky operator corresponding to F, i.e. $R(x) = S^2_{F(\cdot,x(\cdot))}$.

Proposition 1. *If hypotheses $H(F)_1$ hold, then R is lower semicontinuous.*

Proof. By what was said in Section 2, it suffices to show that for every $v \in L^2(T,H)$ the function $x \to d(v,R(x))$ is upper semicontinuous from $C(T,H)$ into $\mathbb{R}_+$. To this end we have to prove that for every $\theta > 0$ the superlevel set

$$U(\theta) = \big\{x \in C(T,H) : d(v,R(x)) \geq \theta\big\}$$

is closed. Let $x_n \in U(\theta)$ $(n \geq 1)$ and assume that $x_n \to x$ in $C(T,H)$. Then by Fatou's Lemma (hypothesis $H(F)_1/(iii)$ permits its use) we have

$$\limsup_{n\to\infty} \int_0^b d\big(v(t), F(t,x_n(t))\big)dt \leq \int_0^b \limsup_{n\to\infty} d\big(v(t), F(t,x_n(t))\big)dt.$$

Because $F(t,\cdot)$ is lower semicontinuous for almost all $t \in T$, $y \to d\big(v(t), F(t,y)\big)$ is upper semicontinuous. So since $x_n(t) \to x(t)$ for all $t \in T$ we have

$$\limsup_{n\to\infty} d\big(v(t), F(t,x_n(t))\big) \leq d\big(v(t), F(t,x(t))\big)$$

a.e. on T. Hence

$$\limsup_{n\to\infty} \int_0^b d\big(v(t), F(t,x_n(t))\big)dt \leq \int_0^b d\big(v(t), F(t,x(t))\big)dt.$$

But we know that

$$\int_0^b d\big(v(t), F(t,x_n(t))\big)dt = d(v,R(x_n)) \quad (n \geq 1)$$
$$\int_0^b d\big(v(t), F(t,x(t))\big)dt = d(v,R(x))$$

(see [13: p. 183]). Therefore $\theta \leq d(v,R(x))$ and we have proved the closedness of $U(\theta)$. So R is lower semicontinuous as claimed by the proposition ∎

Note that the values of R are decomposable subsets of $L^2(T,H)$. So we can apply [13: p. 245/Theorem II.8.7] and obtain a continuous function $u: C(T,H) \to L^2(T,H)$ such that $u(x) \in R(x)$ for all $x \in C(T,H)$.

Now let

$$K_n : W^{1,2}_{per}(T,H) \to W^{1,2}_{per}(T,H)^*$$

be defined by

$$\langle K_n(x), y\rangle = ((\tfrac{1}{n}x', y')) + ((\tfrac{1}{n}x, y)) + ((x' + u(\hat{J}_{\frac{1}{n}}(x)), y))$$

for all $x, y \in W^{1,2}_{per}(T,H)$. Here by $((\cdot,\cdot))$ we denote the inner product for the Hilbert space $L^2(T,H)$, i.e. $((f,g)) = \int_0^b (f(t), g(t))dt$. Also, note that since $W^{1,2}_{per}(T,H) \subset C(T,H)$, from the properties of the resolvent operator (see Section 2) we have $\hat{J}_{\frac{1}{n}}(x)(\cdot) = J_{\frac{1}{n}}(x(\cdot)) \in C(T,H)$ for all $x \in W^{1,2}_{per}(T,H)$ and so $u(\hat{J}_{\frac{1}{n}}(x))$ is well defined.

Proposition 2. *If hypotheses $H(\varphi)$ and $H(F)_1$ hold, then K_n is demicontinuous and of type $(S)_+$.*

Proof. First we show the demicontinuity of K_n. To this end let $x_m \to x$ in $W^{1,2}_{per}(T,H)$. Then $x'_m \to x'$ in $L^2(T,H)$ and from the non-expansiveness of $J_{\frac{1}{n}}(\cdot)$ we have

$$\left\|J_{\frac{1}{n}}(x_m(t)) - J_{\frac{1}{n}}(x(t))\right\| \le \left\|x_m(t) - x(t)\right\|$$

for all $t \in T$ and $m \ge 1$ from which

$$\left\|\hat{J}_{\frac{1}{n}}(x_m) - \hat{J}_{\frac{1}{n}}(x)\right\|_{C(T,H)} \le \left\|x_m - x\right\|_{C(T,H)}$$

follows. But because $x_m \to x$ in $W^{1,2}_{per}(T,H)$ and $W^{1,2}_{per}(T,H)$ is embedded continuously in $C(T,H)$, $\|x_m - x\|_{C(T,H)} \to 0$ as $m \to \infty$, hence $\hat{J}_{\frac{1}{n}}(x_m) \to \hat{J}_{\frac{1}{n}}(x)$ in $C(T,H)$ as $m \to \infty$. Therefore $u(\hat{J}_{\frac{1}{n}}(x_m)) \to u(\hat{J}_{\frac{1}{n}}(x))$ in $L^2(T,H)$ as $m \to \infty$. For every $y \in W^{1,2}_{per}(T,H)$ we have

$$\left.\begin{aligned} ((\tfrac{1}{n}x'_m, y)) &\to ((\tfrac{1}{n}x', y)) \\ ((\tfrac{1}{n}x_m, y)) &\to ((\tfrac{1}{n}x, y)) \\ ((x'_m + u(\hat{J}_{\frac{1}{n}}(x_m)), y)) &\to ((x' + u(\hat{J}_{\frac{1}{n}}(x)), y)) \end{aligned}\right\} \qquad (m \to \infty).$$

It follows that $\langle K_n(x_m), y\rangle \to \langle K_n(x), y\rangle$ as $m \to \infty$, which proves the demicontinuity of K_n.

Next we show that K_n is of type $(S)_+$. So suppose that $x_m \xrightarrow{w} x$ in $W^{1,2}_{per}(T,H)$ and assume $\limsup_{m\to\infty}\langle K_n(x_m), x_m - x\rangle \le 0$. We have to show that $x_m \to x$ in $W^{1,2}_{per}(T,H)$ as $m \to \infty$. From the definition of the operator K_n,

$$\begin{aligned} \langle K_n(x_m), x_m - x\rangle = ((\tfrac{1}{n}x'_m, x'_m - x')) + ((\tfrac{1}{n}x_m, x_m - x)) \\ + ((x'_m + u(\hat{J}_{\frac{1}{n}}(x_m)), x_m - x)). \end{aligned} \tag{2}$$

