Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 4, 985–1004

Bifurcation of Homoclinic Solutions for Hamiltonian Systems

R. Joosten

Abstract. We consider the Hamiltonian system

$$Ju'(x) + Mu(x) - \nabla_u F(x, u(x)) = \lambda u(x).$$

Using variational methods obtained by Stuart on the one hand and by Giacomoni and Jeanjean on the other, we get bifurcation results for homoclinic solutions by imposing conditions on the function F. We study both the case where F is defined globally with respect to u and the case where F is defined locally only.

Keywords: *Hamiltonian systems, homoclinic solutions, bifurcation*

AMS subject classification: 34C23, 34C37, 47J30, 47N20, 70H05

1. Introduction

1.1 Presentation of the problem. The paper continues our study from [10] of homoclinic solutions of first order Hamiltonian systems

$$Ju'(x) + \nabla_s H(\lambda, x, u(x)) = 0$$

where J is a $2N \times 2N$ real matrix such that

$$J = -J^T = -J^{-1}$$

and the Hamiltonian $H : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is defined by

$$H(\lambda, x, s) = \tfrac{1}{2}(M - \lambda I)s \cdot s - F(x, s)$$

where dot denotes the usual scalar product in $\mathbb{R}^{2N}$, M is a $2N \times 2N$ real symmetric matrix such that $\sigma(JM) \cap i\mathbb{R} = \emptyset$ and the function $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is such that $F(x, 0) \equiv 0$ and $\lim_{|s|\to 0} \frac{F(x,s)}{|s|^2} = 0$.
The Hamiltonian system can be written in the form

$$Ju'(x) + Mu(x) - \nabla_s F(x, u(x)) = \lambda u(x).$$

R. Joosten: Inst. de math., Fac. des sci. de base, EPFL, CH-1015 Lausanne, Switzerland
robert.joosten@a3.epfl.ch

ISSN 0232-2064 / $ 2.50

We use the notations $f(x,s) := \nabla_s F(x,s)$ and $f'(x,s) := \nabla_s f(x,s)$.

In [10] we have obtained existence results for homoclinic solutions and we refer the reader to the introduction of that paper for a discussion of our approach and the setting we have chosen. Now, our goal is to obtain bifurcation results for homoclinic solutions for a subset of such systems by imposing conditions on the function F. By a *homoclinic solution* we mean a solution $u(x)$ such that $\lim_{x\to\pm\infty} u(x) = 0$. Stuart has already worked on this question (see [13]). He seems to be the only other author to introduce the real parameter λ in the Hamiltonian system and to have got bifurcation results for Hamiltonian systems of this type by variational methods. But we get more general bifurcation results. Indeed, F may have different behaviours in s at 0 and at infinity which is not the case in [13]. This generates important complications in the proofs.

Since $\lim_{|s|\to 0} \frac{F(x,s)}{|s|^2} = 0$, we have $\nabla_s F(x,0) = 0$ and the system admits the axis of trivial solutions $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$. To obtain non-trivial homoclinic solutions, we seek solutions

$$(\lambda, u) \in \mathbb{R} \times H^1(\mathbb{R}, \mathbb{R}^{2N}) \setminus \{0\}.$$

Indeed, the space $H^1(\mathbb{R})$ has the basic property $\lim_{|x|\to\infty} u(x) = 0$ for all $u \in H^1(\mathbb{R})$. We recall that the space $H^1(\mathbb{R}, \mathbb{R}^{2N})$ will be identified to $[H^1(\mathbb{R})]^{2N}$.

1.2 Our bifurcation theorems. Before stating the first theorem, let us introduce some notions developped in [13]. We will define the operator S and its spectral gap.

We consider $[L^2(\mathbb{R})]^{2N} = L^2(\mathbb{R}, \mathbb{R}^{2N})$ with the scalar product

$$(u,v)_{[L^2]^{2N}} = \sum_{i=1}^{2N} \int_{\mathbb{R}} u_i(x) v_i(x)\, dx.$$

This scalar product will often be denoted by $(\cdot,\cdot)$. Let

$$S : [H^1(\mathbb{R})]^{2N} \subset [L^2(\mathbb{R})]^{2N} \to [L^2(\mathbb{R})]^{2N}, \qquad Su = Ju' + Mu.$$

Knowing that $J^T = J^{-1} = -J$ and $M^T = M$, it is easy to show that S is self-adjoint (see [9: Lemma 1]). As S is self-adjoint, its spectrum $\sigma(S)$ lies on the real line. By [13: Corollary 10.2] we have $\inf \sigma(S) = -\infty$ and $\sup \sigma(S) = \infty$ (S is unbounded). It is possible to show that S has no eigenvalues.

Again by [13: Corollary 10.2] the hypothesis $\sigma(JM) \cap i\mathbb{R} = \emptyset$ is equivalent to $0 \notin \sigma(S)$. We get as easy corollary of this result the equivalence

$$\lambda \in \sigma(S) \quad \Longleftrightarrow \quad \sigma(J(M - \lambda I)) \cap i\mathbb{R} \neq \emptyset$$

(see [9: Lemma 2]).

We denote by $\rho(S) = \mathbb{R} \setminus \sigma(S)$ the regular values of S. As $\rho(S)$ is open, there exists a maximal open interval (a,b) in $\rho(S)$ containing 0 with $-\infty < a < 0 < b < \infty$ and a and b being part of the spectrum. This interval is called a *spectral gap*. The upper bound b will be a bifurcation point under appropriate hypotheses on F.

We introduce the following hypotheses on F:

(F1) F is of Carathéodory type (in the C^2 sense), i.e. $F(\cdot, s) : \mathbb{R} \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}^{2N}$ and, for a.e. $x \in \mathbb{R}$, $F(x, \cdot) \in C^2(\mathbb{R}^{2N}, \mathbb{R})$, $F(x, 0) = 0$, $f(x, 0) = 0$ and $|f'(x, s)| \le a_1|s|^{r_1-1} + a_2|s|^{r_2-1}$ $(s \in \mathbb{R}^{2N})$ where $a_1, a_2 > 0$ and $r_2 \ge r_1 > 1$.

(F2) $F(x, \cdot)$ is convex on $\mathbb{R}^{2N}$ for a.e. $x \in \mathbb{R}$.

(F3) $F(x, s) \ge 0$ for a.e. $x \in \mathbb{R}$ and all $s \in \mathbb{R}^{2N}$.

(F4) $|f(x, s)| \le C_1 F(x, s)^{t_1} + C_2 F(x, s)^{t_2}$ for a.e. $x \in \mathbb{R}$ and all $s \in \mathbb{R}^{2N}$ where $C_1, C_2 > 0$ and $t_1, t_2 \in [\frac{1}{2}, 1)$.

(F5) $pF(x, s) \le f(x, s) \cdot s$ for a.e. $x \in \mathbb{R}$ and all $s \in \mathbb{R}^{2N}$ where $p > 2$.

(F6) $f(x, s) \cdot s \le qF(x, s)$ for a.e. $x \in \mathbb{R}$ and all $s \in \mathbb{R}^{2N}$ where $q > 2$.

(F7) The set $\{x \in \mathbb{R} : F(x, s) = 0 \text{ for some } s \ne 0\}$ is of measure zero.

(F8) There exists $d > 0$ such that the set $\{x \in \mathbb{R} : F(x, s) \ne 0 \text{ if } 0 < |s| \le d\}$ is not of measure zero.