Because $x_m \xrightarrow{w} x$ in $W^{1,2}_{per}(T,H)$, $x_m \xrightarrow{w} x$ in $C(T,H)$ and so $x_m(t) \xrightarrow{w} x(t)$ in H as $m \to \infty$, for all $t \in T$. Since φ is of compact type (hypothesis $H(\varphi)$), $J_{\frac{1}{n}}(\cdot)$ is a compact operator (see [13: p. 412]) and so $J_{\frac{1}{n}}(x_m(t)) \to J_{\frac{1}{n}}(x(t))$ in H as $m \to \infty$, for all $t \in T$. Thus, for each $t \in T$, $\{J_{\frac{1}{n}}(x_m(t))\}_{m\geq 1}$ is relatively compact. Also, we know that $x'_m \xrightarrow{w} x'$ in $L^2(T,H)$ and so $\{x'_m\}_{m\geq 1}$ is uniformly integrable. Hence given $t \in T$ and $\varepsilon > 0$ there exists $0 < \delta = \delta(t,\varepsilon)$ such that

$$\int_t^{t+\delta} \|x'_m(s)\|\, ds < \varepsilon \qquad (m \geq 1).$$

Hence for $\hat{t} \in [t, t+\delta)$

$$\left\|J_{\frac{1}{n}}(x_m(\hat{t})) - J_{\frac{1}{n}}(x_m(t))\right\| \leq \left\|x_m(\hat{t}) - x_m(t)\right\| \leq \int_t^{\hat{t}} \|x'_m(s)\|\, ds < \varepsilon$$

and we see that $\{J_{\frac{1}{n}}(x_m(\cdot))\}_{m\geq 1}$ is also equicontinuous. By the Ascoli-Arzela theorem it follows that $\{\hat{J}_{\frac{1}{n}}(x_m)\}_{m\geq 1} \subset C(T,H)$ is relatively compact. Since u is continuous, we obtain the same conclusion for $\{u(\hat{J}_{\frac{1}{n}}(x_m))\}_{m\geq 1} \subset L^2(T,H)$. Moreover, because $x_m \xrightarrow{w} x$ in $W^{1,2}_{per}(T,H)$, $x_m \xrightarrow{w} x$ in $L^2(T,H)$ and so

$$\lim_{m\to\infty} ((u(\hat{J}_{\frac{1}{n}}(x_m)), x_m - x)) = 0. \tag{3}$$

Also, for all $m \geq 1$

$$((x'_m, x_m - x)) = \int_0^b \tfrac{1}{2}\tfrac{d}{dt}\|x_m(t)\|^2 dt - ((x'_m, x)) = -((x'_m, x))$$

since $x_m(0) = x_m(b)$ and so

$$\begin{aligned} \lim_{m\to\infty} ((x'_m, x_m - x)) &= -\lim_{m\to\infty} ((x'_m, x)) \\ &= -((x', x)) \\ &= -\int_0^b \frac{d}{dt}\|x(t)\|^2 dt \\ &= 0. \end{aligned} \tag{4}$$

We return to (2), pass to the limit as $m \to \infty$ and use (3) - (4) above. So

$$\begin{aligned} & \limsup_{m\to\infty} \left[((\tfrac{1}{n}x'_m, x'_m - x')) + ((\tfrac{1}{n}x_m, x_m - x))\right] \leq 0 \\ \Longrightarrow\quad & \limsup_{m\to\infty} \left[((\tfrac{1}{n}(x'_m - x'), x'_m - x')) + ((\tfrac{1}{n}(x_m - x), x_m - x))\right] \leq 0 \\ \Longrightarrow\quad & \limsup_{m\to\infty} \left[\tfrac{1}{n}\|x'_m - x'\|_2^2 + \tfrac{1}{n}\|x_m - x\|_2^2\right] \leq 0 \\ \Longrightarrow\quad & \|x_m - x\|_{W^{1,2}(T,H)} \to 0 \quad (m \to \infty). \end{aligned}$$

Therefore we have proved that K_n is of type $(S)_+$ ∎

For $\lambda > 0$ let $A_\lambda : H \to H$ be the Yosida approximation of the maximal monotone operator $A = \partial\varphi$. Recall that $A_\lambda = \partial\varphi_\lambda$ (see Section 2). Let

$$\hat{A}_\lambda : L^2(T, H) \to L^2(T, H)$$

be the Nemitsky operator corresponding to A_λ, i.e. $\hat{A}_\lambda(x)(\cdot) = A_\lambda(x(\cdot))$.

Proposition 3. *If hypotheses $H(\varphi)$ and $H(F)_1$ hold, then for $n \geq 1$ large $K_n + \hat{A}_{\frac{1}{n}}$ is coercive.*

Proof. Suppose that $\|x_m\|_{W^{1,2}(T,H)} \to \infty$ as $m \to \infty$. We have

$$\begin{aligned} &\langle K_n(x_m) + \hat{A}_{\frac{1}{n}}(x_m), x_m \rangle \\ &= \tfrac{1}{n}\|x_m\|^2_{W^{1,2}(T,H)} + ((x'_m, x_m)) + \big((\hat{A}_{\frac{1}{n}}(x_m) + u(\hat{J}_{\frac{1}{n}}(x_m)), x_m)\big). \end{aligned} \tag{5}$$

From the definition of the Yosida approximation, $\hat{A}_{\frac{1}{n}} = n(I - \hat{J}_{\frac{1}{n}})$. So

$$\begin{aligned} ((\hat{A}_{\frac{1}{n}}(x_m), x_m)) &= \big((n(I - \hat{J}_{\frac{1}{n}})(x_m), x_m)\big) \\ &= \big((n(I - \hat{J}_{\frac{1}{n}})(x_m), (I - \hat{J}_{\frac{1}{n}})(x_m))\big) + \big((\hat{A}_{\frac{1}{n}}(x_m), \hat{J}_{\frac{1}{n}}(x_m))\big) \\ &= n\big\|(I - \hat{J}_{\frac{1}{n}})(x_m)\big\|_2^2 + \big((\hat{A}_{\frac{1}{n}}(x_m), \hat{J}_{\frac{1}{n}}(x_m))\big). \end{aligned}$$

Also,

$$((x'_m, x_m)) = \tfrac{1}{2}\int_0^b \frac{d}{dt}\|x_m(t)\|^2 dt = 0$$

because $x_m(0) = x_m(b)$. Therefore returning to (5) we can write

$$\begin{aligned} \langle K_n(x_m) + \hat{A}_{\frac{1}{n}}(x_m), x_m \rangle = \tfrac{1}{n}\|x_m\|^2_{W^{1,2}(T,H)} &+ n\big\|(I - \hat{J}_{\frac{1}{n}})(x_m)\big\|_2^2 \\ &+ \big((\hat{A}_{\frac{1}{n}}(x_m) + u(\hat{J}_{\frac{1}{n}}(x_m)), \hat{J}_{\frac{1}{n}}(x_m))\big) \\ &+ \big((u(\hat{J}_{\frac{1}{n}}(x_m)), x_m - \hat{J}_{\frac{1}{n}}(x_m))\big). \end{aligned} \tag{6}$$

We know that

$$\left.\begin{aligned} A_{\frac{1}{n}}(x_m(t)) \in A\big(J_{\frac{1}{n}}(x_m(t))\big) \quad &(t \in T) \\ u(\hat{J}_{\frac{1}{n}}(x_m))(t) \in F\big(t, J_{\frac{1}{n}}(x_m(t))\big) \quad &\text{a.e. on } T \end{aligned}\right. \quad (m \geq 1).$$

Using hypothesis $H(F)_1/(iv)$, we get

$$\big((\hat{A}_{\frac{1}{n}}(x_m) + u(\hat{J}_{\frac{1}{n}}(x_m)), \hat{J}_{\frac{1}{n}}(x_m))\big) \geq c_5\big\|\hat{J}_{\frac{1}{n}}(x_m)\big\|_2^2 - c_6$$

for $m \geq 1$ with $c_5, c_6 > 0$. Also, from hypothesis $H(F)_1/(iii)$ we have

$$\big|\big((u\big(\hat{J}_{\frac{1}{n}}(x_m)\big), x_m - \hat{J}_{\frac{1}{n}}(x_m))\big)\big| \leq \big(c_7 + c_8\big\|\hat{J}_{\frac{1}{n}}(x_m)\big\|_2\big)\big\|x_m - \hat{J}_{\frac{1}{n}}(x_m)\big\|_2 \tag{7}$$

for $m \geq 1$ with $c_7, c_8 > 0$. Using Young's inequality with $\varepsilon > 0$ on the right-hand side we obtain

$$\begin{aligned}&\big(c_7 + c_8\|\hat{J}_{\frac{1}{n}}(x_m)\|_2\big)\|x_m - \hat{J}_{\frac{1}{n}}(x_m)\|_2 \\ &\quad \leq \varepsilon c_6^2 + \varepsilon c_7^2\|\hat{J}_{\frac{1}{n}}(x_m)\|_2^2 + \tfrac{1}{2\varepsilon}\|x_m - \hat{J}_{\frac{1}{n}}(x_m)\|_2^2\end{aligned}$$

from which

$$\begin{aligned}&\big|\big(\big(u(\hat{J}_{\frac{1}{n}}(x_m)), x_m - \hat{J}_{\frac{1}{n}}(x_m)\big)\big)\big| \\ &\quad \leq \varepsilon c_6^2 + \varepsilon c_7^2\|\hat{J}_{\frac{1}{n}}(x_m)\|_2^2 + \tfrac{1}{2\varepsilon}\|x_m - \hat{J}_{\frac{1}{n}}(x_m)\|_2^2\end{aligned} \tag{8}$$

follows. Using (7) and (8) in (6), we obtain

$$\begin{aligned}&\big\langle K_n(x_m) + \hat{A}_{\frac{1}{n}}(x_m), x_m\big\rangle \\ &\quad \geq \tfrac{1}{n}\|x_m\|_{W^{1,2}(T,H)}^2 + n\|(I - \hat{J}_{\frac{1}{n}})(x_m)\|_2^2 \\ &\quad\quad + c_5\|\hat{J}_{\frac{1}{n}}(x_m)\|_2^2 - \varepsilon c_7^2\|\hat{J}_{\frac{1}{n}}(x_m)\|_2^2 - \tfrac{1}{2\varepsilon}\|x_m - \hat{J}_{\frac{1}{n}}(x_m)\|_2^2 - c_8\end{aligned} \tag{9}$$

with $c_8 = c_8(\varepsilon) > 0$. Choose $\varepsilon > 0$ so that $c_5 > \varepsilon c_7^2$. Then based on this choice of $\varepsilon > 0$ choose $n_0 \geq 1$ large enough so that for $n \geq n_0$ we have $n \geq \frac{1}{2\varepsilon}$. With these choices, we see from (9) that for $n \geq n_0$ the operator $K_n + \hat{A}_{\frac{1}{n}}$ is coercive ∎

Using these auxiliary results we can now prove an existence theorem for problem (1).

Theorem 4. *If hypotheses $H(\varphi)$ and $H(F)_1$ hold, then problem* (1) *has a strong solution $x \in W_{per}^{1,2}(T, \mathbb{R}^N)$.*

Proof. The operator $\hat{A}_{\frac{1}{n}}$ is maximal monotone and continuous, hence pseudomonotone. Also, from Proposition 2 we know that K_n is demicontinuous and of type $(S)_+$, thus pseudomonotone. The sum of pseudomonotone operators is pseudomonotone. Therefore $x \to (K_n + \hat{A}_{\frac{1}{n}})(x)$ is pseudomonotone. From Proposition 3 we know that it is also coercive. Hence it is surjective (see Section 2). So for every $n \geq 1$ we can find $x_n \in W_{per}^{1,2}(T, H)$ such that

$$K_n(x_n) + \hat{A}_{\frac{1}{n}}(x_n) = 0.$$

From (9) and the choices of $\varepsilon > 0$ and $n \geq 1$ made there (see the proof of Proposition 3), we have

$$\tfrac{1}{n}\|x_n\|_{W_{per}^{1,2}(T,H)} \leq M_1 \qquad \text{for some } M_1 > 0 \text{ and all } n \geq n_0.$$

Also, as before, from the definition of the Yosida approximation, we have

$$\begin{aligned}&\big(\big(\hat{A}_{\frac{1}{n}}(x_n) + u(\hat{J}_{\frac{1}{n}}(x_n)), x_n\big)\big) \\ &\quad \geq c_3\|\hat{J}_{\frac{1}{n}}(x_n)\|_2^2 + n\|(I - \hat{J}_{\frac{1}{n}})(x_n)\|_2^2 \\ &\quad\quad - \big(c_7 + c_8\|\hat{J}_{\frac{1}{n}}(x_n)\|_2\big)\|x_n - \hat{J}_{\frac{1}{n}}(x_n)\|_2 - c_9\end{aligned} \tag{10}$$

for some $c_9 > 0$. Let $\beta > 1$ be such that $c_3\left(\frac{\beta-1}{\beta}\right)^2 > \frac{c_8}{\beta}$. Then

$$\beta\left\|(I - \hat{J}_{\frac{1}{n}})(x_n)\right\|_2 \leq \|x_n\|_2 \quad \Longrightarrow \quad \beta\left|\,\|x_n\|_2 - \|\hat{J}_{\frac{1}{n}}(x_n)\|_2\right| \leq \|x_n\|_2$$

and so

$$(\beta - 1)\|x_n\|_2 \leq \left\|\hat{J}_{\frac{1}{n}}(x_n)\right\|_2.$$

Because $0 \in \partial\varphi(0)$, $J_\lambda(0) = 0$ and so $\|\hat{J}_{\frac{1}{n}}(x_n)\|_2 \leq \|x_n\|_2$. From (10) we obtain

$$\begin{aligned}
&\left((\hat{A}_{\frac{1}{n}}(x_n) + u(\hat{J}_{\frac{1}{n}}(x_n)), x_n)\right) \\
&\quad \geq c_3\left(\tfrac{\beta-1}{\beta}\right)^2\|x_n\|_2^2 - \tfrac{c_7}{\beta}\|x_n\|_2 - \tfrac{c_8}{\beta}\|\hat{J}_{\frac{1}{n}}(x_n)\|_2\|x_n\|_2 - c_9 \\
&\quad \geq c_3\left(\tfrac{\beta-1}{\beta}\right)^2\|x_n\|_2^2 - \tfrac{c_8}{\beta}\|x_n\|_2^2 - \tfrac{c_7}{\beta}\|x_n\|_2 - c_9 \\
&\quad = \left(c_3\left(\tfrac{\beta-1}{\beta}\right)^2 - \tfrac{c_8}{\beta}\right)\|x_n\|_2^2 - \tfrac{c_7}{\eta}\|x_n\|_2 - c_9.
\end{aligned} \tag{11}$$

On the other hand, if $\beta\left\|(I - \hat{J}_{\frac{1}{n}})(x_n)\right\|_2 \geq \|x_n\|_2$, then since $\left\|(I - \hat{J}_{\frac{1}{n}})(x_n)\right\|_2 \leq 2\|x_n\|_2$, from (10) we have

$$\left((\hat{A}_{\frac{1}{n}}(x_n) + u(\hat{J}_{\frac{1}{n}}(x_n)), x_n)\right) \geq \left(\tfrac{n}{\beta^2} - 2c_8\right)\|x_n\|_2^2 - 2c_7\|x_n\|_2 - c_9. \tag{12}$$