(F9) $F(x, ts) \ge c\, t^{\tilde{p}} F(x, s)$ for a.e. $x \in \mathbb{R}$ if $t \in [0, \varepsilon]$ and $|s| \le \Delta$ where $c, \varepsilon, \Delta > 0$ and $\tilde{p} > 2$.

Remark. Hypothesis (F7) is required in most previous works on the existence of homoclinic solutions by variational methods [2- 5, 8, 12, 14]. An exception is [10] where existence theorems are obtained in the same context as we now use to study bifurcation. Let us emphasize that hypothesis (F7) will not be used here. Our use of test functions means that the much weaker condition (F8) is sufficient.

The number λ_0 is said to be a *bifurcation point on the left* for the system

$$Ju' + Mu - f(x, u) = \lambda u \qquad (u \in [H^1(\mathbb{R})]^{2N})$$

if there exists

$$\{(\lambda_n, u_n)\} \subset \left\{(\lambda, u) \in \mathbb{R} \times [H^1(\mathbb{R})]^{2N} : \ u \ne 0 \text{ and } Ju' + Mu - f(x, u) = \lambda u\right\}$$

such that $\lambda_n < \lambda_0$ for all n, $\lambda_n \to \lambda_0$ and $\lim_{n\to\infty} \|u_n\|_{H^1} = 0$. Moreover, the bifurcation point is said to be of *order* γ if $\lim_{n\to\infty} \frac{\|u_n\|_{H^1}}{(\lambda_0-\lambda_n)^\gamma} = 0$.

We can establish now a first bifurcation theorem. Recall that the following term $\tilde{p}$ occurs in hypothesis (F9).

Theorem 1. *Let us consider* $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ *with hypotheses* (F1) - (F6), (F8), (F9) *and let us consider the Hamiltonian system* $Ju' + Mu - f(x, u) = \lambda u$. *Then:*

If

(1.1) $|f(x, s)| \le a(x)(|s|^{q_1} + |s|^{q_2})$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in \mathbb{R}^{2N}$ *where* $q_2 \ge q_1 > 1$ *and a is measurable such that* $\lim_{|x|\to\infty} a(x) = 0$

(1.2) $\tilde{p} < 4$,

then b is a bifurcation point (on the left) of order γ, for any $\gamma < \frac{1}{q-2}(1-\frac{\tilde{p}}{4})$, for the Hamiltonian system.

If

(2.1) $F(x+1,s) = F(x,s)$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in \mathbb{R}^{2N}$

(2.2) $\tilde{p} < 6$

(2.3) $f(x,s) \cdot s - 2F(x,s)$ *is convex in* s *for a.e.* $x \in \mathbb{R}$,

then b is a bifurcation point (on the left) of order γ, for any $\gamma < \frac{1}{q-2}(1-\frac{\tilde{p}-2}{4})$, for the Hamiltonian system.

If

(3.1) $|f(x,s)| \le a(x)(|s|^{q_1} + |s|^{q_2})$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in \mathbb{R}^{2N}$ *where* $q_2 \ge q_1 > 1$ *and* a *is measurable such that* $\lim_{|x|\to\infty} a(x) = 0$

(3.2) *there exists* $\Delta' \in (0,\Delta]$ *such that* $\min_{|s|=\Delta'} F(x,s) \ge \frac{C}{|x|^\alpha}$ *for a.e.* $|x| \ge M$ *where* $\alpha \in (0,1]$ *and* $M, C > 0$

(3.3) $\tilde{p} < 4 + 2(1-\alpha)$,

then b is a bifurcation point (on the left) of order γ, for any $\gamma < \frac{1}{q-2}(1-\frac{\tilde{p}-2(1-\alpha)}{4})$, for the Hamiltonian system.

We state now a bifurcation theorem following from the result of Giacomoni-Jeanjean [7]. The convexity of $F(x,\cdot)$ and hypothesis (F5) are no longer required.

Theorem 2. *Let us consider* $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ *with hypotheses* (F1), (F3), (F4), (F6), (F8), (F9). *Moreover, let us suppose that*

(1.1) $|f(x,s)| \le a(x)(|s|^{q_1} + |s|^{q_2})$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in \mathbb{R}^{2N}$ *where* $q_2 \ge q_1 > 1$ *and* a *is measurable such that* $\lim_{|x|\to\infty} a(x) = 0$

(1.2) $\tilde{p} < 4$

or

(2.1) $F(x+1,s) = F(x,s)$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in \mathbb{R}^{2N}$

(2.2) $\tilde{p} < 6$.

Then b is a bifurcation point (on the left) for the Hamiltonian system $Ju' + Mu - f(x,u) = \lambda u$.

Remark. In this theorem, we have no information about the order of the bifurcation.

Up to now, most of the hypotheses on F where global with respect to s (hypotheses (F1) - (F7)). Now, we will give a bifurcation theorem where these hypotheses are relaxed.

Instead of F, we work with a function G which is defined locally only. Let $R > 0$. We consider $G : \mathbb{R} \times B(0,R) \to \mathbb{R}$ where $B(0,R)$ is the open ball of radius R centered at 0. Let us introduce some hypotheses on this function. The numerotation of these hypotheses corresponds to that of the hypotheses on F. The letter g will denote the gradient of G.

(G1) G is of Carathéodory type (in the C^2 sense), i.e. $G(\cdot, s) : \mathbb{R} \to \mathbb{R}$ is measurable for all $s \in B(0,R)$ and, for a.e. $x \in \mathbb{R}$, $G(x,\cdot) \in C^2(B(0,R),\mathbb{R})$, $G(x,0) = 0$, $g(x,0) = 0$, $|g'(x,s)| \leq a|s|^{r_1-1}$ $(s \in B(0,R))$ where $a > 0$ and $r_1 > 1$.

(G3) $G(x,s) \geq 0$ for a.e. $x \in \mathbb{R}$ and all $s \in B(0,R)$.

(G4) $|g(x,s)| \leq C_1 G(x,s)^{t_1} + C_2 G(x,s)^{t_2}$ for a.e. $x \in \mathbb{R}$ and all $s \in B(0,R)$ where $C_1, C_2 > 0$ and $t_1, t_2 \in [\frac{1}{2}, 1)$.

(G6) $g(x,s) \cdot s \leq qG(x,s)$ for a.e. $x \in \mathbb{R}$ and all $s \in B(0,R)$ where $q > 2$.

(G8) There exists $d \in (0,R)$ such that the set $\{x \in \mathbb{R} : G(x,s) \neq 0 \text{ if } 0 < |s| \leq d\}$ is not of measure zero.

(G9) $G(x,ts) \geq ct^{\tilde{p}} G(x,s)$ for a.e. $x \in \mathbb{R}$ if $t \in [0,\varepsilon]$ and $|s| \leq \Delta$ where $c > 0$, $0 < \Delta < R$, $\varepsilon > 0$ is such that $\Delta\varepsilon < R$ and $\tilde{p} > 2$.

Remarks.

1) Since we work with $s \in B(0,R)$, it would have been useless to ask $|g'(x,s)| \leq a_1|s|^{r_1-1} + a_2|s|^{r_2-1}$ in hypothesis (G1).

2) We remark that hypotheses (G8) and (G9) were already local for hypotheses (F8) and (F9).

We can state now the bifurcation theorem under local conditions.