From (11) and (12) we see that, for $n \geq n_0$,

$$\left((\hat{A}_{\frac{1}{n}}(x_n) + u(\hat{J}_{\frac{1}{n}}(x_n)), x_n)\right) \geq c_{10}\|x_n\|_2^2 - c_{11}$$

for some $c_{10}, c_{11} > 0$. Thus for $n \geq n_0$

$$\begin{aligned}
0 &= \left\langle K_n(x_n) + \hat{A}_{\frac{1}{n}}(x_n), x_n\right\rangle \\
&\geq \left((\hat{A}_{\frac{1}{n}}(x_n) + u(\hat{J}_{\frac{1}{n}}(x_n)), x_n)\right) \\
&\geq c_{10}\|x_n\|_2^2 - c_{11}.
\end{aligned}$$

Thus $\{x_n\}_{n\geq 1} \subset L^2(T, H)$ is bounded. From this and the fact that $\|\hat{J}_{\frac{1}{n}}(x_n)\|_2 \leq \|x_n\|_2$ $(n \geq 1)$ we deduce that $\left\{\hat{J}_{\frac{1}{n}}(x_n)\right\}_{n\geq 1} \subset L^2(T, H)$ is bounded. From this, the fact that $u(\hat{J}_{\frac{1}{n}}(x_n)) \in R(\hat{J}_{\frac{1}{n}}(x_n))$ and hypothesis $H(F)_1/(iii)$ it follows that

$$\left\{u(\hat{J}_{\frac{1}{n}}(x_n))\right\}_{n\geq 1} \subset L^2(T, H) \quad \text{is bounded.}$$

Note that, since $x_n \in W^{1,2}_{per}(T, H)$ and $\hat{A}'_{\frac{1}{n}}(\cdot)$ is Lipschitz continuous, $\hat{A}_{\frac{1}{n}}(x_n) \in W^{1,2}(T, H)$. Because $K_n(x_n) + \hat{A}_{\frac{1}{n}}(x_n) = 0$ $(n \geq 1)$, by tacking duality brackets with $\hat{A}_{\frac{1}{n}}(x_n)$ we obtain

$$0 = \left\langle K_n(x_n), \hat{A}_{\frac{1}{n}}(x_n)\right\rangle + \left\|\hat{A}_{\frac{1}{n}}(x_n)\right\|_2^2$$

and so

$$0 = \left(\left(\tfrac{1}{n}x_n', \tfrac{d}{dt}\hat{A}_{\frac{1}{n}}(x_n)\right)\right) + \left(\left(\tfrac{1}{n}x_n, \hat{A}_{\frac{1}{n}}(x_n)\right)\right) + \left(\left(x_n' + u(\hat{J}_{\frac{1}{n}}(x_n)), \hat{A}_{\frac{1}{n}}(x_n)\right)\right) + \|\hat{A}_{\frac{1}{n}}(x_n)\|_2^2. \tag{13}$$

Recall that $\hat{A}_{\frac{1}{n}}$ is monotone, and because $0 \in \partial\varphi(0)$ we have $\hat{A}_{\frac{1}{n}}(0) = 0$. So

$$0 \leq \left(\left(\tfrac{1}{n}x_n, \hat{A}_{\frac{1}{n}}(x_n)\right)\right). \tag{14}$$

Also, for all $n \geq 1$

$$\left(\left(\tfrac{1}{n}x_n', \tfrac{d}{dt}\hat{A}_{\frac{1}{n}}(x_n)\right)\right) = \int_0^b \left(\tfrac{1}{n}x_n'(t), \tfrac{d}{dt}A_{\frac{1}{n}}(x_n(t))\right)dt.$$

We know that $A_{\frac{1}{n}}$ is Lipschitz continuous, and so by the generalized Rademacher theorem (see, for example, [5: p. 121]) it is Gateaux differentiable at every $x \in H \setminus D$, with D being a Haar-null subset of H. Then, employing the chain rule of Marcus and Mizel [17],

$$\left(x^*, \tfrac{d}{dt}A_{\frac{1}{n}}(x_n(t))\right) = \tfrac{d}{dt}\left(x^*, A_{\frac{1}{n}}(x_n(t))\right) = \left(x^*, A_{\frac{1}{n}}'(x_n(t))x_n'(t)\right)$$

for all $x^* \in H$ and all $t \in T \setminus N_n(x^*)$ with $|N_n(x^*)| = 0$, where $|\cdot|$ is the Lebesgue measure on T. Let $\{x_m^*\}_{m\geq 1}$ be dense in H and set $N_n = \cup_{m\geq 1} N_n(x_m^*)$. Evidently, $|N_n| = 0$ and for $t \in T \setminus N_n$ and $x^* \in H$ we have

$$\left(x^*, \tfrac{d}{dt}A_{\frac{1}{n}}(x_n(t))\right) = \tfrac{d}{dt}\left(x^*, A_{\frac{1}{n}}(x_n(t))\right) = \left(x^*, A_{\frac{1}{n}}'(x_n(t))x_n'(t)\right)$$

and so

$$\tfrac{d}{dt}A_{\frac{1}{n}}(x_n(t)) = A_{\frac{1}{n}}'(x_n(t))x_n'(t) \qquad \text{a.e. on } T.$$

Therefore

$$\begin{aligned}\left(\left(\tfrac{1}{n}x_n', \tfrac{d}{dt}\hat{A}_{\frac{1}{n}}(x_n)\right)\right) &= \int_0^b \left(\tfrac{1}{n}x_n'(t), \tfrac{d}{dt}A_{\frac{1}{n}}(x_n(t))\right)dt \\ &= \int_0^b \left(\tfrac{1}{n}x_n'(t), A_{\frac{1}{n}}'(x_n(t))x_n'(t)\right)dt.\end{aligned}$$

Exploiting the monotonicity of $A_{\frac{1}{n}}$ we can easily check that

$$\left(\tfrac{1}{n}x_n'(t), A_{\frac{1}{n}}'(x_n(t))x_n'(t)\right) \geq 0 \qquad \text{a.e. on } T.$$

So we deduce

$$0 \leq \left(\left(\tfrac{1}{n}x_n', \tfrac{d}{dt}\hat{A}_{\frac{1}{n}}(x_n)\right)\right). \tag{15}$$

Finally,

$$\begin{aligned}\left(\left(x_n', \hat{A}_{\frac{1}{n}}(x_n)\right)\right) &= \int_0^b \left(x_n'(t), A_{\frac{1}{n}}(x_n(t))\right)dt \\ &= \int_0^b \left(x_n'(t), \partial\varphi_{\frac{1}{n}}(x_n(t))\right)dt \\ &= \int_0^b \tfrac{d}{dt}\varphi_{\frac{1}{n}}(x_n(t))\,dt \\ &= 0\end{aligned}$$

(for the last two equalities see [13: p. 357]) and recall that $x_n(0) = x_n(b)$). Using (14) - (16) in (13) we obtain

$$\|\hat{A}_{\frac{1}{n}}(x_n)\|_2^2 \le \|u(\hat{J}_{\frac{1}{n}}(x_n))\|_2 \|\hat{A}_{\frac{1}{n}}(x_n)\|_2.$$

We already know that $\{u(\hat{J}_{\frac{1}{n}}(x_n))\}_{n\ge1} \subset L^2(T,H)$ is bounded. Therefore the sequence $\{\hat{A}_{\frac{1}{n}}(x_n)\}_{n\ge1} \subset L^2(T,H)$ is bounded.

For every $n \ge 1$, $x_n \in (K_n + \hat{A}_{\frac{1}{n}})^{-1}(0)$. From Proposition 2 we know that $K_n + \hat{A}_{\frac{1}{n}}$ is coercive. Therefore $\{x_n\}_{n\ge1} \subset W^{1,2}_{per}(T,H)$ is bounded. By passing to a subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W^{1,2}_{per}(T,H)$. Arguing as in the proof of Proposition 2, we obtain that $\{\hat{J}_{\frac{1}{n}}(x_n)\}_{n\ge1} \subset L^2(T,H)$ is relatively compact and so we may asume that $\hat{J}_{\frac{1}{n}}(x_n) \to y$ in $L^2(T,H)$. Recall that $\hat{A}_{\frac{1}{n}}(x_n) = n(I - \hat{J}_{\frac{1}{n}})(x_n)$ and $\{\hat{A}_{\frac{1}{n}}(x_n)\}_{n\ge1} \subset L^2(T,H)$ is bounded. So $\|x_n - \hat{J}_{\frac{1}{n}}(x_n)\|_2 \to 0$ as $n \to \infty$, hence $x_n \to x = y$ in $L^2(T,H)$. Also, we may assume that $\hat{A}_{\frac{1}{n}}(x_n) \xrightarrow{w} v$ in $L^2(T,H)$.

Let $\Phi : L^2(T,H) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ be defined by

$$\Phi(y) = \begin{cases} \int_0^b \varphi(y(t))\,dt & \text{if } \varphi(y(\cdot)) \in L^1(T) \\ +\infty & \text{otherwise.} \end{cases}$$

We know that $\Phi \in \Gamma_0(L^2(T,H))$ and $\hat{A}_{\frac{1}{n}}(x_n) = \partial\Phi_{\frac{1}{n}}(x_n) \in \partial\Phi(\hat{J}_{\frac{1}{n}}(x_n))$ (see [13: p. 349]). The subdifferential $\partial\Phi$ ia a maximal monotone operator in the Hilbert space $L^2(T,H)$ and $\hat{J}_{\frac{1}{n}}(x_n) \to x$ in $L^2(T,H)$ and $\hat{A}_{\frac{1}{n}}(x_n) \xrightarrow{w} v$ in $L^2(T,H)$. Recalling that $\mathrm{Gr}\,\partial\Phi$ is closed in $L^2(T,H) \times L^2(T,H)_w$, $v \in \partial\Phi(x)$ and so $v(t) \in \partial\varphi(x(t))$ a.e. on T.

Also, since $\{x_n\}_{n\ge1} \subset W^{1,2}_{per}(T,H)$ is bounded, $\frac{1}{n}x_n, \frac{1}{n}x'_n \to 0$ in $L^2(T,H)$. Recall that for every $z \in L^2(T,H)$ and all $n \ge 1$

$$((\tfrac{1}{n}x'_n, z)) + ((\tfrac{1}{n}x_n, z)) + ((x'_n + u(\hat{J}_{\frac{1}{n}}(x_n)), z)) + ((\hat{A}_{\frac{1}{n}}(x_n), z)) = 0. \tag{17}$$

Passing to the limit as $n \to \infty$ and since $u(\hat{J}_{\frac{1}{n}}(x_n)) \to u(x)$ in $L^2(T,H)$ (because u is continuous) we obtain

$$\begin{aligned} & ((x' + u(x), z)) + ((v, z)) = 0 \\ \Longrightarrow\ & x' + u(x) + v = 0 \\ \Longrightarrow\ & -x'(t) \in \partial\varphi(x(t)) + F(t, x(t)) \ \text{ a.e. on } T \\ & x(0) = x(b) \end{aligned}$$

because $u(x) \in R(x)$. This proves that $x \in W^{1,2}_{per}(T,H)$ is a strong solution of problem (1) ∎

4. Convex problem

In this section we prove an existence theorem for the "convex" version of problem (1). Our hypothesis on the orientor field $F(t,x)$ is the following:

H(F)$_2$ $F : T \times H \to P_{fc}(H)$ is a multifunction such that the following conditions are satisfied:

- **(i)** $(t,x) \to F(t,x)$ is measurable.
- **(ii)** For almost all $t \in T$, $x \to F(t,x)$ is h-upper semicontinuous.
- **(iii)** For almost all $t \in T$, all $x \in H$ and all $v \in F(t,x)$, $\|v\| \le c_1(t) + c_2(t)\|x\|$ with $c_1, c_2 \in L^2(T)_+$.
- **(iv)** For almost all $t \in T$, all $x \in H$ and all $v \in F(t,x)$, $(v,x) \ge c_3\|x\|^2 - c_4(t)$ with $c_3 > 0$ and $c_4 \in L^1(T)_+$.
- **(v)** There exists $r > 0$ such that, for almost all $t \in T$, all $x \in H$ with $\|x\| = r$ and all $v \in F(t,x)$, $(v,x) \ge 0$.

Remark. If, for example, in hypothesis $H(F)_2/(iv)$ above $c_4 \in L^\infty(T)_+$, then hypothesis $H(F)_2/(v)$ follows from hypothesis $H(F)_2/(iv)$.

Theorem 5. *If hypotheses $H(\varphi)$ and $H(F)_2$ hold, then the set of strong solutions of problem* (1) *is non-empty and compact in $C(T,H)$.*

Proof. Let $r > 0$ be as in hypothesis $H(F)_2/(v)$ and let $p_r : H \to H$ denote the r-radial retraction on H, i.e.

$$p_r(x) = \begin{cases} x & \text{if } \|x\| \le r \\ \frac{rx}{\|x\|} & \text{if } \|x\| > r. \end{cases}$$

Denote by F_1 the modification of F given by

$$F_1 : T \times H \to P_{fc}(H), \qquad F_1(t,x) = F(t, p_r(x)).$$

So

- $(t,x) \to F_1(t,x)$ is measurable
- for a.a. $t \in T$, $x \to F_1(t,x)$ is h-upper semicontinuous
- for a.a. $t \in T$, all $x \in H$ and all $v \in F_1(t,x)$, $\|v\| \le c(t)$ with $c \in L^2(T)_+$.