Theorem 3. *Let us consider* $G : \mathbb{R} \times B(0,R) \to \mathbb{R}$ *with hypotheses* (G1), (G3), (G4), (G6), (G8), (G9). *Moreover, let us suppose*

(1.1) $|g(x,s)| \leq a(x)|s|^{q_1}$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in B(0,R)$ *where* $q_1 > 1$ *and* a *is measurable with* $\lim_{|x|\to\infty} a(x) = 0$

(1.2) $\tilde{p} < 4$

or

(2.1) $G(x+1,s) = G(x,s)$ *for a.e.* $x \in \mathbb{R}$ *and all* $s \in B(0,R)$

(2.2) $\tilde{p} < 6$.

Then b *is a bifurcation point* (*on the left*) *of the Hamiltonian system*

$$Ju' + Mu - g(x,u) = \lambda u \qquad (u \in [H^1(\mathbb{R})]^{2N})$$

and $\|u\|_\infty < R$.

Remark. Since $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, the notation $g(x,u(x))$ makes sense for $u \in [H^1(\mathbb{R})]^{2N}$ such that $\|u\|_{H^1}$ is small enough.

1.3 Plan of the article. In Section 2 we give examples of Hamiltonian systems satisfying the hypotheses of the theorems. In Section 3 we transform the Hamiltonian system into an equivalent functional equation about which there are existence results in the monograph of Stuart [13] and in the article of Giacomoni and Jeanjean [7]. Up to our knowledge, we are the first to apply the abstract results of [7] in a particular case.

We first introduce the space $H = D(|S|^{1/2})$ with scalar product

$$\langle u, v\rangle_H = (u, v)_{L^2} + \big(|S|^{1/2}u, |S|^{1/2}v\big)_{L^2}.$$

This space coincides with the fractional Sobolev space $[H^{1/2}(\mathbb{R})]^{2N}$ and their norms are equivalent. We introduce the self-adjoint operators $A, L \in \mathcal{B}(H)$ such that

$$\big((S - \lambda I)u, v\big)_{L^2} = \langle (A - \lambda L)u, v\rangle_H$$

for $u \in [H^1(\mathbb{R})]^{2N}$ and $v \in H$. Setting

$$\sigma(A, L) = \Big\{\lambda \in \mathbb{R} \Big| A - \lambda L : H \to H \text{ is not an isomorphism}\Big\}$$

we have that $\sigma(A, L)$ coincides with the spectrum of S and the spectral gaps coincide as well.

We introduce the functional

$$\varphi : H \to \mathbb{R}, \qquad \varphi(u) = \int F(x, u(x))\, dx \quad (u \in H)$$

and have $\lim_{\|u\|\to 0} \frac{\varphi(u)}{\|u\|^2} = 0$ and that φ is of class C^2. Further, we introduce the operator

$$N : H \to H, \qquad \langle N(u), v\rangle_H = \varphi'(u)v \quad (u, v \in H).$$

This operator is bounded, of class C^1 and weakly sequentially continuous. We get

$$(f(x, u), v)_{L^2} = \langle N(u), v\rangle_H \qquad (u, v \in H).$$

Altogether

$$\big(Ju' + Mu - \lambda u - f(x, u), v\big)_{L^2} = \langle (A - \lambda L)u - N(u), v\rangle_H$$

for all $u \in [H^1(\mathbb{R})]^{2N}$ and $v \in H$. The functional equation $(A - \lambda L)u - N(u) = 0$ is equivalent to the Hamiltonian system. Indeed, λ_0 is a bifurcation point of the functional equation if and only if it is a bifurcation point of the Hamiltonian system.

In Section 4 we state two bifurcation theorems contained in [13] and [7]. Under certain hypotheses on A, L and φ, the first theorem states that b is a bifurcation point of a certain order for the functional equation $(A - \lambda L)u - N(u) = 0$. In [7] we considered the functional ϕ defined on a ball around 0 instead of the functional φ. The operator N given by $\langle N(u), v\rangle = \phi'(u)v \quad (u, v \in H)$ is defined on a ball only. Under weaker hypotheses on A, L and ϕ, the second theorem states that b is a bifurcation point for the functional equation $(A - \lambda L)u - N(u) = 0$.

In Section 5 we give sufficient conditions for satisfying the hypotheses of the two theorems of Section 4. In Section 6, using what we have done in Section 5 we prove Theorems 1 and 2. In Section 7 we prove the bifurcation theorem under local conditions (Theorem 3). Starting from the function G defined locally, we construct a function F defined globally, satisfying the hypotheses of Theorem 2 such that F and G coincide for s close to 0. The bifurcation theorem under local conditions is an easy consequence of Theorem 2.

2. Examples

Let us give examples to illustrate the bifurcation theorems. We consider $J = \begin{pmatrix} 0 & -I \\ I & o \end{pmatrix}$ and $M = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}$ where $B^T = B$, $\det B \neq 0$ and

$$F(x,s) = a_1(x)|s|^{p_1} + a_2(x)|s|^{p_2}$$

with $2 < p_1 \leq p_2$ where $a_1, a_2 \in L^\infty(\mathbb{R}, \mathbb{R})$ and $a_1, a_2 \geq 0$ for a.e. $x \in \mathbb{R}$ and $a_1 + a_2 \not\equiv 0$. Clearly, $J = -J^T = -J^{-1}$ and $M = M^T$. As $JM = \begin{pmatrix} -B & 0 \\ 0 & B \end{pmatrix}$ is symmetric, there are only real eigenvalues. We get $b = \min\{|\lambda| : \lambda \in \sigma(B)\}$ and $a = -b$ (see [13: Lemma 10.12]). On the other hand, as $\det B \neq 0$, 0 is not an eigenvalue of JM. Thus $\sigma(JM) \cap i\mathbb{R} = \emptyset$. The function F satisfies the hypotheses of Theorem 1. The exponents appearing herein are $\tilde{p} = q = p_2, p = p_1, r_i = p_i - 1, t_i = \frac{p_i - 1}{p_i}$ $(i = 1, 2)$. Now,

- taking $F(x,s) = \frac{1}{\cosh x}(|s|^{\frac{5}{2}} + |s|^3)$, b is a bifurcation point (since $p_2 < 4$) being of any order $\gamma < \frac{1}{4}$,
- taking $F(x,s) = \sin^2(2\pi x)(|s|^{\frac{5}{2}} + |s|^5)$, b is a bifurcation point (since $p_2 < 6$) being of any order $\gamma < \frac{1}{12}$, amd finally
- taking $F(x,s) = \frac{1}{1+|x|^{\frac{1}{4}}}(|s|^{\frac{5}{2}} + |s|^5)$, b is a bifurcation point (since $p_2 < \frac{11}{2}$) being of any order $\gamma < \frac{1}{24}$.

We define the function

$$G(x,s) : \mathbb{R} \times B(0,1) \to \mathbb{R}, \qquad G(x,s) = |s|^3 e^{|s|}.$$

This function satisfies all the hypotheses of the bifurcation theorem under local conditions (periodic case). Indeed, the exponents can be chosen as $r_1 = 2, t_1 = t_2 = \frac{2}{3}, q = 4, \tilde{p} = 3$. On the other hand, $F(x,s) := |s|^3 e^{|s|}$ defined on $\mathbb{R} \times \mathbb{R}^{2N}$ does not satisfy the hypotheses of the global bifurcation theorems.