Now consider the periodic evolution inclusion

$$\left.\begin{aligned} & -\dot{x}(t) \in \partial\varphi(x(t)) + \big(x(t) - p_r(x(t))\big) + F_1(t,x(t)) \quad \text{a.e. on } T \\ & x(0) = x(b) \end{aligned}\right\}. \tag{18}$$

Suppose we were able to obtain a strong solution $x \in W^{1,2}_{per(T,H)}$ of it. Then we claim that $\|x\|_{C(T,H)} \le r$. Suppose that this is not the case. Then $\|x(t)\| > r$ for all $t \in (\beta,\gamma)$ and $\|x(\beta)\| = \|x(\gamma)\| = r$. We know that

$$-\dot{x}(t) = v(t) + h(t) \qquad \text{a.e. on } T$$

with $v \in S^2_{\partial\varphi(x(\cdot))}$ and

$$h(t) = \big(x(t) - p_r(x(t))\big) + f(t) \qquad \text{a.e. on } T \tag{19}$$

where $f \in S^2_{F_1(\cdot,x(\cdot))}$. Since $0 \in \partial\varphi(0)$, $(v(t), x(t)) \geq 0$ a.e. on T. So

$$\big(\dot{x}(t), x(t)\big) + \big(h(t), x(t)\big) \leq 0 \ \text{ a.e. on } T$$

and thus

$$\tfrac{1}{2}\tfrac{d}{dt}\|x(t)\|^2 + \big(h(t), x(t)\big) \leq 0 \ \text{ a.e. on } T.$$

Using (19) we see that, for almost all $t \in [\beta, \gamma]$,

$$\begin{aligned} &\big(h(t), x(t)\big) = \|x(t)\|^2 - r\|x(t)\| + \tfrac{\|x(t)\|}{r}\,\big(f(t), p_r(x(t))\big) \\ \Longrightarrow\ \ & 0 < \big(h(t), x(t)\big) \ \text{ a.e. on } [\beta,\gamma] \\ &\quad (\text{hypothesis } H(F)_2/(v), \text{ and recall that } r < \|x(t)\| \text{ on } (\beta,\gamma)) \\ \Longrightarrow\ \ & \tfrac{1}{2}\tfrac{d}{dt}\|x(t)\|^2 < 0 \ \text{ a.e. on } (\beta,\gamma) \\ \Longrightarrow\ \ & \|x(\gamma)\|^2 < \|x(\beta)\|^2. \end{aligned}$$

The last relation is a contradiction. So every strong solution $x \in W^{1,2}_{per}(T,H)$ of problem (18) satisfies $\|x\|_{C(T,H)} \leq r$ and it is obvious that every solution of problem (18) is indeed a solution of problem (1).

Hence, in the sequel we will seek for strong solutions of problem (18). To this end we invoke [13: p. 48/Lemma 3.1] (see also [7]) and we can find a decreasing sequence of multifunctions $F_1^n : T \times H \to P_{fc}(H)$ such that:

(a) For all $x \in H$, $t \to F_1^n(t,x)$ is measurable.

(b) For a.e. $t \in T$, $x \to F_1^n(t,x)$ is locally h-Lipschitz.

(c) For a.a. $t \in T$, all $x \in H$ and all $v \in F_1^n(t,x)$, $\|v\| \leq c(t)$.

(d) For a.a. $t \in T$ and all $x \in H$, $F_1^n(t,x) \xrightarrow{h} F_1(t,x)$ as $n \to \infty$.

Consider the following approximation to problem (18):

$$\left.\begin{aligned} &-\dot{x}(t) \in \partial\varphi(x(t)) + \big(x(t) - p_r(x(t))\big) + F_1^n(t,x(t)) \text{ a.e. on } T \\ &x(0) = x(b) \end{aligned}\right\}. \tag{20}$$

Note that for almost all $t \in T$ and all $x \in H$

$$F_1^n(t,x) \subset F_1(t,x) + 2c(t)\bar{B}_1 \qquad \text{where } \bar{B}_1 = \big\{x \in H : \|x\| \leq 1\big\}.$$

So if $v \in F_1^n(t,x)$, then $v = \hat{v} + 2c(t)e$ with $\hat{v} \in F_1(t,x)$ and $e \in \bar{B}_1$. Now suppose $\|x\| \leq r$. Then

$$\begin{aligned} (v,x) &= (\hat{v} + 2c(t)e, x) \\ &\geq (\hat{v}, x) - 2c(t)\|x\| \\ &\geq c_3\|x\|^2 - c_4(t) - 2c(t)\|x\| \\ &\geq c_3\|x\|^2 - 2c(t)r - c_4(t) \end{aligned}$$

provided $(t,x) \in T \times H$ is such that hypothesis $H(F)_2/(iv)$ holds. Now if $\|x\| > r$, then

$$\begin{aligned}((x - p_r(x)) + v, x) &= \|x\|^2 - r\|x\| + (\hat{v} + 2c(t)e, x)\\ &\geq \|x\|^2 - r\|x\| - 2c(t)\|x\|\\ &\geq \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\big(r + 2c(t)\big)^2\end{aligned}$$

where we have used hypothesis $H(F)_2/(v)$. From these observations and the fact that $(w,x) \geq 0$ for all $w \in \partial\varphi(x)$ (since $0 \in \partial\varphi(0)$) it follows that there exists $\bar{c}_3 > 0$ and $\bar{c}_4 \in L^1(T)_+$ such that the mapping $(t,x) \mapsto x - p_r(x) + F_1^n(t,x)$ satisfies hypothesis $H(F)_1/(iv)$.

We are now in a position to apply Theorem 4 and we obtain a strong solution $x_n \in W^{1,2}_{per}(T,H)$ of problem (20) for every $n \geq 1$. We have

$$\left.\begin{aligned}-\dot{x}_n(t) &= v_n(t) + \big(x_n(t) + p_r(x_n(t))\big) + f_n(t) \quad \text{a.e. on } T\\ x_n(0) &= x_n(b)\end{aligned}\right\}$$

with $v_n \in S^2_{\partial\varphi(x_n(\cdot))}$ and $f_n \in S^2_{F_1^n(\cdot,x_n(\cdot))}$. Taking the inner product with $\dot{x}_n(t)$, we obtain

$$\|\dot{x}_n(t)\|^2 + \big(v_n(t), \dot{x}_n(t)\big) + \big(x_n(t) + p_r(x_n(t)), \dot{x}_n(t)\big) + \big(f_n(t), \dot{x}_n(t)\big) = 0$$

a.e. on T. From [13: p. 357] we know that

$$\big(v_n(t), \dot{x}_n(t)\big) = \frac{d}{dt}\varphi(x_n(t)) \qquad \text{a.e. on } T.$$

Therefore

$$\begin{aligned}&\|\dot{x}_n(t)\|^2 + \tfrac{d}{dt}\varphi(x_n(t)) + \big(x_n(t) + p_r(x_n(t)), \dot{x}_n(t)\big) + \big(f_n(t), \dot{x}_n(t)\big) = 0 \text{ a.e. on } T\\ \Longrightarrow\quad &\|\dot{x}_n\|_2^2 = \int_0^b \big(p_r(x_n(t)), \dot{x}_n(t)\big)dt - \int_0^b \big(f_n(t), \dot{x}_n(t)\big)dt\\ &\big(\text{since } x_n(0) = x_n(b) \text{ and } \varphi(x_n(0)) = \varphi(x_n(b))\big)\\ \Longrightarrow\quad &\|\dot{x}_n\|_2 \leq \sqrt{b}\,r + \|c\|_2 = M_1.\end{aligned}$$

Then for all $n \geq 1$ and all $s,t \in T$ with $s < t$

$$\|x_n(t) - x_n(s)\| \leq \int_s^t \|\dot{x}_n(\tau)\|\,d\tau \leq M_1\sqrt{t-s}$$

from which there follows that $\{x_n\}_{n\geq 1} \subset C(T,H)$ is equicontinuous.