3. Transforming the Hamiltonian system into a functional equation

Now we will turn to the formalism introduced in [13]. We would like to get from the unbounded and densely defined self-adjoint operator S to a bounded and everywhere defined self-adjoint operator. This new operator will be defined on another Hilbert space that will be introduced below. Our goal is to transform the Hamiltonian system into an equivalent functional equation about which there are existence results.

3.1 Introduction of the space H. To the self-adjoint operator

$$S : [H^1(\mathbb{R})]^{2N} \subset [L^2(\mathbb{R})]^{2N} \to [L^2(\mathbb{R})]^{2N}, \qquad Su = Ju' + Mu$$

with $\sigma(JM) \cap i\mathbb{R} = \emptyset$ we can associate

$$|S|^{1/2} : D(|S|^{1/2}) \subset [L^2(\mathbb{R})]^{2N} \to [L^2(\mathbb{R})]^{2N}$$

by means of the spectral resolution of S (see [13: p. 30]). Let us set $H = D(|S|^{1/2})$. With the scalar product $\langle u, v\rangle_H = (u,v)_{L^2} + (|S|^{1/2}u, |S|^{1/2}v)_{L^2}$, H is a Hilbert space. The induced norm $\|u\|_H = \langle u, u\rangle_H^{1/2}$ is the graph norm of $|S|^{1/2}$. In this norm, $D(S) = [H^1(\mathbb{R})]^{2N}$ is dense in $D(|S|^{1/2}) = H$. Sometimes, H is called the form domain of S and $(H, \langle\cdot,\cdot\rangle)$ the form space (see [6]). Often we will write $\|\cdot\|$ instead of $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_H$.

Using the hypotheses on J and M, it has been shown in [13: Lemma 10.3] that the linear space H corresponds to the fractional Sobolev space $[H^{1/2}(\mathbb{R})]^{2N}$ and that the norm of H is equivalent to the usual norm of $[H^{1/2}(\mathbb{R})]^{2N}$, i.e. $\|u\|_{H^{1/2}} = \left(\int_{\mathbb{R}} \sqrt{1+\xi^2}\,|\hat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$. Here are some important properties of this space:

1. $H^1(\mathbb{R}) \hookrightarrow H^{1/2}(\mathbb{R})$
2. $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for all $q \in [2, \infty)$
3. $H^{\frac{1}{2}}(\mathbb{R}) \not\subset L^\infty(\mathbb{R})$
4. $H^{\frac{1}{2}}(\mathbb{R}) \not\subset C(\mathbb{R})$ (in the sense that there exists $u \in H^{\frac{1}{2}}(\mathbb{R})$ non-equivalent to a continuous function).

Assertions 1 and 2 can be found in any book about Sobolev spaces (see, for instance, [1]). In [9] an example is given showing assertions 3 and 4.

3.2 The linear part. In this subsection we will transform the linear part of our Hamiltonian system. Using the Riesz lemma, it is shown in [13: p. 31] that there exists a unique operator $A \in \mathcal{B}(H,H)$ such that $(Su, v)_{L_2} = \langle Au, v\rangle_H$ for all $u \in [H^1(\mathbb{R})]^{2N}$ and $v \in H$. A is self-adjoint and one can show that $0 \notin \sigma(A)$ [13: p. 32].

On the other hand, there exists a unique operator $L \in \mathcal{B}(H,H)$ such that $(u,v)_{L_2} = \langle Lu, v\rangle_H$ for all $u, v \in H$. Clearly, L is self-adjoint and strictly positive, i.e. $\langle Lu, u\rangle > 0$ for all $u \in H \setminus \{0\}$.

Combining these results, we get

$$\big((S-\lambda I)u, v\big)_{L^2} = \langle (A-\lambda L)u, v\rangle_H$$

for all $u \in [H^1(\mathbb{R})]^{2N}$ and $v \in H$. As $\|v\|_{L^2} \le \|v\|_H$ for all $v \in H$, $\|(A-\lambda L)u\|_H \le \|(S-\lambda I)u\|_{L^2}$ for all $u \in [H^1(\mathbb{R})]^{2N}$ (see [13: p. 33]). Denoting

$$\rho(A,L) = \Big\{\lambda \in \mathbb{R} \Big|\, A-\lambda L : H \to H \text{ is an isomorphism}\Big\}$$
$$\sigma(A,L) = \mathbb{R} \setminus \rho(A,L)$$

we have $\sigma(S) = \sigma(A,L)$ and consequently (a,b) is also a spectral gap of $\sigma(A,L)$. At [13: page 33] it has been proved that $\rho(A,L) \subset \rho(S)$. In [9: Proposition 4] we proved the reverse inclusion $\rho(S) \subset \rho(A,L)$.

3.3 The nonlinear part. Let us study now the nonlinear part of the Hamiltonian system. We define

$$\varphi : H \to \mathbb{R}, \qquad \varphi(u) = \int F(x, u(x))\, dx$$

and have the following properties:

Proposition 1. *Suppose that F satisfies hypothesis* (F1). *Then:*

1. φ *is well-defined.*

2. $\lim_{\|u\|\to 0} \frac{\varphi(u)}{\|u\|^2} = 0$.

3. $\varphi \in C^2(H,\mathbb{R})$.

4. $u \in H$ *implies* $f(\cdot,u(\cdot)) \in [L^2(\mathbb{R})]^{2N}$.

5. $\varphi'(u)v = \int_{\mathbb{R}} f(x,u(x)) \cdot v(x)\,dx$ *for all* $u,v \in H$.

Proof. The proof is quite long and can be found in [9: Sections 6.3 and 6.4]. It uses results about Nemyckii operators ∎

By the Riesz lemma, there exists a unique operator $N : H \to H$ such that $\langle N(u),v\rangle_H = \varphi'(u)v$ for all $u,v \in H$. Usually, we write $N = \nabla\varphi$. This operator has the following properties:

Proposition 2. *Suppose that F satisfies hypothesis* (F1). *Then:*

1. N *is bounded.*

2. $N \in C^1(H,H)$.

3. $N : H \to H$ *is weakly sequentially continuous.*

Proof. The proof can be found in [9] ∎

Remark. $N : H \to H$ is said to be *weakly sequentially continuous* if $u_n \rightharpoonup u$ implies $N(u_n) \rightharpoonup N(u)$, where $\rightharpoonup$ denotes the weak convergence in H.

3.4 Relation between the Hamiltonian system and the functional equation. Combining the preceeding results, we have

$$\big(Ju' + Mu - \lambda u - f(x,u), v\big)_{L^2} = \big\langle (A-\lambda L)u - N(u), v\big\rangle_H$$

for all $u \in [H^1(\mathbb{R})]^{2N}$ and all $v \in H$. Let us define the concept of bifurcation for the functional equation. The number λ_0 is said to be a *bifurcation point on the left* for the equation $(A-\lambda L)u - N(u) = 0 \qquad (u \in H)$ if there exists

$$\{(\lambda_n,u_n)\} \subset \Big\{(\lambda,u) \in \mathbb{R}\times H : \, u \neq 0 \;\text{ and }\; (A-\lambda L)u - N(u) = 0\Big\}$$

such that $\lambda_n < \lambda_0$ for all n, $\lambda_n \to \lambda_0$ and $\lim_{n\to\infty} \|u_n\|_H = 0$. Moreover, the bifurcation point is said to be of *order* γ if $\lim_{n\to\infty} \frac{\|u_n\|_H}{(\lambda_0-\lambda_n)^\gamma} = 0$. Using the continuous embedding $[H^1(\mathbb{R})]^{2N} \hookrightarrow H$, if λ_0 is a bifurcation point of order γ of the Hamiltonian system, then λ_0 is also a bifurcation point of the same order of the functional equation.