Also, let $\{S(t)\}_{t\in T}$ be the nonlinear semigroup of contractions generated by the maximal monotone operator $\partial\varphi$. From [13: p. 408] we know that, for all $n \geq 1, t \in [0,b)$ and $\lambda > 0$ with $t + \lambda \leq b$,

$$\begin{aligned}&\big\|J_\lambda(x_n(t)) - x_n(t)\big\|\\ &\leq \tfrac{4}{\lambda}\int_0^\lambda \big\|S(\tau)x_n(t) - x_n(t)\big\|d\tau\\ &\leq \tfrac{4}{\lambda}\int_0^\lambda \big\|S(\tau)x_n(t) - x_n(t+\tau)\big\|d\tau + \tfrac{4}{\lambda}\int_0^\lambda \big\|x_n(t+\tau) - x_n(t)\big\|d\tau.\end{aligned}$$

From [13: p. 408] we also have

$$\|S(\tau)x_n(t) - x_n(t+\tau)\| \le \int_t^{t+\tau} c(s)\, ds.$$

Therefore exploiting also the equicontinuity of $\{x_n\}_{n\ge1}$, we see that there exists 0_t, a non-decreasing and continuous function on T such that $0_t(0) = 0$ and

$$\|J_\lambda(x_n(t)) - x_n(t)\| \le \tfrac{4}{\lambda}\lambda\, 0_t(\lambda) = 4\, 0_t(\lambda) \to 0 \qquad (\lambda \downarrow 0)$$

from which

$$\sup_{n\ge1} \|J_\lambda(x_n(t)) - x_n(t)\| \to 0 \qquad (\lambda \downarrow 0)$$

follows.

Also, if $t = b$, then

$$\|J_\lambda(x_n(b)) - x_n(b)\| = \|J_\lambda(x_n(0)) - x_n(0)\|$$

and the above argument is still valid. Because J_λ is compact (since φ is of compact type), it follows that, for all $t \in T$, $\overline{\{x_n(t)\}}_{n\ge1}$ is compact in H. So by the Arzela-Ascoli theorem $\{x_n\}_{n\ge1} \subset C(T,H)$ is relatively compact. Thus we may assume that $x_n \to x$ in $C(T,H)$. Evidently, $x \in W^{1,2}_{per}(T,H)$ and $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^2(T,H)$. Also, there exists $f \in L^2(T,H)$ such that $f_n \xrightarrow{w} f$ in $L^2(T,H)$. From [13: p. 694/Proposition vii.3.9] and the properties of the sequence $\{F_1^n\}_{n\ge1}$ we have

$$f(t) \in \overline{\mathrm{conv}}\, w\text{-}\limsup_{n\to\infty} F_1^n(t, x_n(t)) \subset F_1(t, x(t)) \;\text{ a.e. on } T$$

and thus $f \in S^2_{F_1(\cdot,x(\cdot))}$. Since

$$-\dot{x}_n - (x_n + p_r(x_n)) - f_n \in \partial\Phi(x_n) \qquad (n \ge 1)$$

we have

$$-\dot{x} - (x + p_r(x)) - f \in \partial\Phi(x)$$

and so $x \in W^{1,2}_{per}(T,H)$ is a strong solution of problem (18). As already observed it follows that x is a strong solution of problem (1). Finally, from the above argument it is clear that the set of solutions of problem (1) is compact in $C(T,H)$ ∎

5. Examples

In this section we present three examples illustrating the applicability of our work.

(a) We start with a nonlinear parabolic variational inequality with discontinuous forcing term. So let $Z \subset \mathbb{R}^N$ be a bounded domain with C^1-boundary Γ. We consider the parabolic variational inequality

$$\left.\begin{aligned} &\frac{\partial x}{\partial t} - \operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) + \beta(x(t,z)) \ni f(t,z,x(t,z)) \\ &x|_{T\times\Gamma} = 0,\ x(0,z) = x(b,z) \text{ a.e. on } Z \end{aligned}\right\} \tag{21}$$

where $2 \le p < \infty$. The right-hand side term $f(t,z,x)$ is discontinuous in $x \in \mathbb{R}$. So following Chang [4], to obtain an existence theorem of problem (21) we pass to a multi-valued forcing term by, roughly speaking, filling in the jumps at the discontinuity points of $f(t,z,\cdot)$. To this end we introduce

$$f_1(t,z,x) = \liminf_{x'\to x} f(t,z,x')$$
$$f_2(t,z,x) = \limsup_{x'\to x} f(t,z,x').$$

Then instead of (21) we consider the prob

$$\left.\begin{aligned} &\frac{\partial x}{\partial t} - \operatorname{div}(\|Dx(z)\|^{p-2}Dx(z) + \beta(x(t,z)) \\ &\qquad - \big[f_1(t,z,x(t,z)), f_2(t,z,x(t,z))\big] \ni 0 \\ &x|_{T\times\Gamma} = 0,\ x(0,z) = x(b,z) \text{ a.e. on } Z \end{aligned}\right\} \tag{22}$$

with $2 \le p < \infty$. We solve this new problem. The hypotheses of the data are the following ones:

H(β)$_1$ $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ is a maximal monotone map with $0 \in \beta(0)$ (hence $\beta = \partial j$ with $j \in \Gamma_0(\mathbb{R})$).

H(f)$_1$ $f : T \times Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that $|f(t,z,x)| \le c_1(t,z) + c_2(t,z)|x|$ a.e. on $T \times Z$ with $c_1, c_2 \in L^2(T \times Z)_+$, f_1, f_2 are both jointly measurable and, for almost all $(t,z) \in T \times Z$ and all $x \in \mathbb{R}$, $f(t,z,x)x \ge c_3(z)|x|^2 - c_4(t,z)$ with $c_3 \in L^\infty(Z)$ and $c_4 \in L^\infty(T \times Z)$.

Let $H = L^2(Z)$ and

$$\varphi(x) = \begin{cases} \frac{1}{p}\|Dx\|_p^p + \int_Z j(x(z))\,dz & \text{if } x \in W^{1,2}(Z, j(x(\cdot)) \in L^1(Z) \\ +\infty & \text{otherwise.} \end{cases}$$

Evidently, $\varphi \in \Gamma_0(H)$ (see [20: p. 194]) and $\partial\varphi(x) = -\operatorname{div}(\|Dx\|^{p-2}Dx) + S^2_{\beta(x(\cdot))}$ (see [20: p. 195]). Note that $0 \in \partial\varphi(0)$ and by virtue of the Sobolev embedding theorem it is of compact type. Also, set

$$-F(t,x) = \Big\{ -h \in H : f_1(t,z,x(z)) \le h(z) \le f_2(t,z,x(z)) \text{ a.e. on } Z \Big\}.$$

Using hypotheses $H(f)_2$, we can easily check that hypothesis $H(F)_2$ holds. Now rewrite (22) in the equivalent abstract evolution inclusion form (1) and apply Theorem 5 to deduce

Proposition 6. *If hypotheses $H(\beta)_1$ and $H(f)_1$ hold, then problem* (22) *has a solution $x \in C(T, L^2(Z))$ such that $\frac{\partial x}{\partial t} \in L^2(T \times Z)$.*

(b) We consider a semilinear parabolic control system with a priori feedback and non-homogeneous, multi-valued Neumann boundary conditions. So $Z \subset \mathbb{R}^N$ is as before and, for $x \in W^{1,2}(Z, \mathbb{R}^N)$, $Lx = (\Delta x_k)_{k=1}^N$ with $\hat{x} = (x_k)_{k=1}^N$. We consider the problem

$$\left.\begin{aligned} &\frac{\partial x}{\partial t} - Lx(t,z) = f(t,z,x(t,z),u(t,z)) \\ &x|_{T\times\Gamma} = 0, \ x(0,z) = x(b,z) \text{ a.e. on } Z \\ &u(t,z) \in U(t,z,x(t,z)) \text{ a.e. on } T \times Z \end{aligned}\right\}. \tag{23}$$

The hypotheses on the data are the following ones:

H(β)₂ $\beta = \partial j$ with $j \in \Gamma_0(\mathbb{R}^N)$ and $j(0) = \inf_{\mathbb{R}^N} j \geq 0$.