In fact, the opposite is true in our situation as stated in the following

Proposition 3. *Suppose F satisfies hypothesis* (F1). *If $\lambda_0 \in \mathbb{R}$ is a bifurcation point* (*on the left*) *of order $\gamma \geq 0$ of the equation*

$$(A - \lambda L)u - N(u) = 0 \qquad (u \in H),$$

then λ_0 is also a bifurcation point (*on the left*) *of the same order of the equation*

$$Ju' + Mu - f(x,u) = \lambda u \qquad (u \in [H^1(\mathbb{R})]^{2N}).$$

Remark. This result is not trivial: the definition of bifurcation point depends on the norm.

It is thus sufficient to consider the bifurcation points of the functional equation $(A - \lambda L)u - N(u) = 0$. This equation has been treated in [7, 13].

4. The bifurcation results

In this section we will present the bifurcation results for the functional equation contained in [7, 13]

4.1 The hypothesis from [13]. The author of [13] considered a Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and introduced operators A, L and a functional φ by means of the following hypotheses:

- **(H1)** $A \in \mathcal{B}(H,H)$, $A = A^*$ and $0 \notin \sigma(A)$.
- **(H2)** $L \in \mathcal{B}(H,H)$, $L = L^*$ and $\langle Lu, u \rangle > 0$ for all $u \in H \setminus \{0\}$.
- **(H3)** $\varphi \in C^2(H, \mathbb{R})$, $\lim_{\|u\| \to 0} \frac{\varphi(u)}{\|u\|^2} = 0$ and φ is convex.

Supposing $A, L : H \to H$ and $\varphi : H \to \mathbb{R}$ are given, a functional $J : \mathbb{R} \times H \to \mathbb{R}$ is defined by

$$J(\lambda, u) = \tfrac{1}{2}\langle (A - \lambda L)u, u \rangle - \varphi(u).$$

When hypotheses (H1) - (H3) are satisfied, $J \in C^2(\mathbb{R} \times H, \mathbb{R})$ and the equation $\nabla_u J(\lambda, u) = 0$ is equivalent to the equation $Au - \lambda Lu - N(u) = 0$ where $N = \nabla \varphi$. Furthermore, supposing $\varphi \in C^1(H, \mathbb{R})$, the following hypotheses are introduced:

- **(H4)** There exist $C, D > 0$ such that $\|N(u)\| \leq C + D\varphi(u)$ for all $u \in H$.
- **(H5)** There exist $\varepsilon, K > 0$ such that $\|N(u)\| \leq K\varphi(u)^{1/2}$ for all $u \in H$ such that $\varphi(u) < \varepsilon$.
- **(P)** There exist $q \geq p > 2$ such that $q\varphi(u) \geq \varphi'(u)u \geq p\varphi(u) \geq 0$ for all $u \in H$.

Supposing hypothesis (H1) is satisfied, H can be written as an orthogonal sum of closed subspaces $H = V \oplus W$ such that $A(V) \subset V$, $\langle Av, v \rangle \geq \beta \|v\|^2$ for all $v \in V$ and $\langle Aw, w \rangle \leq -\alpha \|w\|^2$ for all $w \in W$ with $\alpha, \beta > 0$. The projection of H onto V is denoted by $P : H \to H$.

When hypotheses (H1) and (H2) are true, an interval (a,b) is introduced by

$$a = \begin{cases} \sup\left\{\frac{\langle Aw,w\rangle}{\langle Lw,w\rangle} : w \in W \setminus \{0\}\right\} & \text{if } W \neq \{0\} \\ -\infty & \text{if } W = \{0\} \end{cases}$$

$$b = \begin{cases} \inf\left\{\frac{\langle Av,v\rangle}{\langle Lv,v\rangle} : v \in V \setminus \{0\}\right\} & \text{if } V \neq \{0\} \\ \infty & \text{if } V = \{0\}. \end{cases}$$

By [13: Lemma 2.1], $a < 0 < b$ and $(a,b) \subset \rho(A,L)$. Moreover, when $PL = LP$, $\{a,b\} \cap \rho(A,L) = \emptyset$. Thus, if P and L commute, (a,b) is the maximal spectral gap of $\sigma(A,L)$ containing 0.

Supposing hypotheses (H1) - (H2) are satisfied and φ is defined, we introduce now for $\delta > 0$ the

T(δ) $PL = LP$ and there exists a sequence $\{u_n\} \subset H$ such that $\|u_n\| = 1$, $\varphi(u_n) > 0$ and

$$\lim_{n\to\infty} \frac{\langle (A-bL)u_n, u_n\rangle}{\varphi(u_n)^\delta} = \lim_{n\to\infty} \frac{\|(A-bL)u_n\|^2}{\varphi(u_n)^\delta} = 0.$$

Remark. This formulation only makes sense when b is finite, that means $V \neq \{0\}$.

Before stating the bifurcation theorem, we have to introduce the notion of weak G-compacity (see [13: p. 15]). We consider $O(H)$ — the group of isometric isomorphisms of H, and a subgroup G of $O(H)$. We denote by $\theta(u) = \{Tu : T \in G\}$ the orbit of u generated by G.

Definition. The functional $K \in C^1(H,\mathbb{R})$ is said to be *weakly G-compact* if the following is satisfied:

(1) K is G-invariant, i.e. $K(Tu) = K(u)$ for all $u \in H$ and all $T \in G$)

(2) When $\{u_n\} \subset H$ is a bounded sequence such that $K(u_n) \to c \neq K(0)$ and $\|\nabla K(u_n)\| \to 0$, then there exist subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $v_{n_i} \in \theta(u_{n_i})$ such that $v_{n_i} \rightharpoonup v$ in H with $v \neq 0$ and $\nabla K(v) = 0$.

Lemma 1. *If $K \in C^1(H,\mathbb{R})$ is G-invariant, then $\|\nabla K(\cdot)\|$ is also G-invariant.*

Proof. See [13: p. 15] ∎

4.2 The bifurcation result of [13]. We can state now the bifurcation theorem contained in [13].

Theorem 4 [13: Theorem 7.2]. *Let hypotheses* (H1) - (H5) *and* (P) *be satisfied. Suppose also that condition $T(\delta)$ is satisfied for a number $\delta \geq 1$ and that either*

(i) *$N : H \to H$ is compact*

or

(ii) *there is a subgroup G of $O(H)$ such that $J(\lambda,\cdot) : H \to \mathbb{R}$ is weakly G-compact for all $\lambda \in (a,b)$ and $\psi = \langle N(\cdot),\cdot\rangle - 2\varphi : H \to \mathbb{R}$ is weakly sequentially lower semi-continuous.*

Set $\theta = \frac{2}{q-2}[1 - \frac{1}{\delta}]$ *where* q *is the constant in hypothesis* (P). *Then there is a sequence*

$$\{(\lambda_n, u_n)\} \subset \Big\{(\lambda, u) \in \mathbb{R} \times H : u \neq 0 \ \ \textit{and} \ \ Au - \lambda Lu = N(u)\Big\}$$

such that $\lambda_n < b$ *for all* $n \in \mathbb{N}$, $\lambda_n \to b$ *and* $\lim_{n\to\infty}(b - \lambda_n)^{-\theta/2}\|u_n\| = 0$, *i.e.* b *is a bifurcation point of order* $\frac{\theta}{2}$.

4.3 The hypotheses of [7]. The functional equation $(A - \lambda L)u - N(u) = 0$ has also been treated with weaker hypotheses in [7]. The bifurcation Theorem 1.1 there implies our Theorem 2. The authors considered a Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and two self-adjoint operators $A, L \in \mathcal{B}(H,H)$ which satisfy the hypothesis

(A1) $\langle Lu, u\rangle > 0 \ \ (u \in H \setminus \{0\})$, $\sigma(A) \cap \mathbb{R}^+ \neq \emptyset$, $\sigma(A) \cap \mathbb{R}^- \neq \emptyset$ and $0 \notin \sigma(A)$.

This hypothesis is almost the same as hypotheses (H1) and (H2) introduced before – it is only a little stronger because of the conditions $\sigma(A)\cap\mathbb{R}^+ \neq \emptyset$ and $\sigma(A)\cap\mathbb{R}^- \neq \emptyset$. As before, we have the orthogonal decomposition $H = V \oplus W$ with the projection on V denoted P. This decomposition generates an interval $(a, b) \subset \rho(A, L)$ such that $a < 0 < b$. By the theory of bounded self-adjoint operators, $\sigma(A) \cap \mathbb{R}^+ \neq \emptyset$ and $\sigma(A) \cap \mathbb{R}^- \neq \emptyset$ imply that $V \neq \{0\}$ and $W \neq \{0\}$ (see [7: Section 2]). Consequently, $a, b \in \mathbb{R}$. Further, in [7] a positive functional $\phi \in C^2(B_{\varepsilon_0}, \mathbb{R})$ was introduced where $B_{\varepsilon_0} = \{u \in H : \|u\| \leq \varepsilon_0\}$ which satisfies the hypothesis

(A2) $\lim_{\|u\|\to 0} \frac{\phi(u)}{\|u\|^2} = 0$.

Supposing hypotheses (A1) and (A2) are satisfied, there is defined the functional

$$J : \mathbb{R} \times B_{\varepsilon_0} \to \mathbb{R}, \qquad J(\lambda, u) = \frac{1}{2}\langle(A - \lambda L)u, u\rangle - \phi(u).$$

Denoting $N = \nabla\phi$, we have as before that $(\lambda, u) \in \mathbb{R} \times B_{\varepsilon_0}$ is a solution of $(A - \lambda L)u - N(u) = 0$ if and only if (λ, u) is a critical point of $J(\lambda, \cdot)$.

Remark. $N = \nabla\phi$ is only defined on B_{ε_0}. To avoid confusion with $N = \nabla\varphi$ defined on H, we will sometimes denote $N_{GJ} = \nabla\phi$.

The following hypothesis is further introduced :

(A3) There exists $q > 2$ such that $\langle N_{GJ}(u), u\rangle \leq q\phi(u)$ for all $u \in B_{\varepsilon_0}$.

This hypothesis implies $\phi(tu) \geq t^q\phi(u)$ for all $t \in [0, 1]$ and all $u \in B_{\varepsilon_0}$ (see [7: (2.1)]). In [7] a hypothesis is used like $T(\delta)$ for $\delta > 0$ but as it is not the same as the one introduced before, it will be denoted by

$\mathbf{T(\delta)_{GJ}}$ $PL = LP$ and there exist $\varepsilon \in (0, \varepsilon_0]$ and $\{u_n\} \subset H$ with $\|u_n\| = \varepsilon$ such that $\phi(u_n) > 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{\langle(A - bL)u_n, u_n\rangle}{\phi(u_n)^\delta} = \lim_{n\to\infty} \frac{\|(A - bL)u_n\|^2}{\phi(u_n)^\delta} = 0.$$

Before stating the main result of [7] we can restrict the notion of weakly G-compact functionals to functionals defined on a ball. Moreover, we need only weakly upper G-compacity.

Definition. Let $r > 0$. A functional $K \in C^1(B_r, \mathbb{R})$ is said to be *weakly upper G-compact* if the following holds:

(1) K is G-invariant, i.e. $K(Tu) = K(u)$ for all $u \in B_r$ and all $T \in G$.

(2) If a sequence $\{u_n\} \subset B_r$ is such that $K(u_n) \to c > K(0)$ and $\|\nabla K(u_n)\| \to 0$, then there exists a subsequences $\{u_{n_i}\} \subset \{u_n\}$ and $v_{n_i} \in \theta(u_{n_i})$ such that $v_{n_i} \rightharpoonup v$ in H with $v \neq 0$ and $\nabla K(v) = 0$.

Remarks.

1) As T is an isometry, $Tu \in B_r$ when $u \in B_r$. Condition (1) of the definition makes thus sense.

2) We do not have explicitly to ask $\{u_n\}$ to be bounded because B_r is already bounded.

3) The weak convergence $v_{n_i} \rightharpoonup v$ implies $\liminf_{n\to\infty} \|v_{n_i}\| \geq \|v\|$ and thus $v \in B_r$. The expression $\nabla K(v)$ in condition (2) is well-defined.

4.4 The bifurcation result of [7]. Here is the main result of [7] which states the existence of a bifurcation point:

Theorem 5 [7: Theorem 1.1]. *Suppose that hypotheses* (A1) - (A3) *hold and that condition $T(\delta)_{GJ}$ is satisfied for some $\delta \geq 1$. Assume also the following:*

(A4) *There exists $K > 0$ such that $\|N(u)\| \leq K\phi(u)^{1-\frac{\delta}{2}}$ for all $u \in B_{\varepsilon_0}$.*

(A5) *Either*

(i) *N is compact*

or

(ii) *for a subgroup G of $O(H)$ and for $\lambda < b$ close to b, $J(\lambda, \cdot)$ is weakly upper G-compact in B_{ε_0}.*

Then there exists a sequence $\{(\lambda_n, u_n)\} \subset (a, b) \times H$ of non-trivial solutions of $(A - \lambda L)u - N(u) = 0$ such that $\lambda_n \to b^-$ and $\|u_n\| \to 0$ as $n \to \infty$. In particular, b is a bifurcation point for $(A - \lambda L)u - N(u) = 0$.

5. Verification of the hypotheses

In this section we will give the conditions to check the hypotheses of Theorems 4 and 5.

5.1 Hypotheses (H1) - (H2). We recall that J and M are $2N \times 2N$ real matrices such that $J^T = J^{-1} = -J$, $M^T = M$ and $\sigma(JM) \cap i\mathbb{R} = \emptyset$. By Subsection 3.2, hypotheses (H1) and (H2) are already true.

Lemma 2. *The operators P and L commute.*

Proof. See [9: Lemma 17] ∎

The interval (a,b) defined by means of hypotheses (H1) and (H2) in Subsection 4.1 is the same as the spectral gap of $\sigma(S)$ containing 0. Indeed, P and L commute so that (a,b) is the maximal spectral gap $\sigma(A,L)$ containing 0 (see Subsection 4.1). We conclude by the fact that $\sigma(S)=\sigma(A,L)$ (see Subsection 3.2). This interval is bounded, i.e. $-\infty < a < 0 < b < \infty$. Indeed, [13: Corollary 10.2] tells us that $\inf\sigma(S)=-\infty$ and $\sup\sigma(S)=\infty$.