H(f)₂ $f : T \times Z \times \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ is a function such that the following conditions are satisfied:

- **(i)** For all $(x,u) \in \mathbb{R} \times \mathbb{R}^m$, $(t,z) \to f(t,z,x,u)$ is measurable.
- **(ii)** For all $(t,z) \in T \times Z$, $(x,u) \to f(t,z,x,u)$ is continuous.
- **(iii)** For all $r > 0$ there exist $c_{1r}, c_{2r} \in L^2(T \times Z)$ such that $\|f(t,z,x,u)\| \leq c_{1r}(t,z) + c_{2r}(t,z)\|x\|$ for a.a. $(t,z) \in T \times Z$, all $x \in \mathbb{R}$ and all $\|u\| \leq r$.
- **(iv)** For a.a. $(t,z) \in T \times Z$, all $x \in \mathbb{R}^N$ and all $u \in \mathbb{R}^m$ with $\|u\| \leq r$, $\big(f(t,z,x,u),x\big)_{\mathbb{R}^N} \geq c_3\|x\|^2 - c_4(t,z)$ with $c_3 > 0$ and $c_4 \in L^1(T \times Z)$.

H(U) $U : T \times Z \times \mathbb{R}^N \to P_k(\mathbb{R}^m)$ is measurable, for almost all $(t,z) \in T \times Z$, $U(t,z,\cdot)$ is lower semicontinuous and, for a.a. $(t,z) \in T \times Z$, all $x \in \mathbb{R}^N$ and all $u \in U(t,z,x)$, $\|u\| \leq M$.

Let $H = L^2(Z, \mathbb{R}^N)$ and let $\varphi : H \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be defined by

$$\varphi(x) = \begin{cases} \frac{1}{p}\|Dx\|_2^2 + \int_\Gamma j(x(z))\,d\sigma & \text{if } x \in W^{1,2}(Z,\mathbb{R}^N), \ j(x(\cdot)) \in L^1(\Gamma) \\ +\infty & \text{otherwise.} \end{cases}$$

From [3: p. 63] we know that $\varphi \in \Gamma_0(H)$ and $\partial\varphi(x) = -Lx$ with domain

$$D(\partial\varphi) = \Big\{x \in W^{2,2}(Z,\mathbb{R}^N) : -\frac{\partial x}{\partial n} \in \beta(x(z)) \text{ a.e. on } \Gamma\Big\}.$$

Set $\hat{f}(t,z,x) = f(t,z,x,U(t,z,x))$ and $F(t,x) = S^2_{\hat{f}(t,\cdot,x(\cdot))}$ for all $(t,x) \in T \times H$. Using hypotheses $H(f)_2$ and $H(U)$ it is routine to check that hypotheses $H(F)_1$ hold. So we can apply Theorem 4 and deduce

Proposition 7. *If hypotheses $H(\beta)_2, H(f)_2$ and $H(U)$ hold, then problem* (23) *has a solution $x \in C(T, L^2(Z, \mathbb{R}^N))$ with $\frac{\partial x}{\partial t} \in L^2(T \times Z, \mathbb{R}^N)$.*

(c) Our formulation also incorporates differential variational inclusions

$$\left.\begin{aligned} &-\dot{x}(t) \in N_K(x(t)) + F(t,x(t)) \ \text{ a.e. on } T \\ &x(0) = x(b) \end{aligned}\right\}. \tag{24}$$

Herein $N_K(x)$ is the normal cone to K at $x \in K$, with $K \in P_{fc}(\mathbf{R}^N)$. Problems like (24) arise in theoretical mechanics and economics (see [14]). In fact, (24) is equivalent to the projected system

$$\left.\begin{aligned} & -\dot{x}(t) \in \operatorname{proj}\big(-F(t,x(t));\, T_K(x(t))\big) \text{ a.e. on } T \\ & x(0) = x(b) \end{aligned}\right\}. \tag{25}$$

Here $T_K(x)$ is the tangent cone to K at x. We know $N_K(x)^- = T_K(x)$. Inclusions like (25) arise in the study of systems with constraints (see, for example, [15]). The hypotheses on $F(t,x)$ are:

H(F)$_3$ $F : T \times \mathbf{R}^N \to P_k(\mathbf{R}^N)$ is a multifunction such that the following conditions are satisfied:

- **(i)** $(t,x) \to F(t,x)$ is graph measurable.
- **(ii)** For a.a. $t \in T$, $x \to F(t,x)$ is lower semicontinuous.
- **(iii)** For a.a. $t \in T$, all $x \in \mathbf{R}^N$ and all $v \in F(t,x)$, $\|v\| \le c_1(t) + c_2(t)\|x\|$ with $c_1, c_2 \in L^2(T)_+$.
- **(iv)** For a.a. $t \in T$, all $x \in \mathbf{R}^N$, all $w \in N_K(x)$ and all $v \in F(t,x)$, $(w+v,x)_{\mathbf{R}^N} \ge c_3\|x\|^2 - c_4(t)$ with $c_3 > 0$ and $c_4 \in L^1(T)_+$.

Using Theorem 4 with

$$\varphi(x) = \delta_K(x) = \begin{cases} 0 & \text{if } x \in K \\ +\infty & \text{otherwise} \end{cases}$$

we obtain

Proposition 8. *If hypotheses $H(F)_3$ hold and $K \subset \mathbf{R}^N$ is non-empty, closed convex with $0 \in K$, then problem (24) (equivalently problem (25)) has a solution $x \in W^{1,2}(T,\mathbf{R}^N)$.*

In particular, if $K = \{x \in \mathbf{R}^N : 0 \le x \le \xi\}$ with $\xi \in \mathbf{R}^N_+$, we obtain

$$\left.\begin{aligned} & -\dot{x}(t) \in F(t,x(t)) \text{ a.e. on } \{t \in T : 0 < x(t) < \xi\} \\ & -\dot{x}(t) \in F(t,x(t)) - \mathbb{R}^N_+ \text{ a.e. on } \{t \in T : x(t) = 0\} \\ & -\dot{x}(t) \in F(t,x(t)) + \mathbb{R}^N_+ \text{ a.e. on } \{t \in T : x(t) = b\} \\ & x \in W^{1,2}(T,\mathbb{R}^N),\, x(0) = x(b) \\ & 0 \le x(t) \le \xi \text{ for all } t \in T \end{aligned}\right\}.$$

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