5.2 Hypothesis (H3). Assuming F satisfies hypothesis (F1), we get by Proposition 1 that $\varphi \in C^2(H,\mathbb{R})$ and $\lim_{\|u\|\to 0}\frac{\varphi(u)}{\|u\|^2}=0$. When $F(x,\cdot)$ is convex (hypothesis (F2)), φ is clearly also convex. In this case, hypothesis (H3) is satisfied.

5.3 Hypotheses (H4) - (H5) and (P). In the following lemma we will give an estimation of $\|N(u)\|$ which will be used several times in the sequel.

Lemma 3. *Suppose that F satisfies hypothesis* (F1). *Moreover, suppose that F satisfies the following conditions:*

1. $F(x,s)\geq 0$ *for a.e.* $x\in\mathbb{R}$ *and all* $s\in\mathbb{R}^{2N}$ (*hypothesis* (F3)).
2. $|f(x,s)|\leq C_1F(x,s)^{t_1}+C_2F(x,s)^{t_2}$ *for a.e.* $x\in\mathbb{R}$ *and all* $s\in\mathbb{R}^{2N}$ *where* $C_1,C_2>0$ *and* $t_1,t_2\in[\frac{1}{2},1)$ (*hypothesis* (F4)).

Then $\|N(u)\|\leq D_1\varphi(u)^{t_1}+D_2\varphi(u)^{t_2}$ *with* $D_1,D_2>0$.

Proof. See [9: Lemma 18] ∎

When we suppose that F is positive and satisfies hypothesis (F4), hypotheses (H4) - (H5) are true.

Corollary 1. *Suppose that F satisfies hypotheses* (F1), (F3), (F4). *Then there exists a constant $C>0$ such that $\|N(u)\|\leq C(1+\varphi(u))$ for all $u\in H$.*

Proof. As $t_i\in[\frac{1}{2},1)$ and $\varphi(u)\geq 0$, $\varphi(u)^{t_i}\leq 1+\varphi(u)$. Thus, by Lemma 3, $\|N(u)\|\leq(\sum_{i=1}^2 D_i)(1+\varphi(u))$ ∎

Corollary 2. *Suppose F satisfies hypothesis* (F1). *Moreover, suppose the following:*

1. $F(x,s)\geq 0$ *a.e. on* $\mathbb{R}$ *for all* $s\in\mathbb{R}^{2N}$.
2. $|f(x,s)|\leq C_1F(x,s)^{t_1}+C_2F(x,s)^{t_2}$ *for a.e.* $x\in\mathbb{R}$ *for all* $s\in\mathbb{R}^{2N}$ *where* $C_1,C_2>0$ *and* $t_1,t_2\in[\frac{1}{2},1)$.

Then there exists a constant $D>0$ such that $\|N(u)\|\leq D\varphi(u)^{1/2}$ for all $u\in H$ with $\varphi(u)<1$.

Proof. By Lemma 3 we have

$$\|N(u)\|\leq\sum_{i=1}^{2}D_i\varphi(u)^{t_i}.$$

Let us consider $u\in H$ such that $\varphi(u)<1$. Since $t_i\geq\frac{1}{2}$,

$$\sum_{i=1}^{2}D_i\varphi(u)^{t_i}\leq\left(\sum_{i=1}^{2}D_i\right)\varphi(u)^{1/2}$$

and the assertion is proved ∎

The lemma below gives us conditions to check hypothesis (P):

Lemma 4. *Suppose that F satisfies hypothesis* (F1). *Then:*

1. *If $f(x,s)\cdot s \le qF(x,s)$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ with $q \in \mathbb{R}$, then $\varphi'(u)u \le q\varphi(u)$ for all $u \in H$.*

2. *If $pF(x,s) \le f(x,s)\cdot s$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ with $p \in \mathbb{R}$, then $p\varphi(u) \le \varphi'(u)u$ for all $u \in H$.*

Proof. The result is trivial when we use the fact that $\varphi'(u)v = \int_{\mathbb{R}} f(x,u(x))\cdot v(x)\,dx$ (see Proposition 1/item 5) ∎

Supposing F is positive, hypothesis (P) is an easy consequence of hypotheses (F5) - (F6).

5.4 Hypothesis $T(\delta)$. Let us give sufficient conditions to check $T(\delta)$.

Lemma 5. *Suppose that F satisfies hypothesis* (F1). *Moreover, suppose the following:*

1. *$F(x,s) \ge 0$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ (hypothesis* (F3)*).*

2. *There exists $d \ge 0$ such that the set $\{x \in \mathbb{R} : F(x,s) \ne 0 \text{ if } 0 < |s| \le d\}$ is not of measure zero (hypothesis* (F8)*).*

3. *$F(x,ts) \ge c\,t^{\tilde{p}} F(x,s)$ for a.e. $x \in \mathbb{R}$ if $t \in [0,\varepsilon]$ and $|s| \le \Delta$ where $c,\varepsilon,\Delta > 0$ and $\tilde{p} > 2$ (hypothesis*(F9)*).*

Then condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{\tilde{p}}$. Moreover, if $F(x+1,s) = F(x,s)$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$, then condition $T(\delta)$ is true for all $\delta < \frac{4}{\tilde{p}-2}$.

Proof. The proof of this result can be found in [10]. It uses the theory of almost periodic functions of Stepanov ∎

In the non-periodic case, we can get a better condition $T(\delta)$ with an extra hypothesis:

Lemma 6. *Suppose that F satisfies hypothesis* (F1). *Moreover, suppose the following:*

1. *$F(x,s) \ge 0$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ (hypothesis* (F3)*).*

2. *$F(x,ts) \ge c\,t^{\tilde{p}} F(x,s)$ for a.e. $x \in \mathbb{R}$ if $t \in [0,\varepsilon]$ and $|s| \le \Delta$ where $c,\varepsilon,\Delta > 0$ and $\tilde{p} > 2$ (hypothesis* (F9)*).*

3. *there exists $\Delta' \in (0,\Delta]$ such that $\min_{|s|=\Delta'} F(x,s) \ge \frac{C}{|x|^\alpha}$ for a.e. $|x| \ge M$ where $\alpha \in (0,1]$ and $M, C > 0$.*

Then condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{\tilde{p}-2(1-\alpha)}$.

Proof. The proof can be found in [9] ∎

Remarks.

1) Hypothesis (F8) is no longer imposed explicitly since it is a consequence of hypotheses 2 and 3.

2) For $\alpha \in (0,1)$, this result is a real improvement with respect to Lemma 5.

5.5 The other hypotheses. Condition 1.1 in Theorem 1 implies the compacity of N.

Lemma 7. *Suppose that F satisfies hypothesis* (F1). *Moreover, suppose* $|f(x,s)| \leq a(x)(|s|^{q_1} + |s|^{q_2})$ *for a.e.* $x \in \mathbb{R}$ *for all* $s \in \mathbb{R}^{2N}$ *where* $q_2 \geq q_1 > 1$ *and* a *is measurable such that* $\lim_{|x|\to\infty} a(x) = 0$. *Then* $N : H \to H$ *is compact.*

Proof. The proof can be found in [9: Lemma 21] ∎

For $k \in \mathbb{Z}$ we define the translation operator $T_k : H \to H$ by $T_k u(x) = u(x-k)$. This operator is an isometry:

Lemma 8. *For* $k \in \mathbb{Z}$,

1. $T_k : H \to H$ *is well-defined and* $T_k \in O(H)$.

2. $T_k(H^1(\mathbb{R})]^{2N}) \subset [H^1(\mathbb{R})]^{2N}$.

3. $T_k S u = S T_k u$ *for all* $u \in [H^1(\mathbb{R})]^{2N}$.

Proof. The proof can be found in [9: Lemma 23] ∎

Lemma 9. *Suppose that F satisfies hypothesis* (F1). *Moreover, suppose the following:*

1. $F(x,s) \geq 0$ *a.e. on* $\mathbb{R}$ *for all* $s \in \mathbb{R}^{2N}$.

2. $F(x+1,s) = F(x,s)$ *a.e on* $\mathbb{R}$ *for all* $s \in \mathbb{R}^{2N}$.

3. $pF(x,s) \leq f(x,s)\cdot s \leq qF(x,s)$ *a.e. on* $\mathbb{R}$ *for all* $s \in \mathbb{R}^{2N}$ *where* $p, q \in \mathbb{R}$.

Then $J(\lambda,\cdot) : H \to \mathbb{R}$ *is weakly G-compact for all* $\lambda \in \mathbb{R}$.

Proof. The proof of this result is quite long and can be found in [9: Subsection 8.5] ∎

To check the weakly sequentially lower semi-continuity of $\psi(u) = \langle N(u), u\rangle - 2\varphi(u)$, we use the following lemma:

Lemma 10. *Suppose that F satisfies hypothesis* (F1) *and that* $f(x,s)\cdot s - 2F(x,s)$ *is convex in* s *a.e. on* $\mathbb{R}$. *Then* $\psi : H \to \mathbb{R}$ *is weakly sequentially lower semi-continuous.*

Proof. The functional ψ is convex by the convexity of $f(x,s)\cdot s - 2F(x,s)$. We conclude by using [15: Theorem 8.10] which states that any finite convex functional defined on an open convex set (in a normed space) is weakly lower semi-continuous ∎

5.6 Verification of the hypotheses of [7]. Now, we will reduce the Hamiltonian system to the functional equation treated in [7]. The operators A and L are chosen like before. We set $\varepsilon_0 = 1$ and $\phi = \varphi|_{B_{\varepsilon_0}}$. Assuming F satisfies hypothesis (F1), $\phi \in C^2(B_{\varepsilon_0}, \mathbb{R})$ by Proposition 1. Clearly, $N_{GJ} = \nabla\phi$ is the restriction to B_{ε_0} of $N = \nabla\varphi$.

The following lemma gives conditions to check hypotheses (A3) and (A4).

Lemma 11. *Suppose that F satisfies hypothesis* (F1) *and that $f(x,s)\cdot s \le qF(x,s)$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ where $q > 2$. Then $< N_{GJ}(u), u > \le q\phi(u)$ for all $u \in B_{\varepsilon_0}$.*

Moreover, if $F(x,s) \ge 0$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ and $|f(x,s)| \le C_1 F(x,s)^{t_1} + C_2 F(x,s)^{t_2}$ for a.e. $x \in \mathbb{R}$ where $C_1, C_2 > 0$ and $t_1, t_2 \in [\frac{1}{2}, 1)$, then for $\delta \ge 1$ there exists $K > 0$ such that $\|N(u)\| \le K\phi(u)^{1-\frac{\delta}{2}}$ for all $u \in B_{\varepsilon_0}$.

Proof. See [9: Lemma 35] ∎

For the weakly upper G-compacity (where G is defined in Subsection 5.5) we do not need hypothesis (F5):

Lemma 12. *Suppose the following:*

1. *$F(x,s) \ge 0$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$.*

2. *$F(x+1,s) = F(x,s)$ a.e on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$.*

3. *$f(x,s)\cdot s \le qF(x,s)$ a.e. on $\mathbb{R}$ for all $s \in \mathbb{R}^{2N}$ where $q \in \mathbb{R}$.*

Then $J(\lambda,\cdot) : H \to \mathbb{R}$ is weakly upper G-compact for all $\lambda \in \mathbb{R}$.

Proof. See [9: Lemma 36] ∎

6. The proofs of the bifurcation theorems

In this section we will give the proofs of our bifurcation Theorems 1 and 2.

Proof of Theorem 1. In Subsection 3.2 we have seen that hypotheses (H1) and (H2) are true, in Subsection 5.2 that hypothesis (H3) is satisfied, by Corollaries 1 and 2 that Hypotheses (H4) and (H5) are satisfied, and Hypothesis (P) is satisfied by Lemma 4.

`Case (1).` By Lemma 7, N is compact. Since $\tilde{p} < 4$, $\frac{4}{\tilde{p}} > 1$ and condition $T(\delta)$ is true for a $\delta > 1$ by Lemma 5. Setting $\theta(\delta) = \frac{2}{q-2}(1-\frac{1}{\delta})$, Theorem 4 states that there is a sequence

$$\{(\lambda_n, u_n)\} \subset \Big\{(\lambda, u) \in \mathbb{R}\times H : u \neq 0 \ \text{ and } \ Au - \lambda Lu = N(u)\Big\}$$

such that $\lambda_n < b$ $(n \in \mathbb{N})$, $\lambda_n \to b$ and $\lim_{n\to\infty}(b-\lambda_n)^{-\frac{\theta(\delta)}{2}}\|u_n\| = 0$. This is true for all $\theta(\delta)$ such that $\delta < \frac{4}{\tilde{p}}$, i.e for all $\theta < \frac{2}{q-2}(1-\frac{\tilde{p}}{4})$. Setting $\gamma = \frac{\theta}{2}$, we have a bifurcation point of order γ for all $\gamma < \frac{1}{q-2}(1-\frac{\tilde{p}}{4})$ and we conclude by Proposition 3.

`Case (2).` By Lemma 9, $J(\lambda,\cdot)$ is weakly G-compact for all $\lambda \in \mathbb{R}$. Since $\tilde{p} < 6$, $\frac{4}{\tilde{p}-2} > 1$ and condition $T(\delta)$ is true for a $\delta > 1$ by Lemma 5. By Lemma 10, ψ is weakly sequentially lower semi-continuous. Setting $\theta(\delta)$ as before, Theorem 4 and Proposition 3 imply that b is a bifurcation point of order $\gamma = \frac{\theta}{2}$ and this is true for all $\theta(\delta)$ such that $\delta < \frac{4}{\tilde{p}-2}$. Thus, b is a bifurcation point of order γ for all $\gamma < \frac{1}{q-2}(1-\frac{\tilde{p}-2}{4})$.

`Case (3).` By Lemma 7, N is compact. Since $\tilde{p} < 4 + 2(1-\alpha)$, $\frac{4}{\tilde{p}-2(1-\alpha)} > 1$. Thus condition $T(\delta)$ is true for a $\delta > 1$ by Lemma 6. Setting $\theta(\delta)$ as before, Theorem 4 and Proposition 3 imply that b is a bifurcation point of order $\gamma = \frac{\theta}{2}$ and this is true for all $\theta(\delta)$ such that $\delta < \frac{4}{\tilde{p}-2(1-\alpha)}$. Thus, b is a bifurcation point of order γ for all $\gamma < \frac{1}{q-2}(1 - \frac{\tilde{p}-2(1-\alpha)}{4})$ ∎

Proof of Theorem 2. This theorem is an application of [7: Theorem 1.1].

Let us show first that hypothesis (A1) is true. By Subsection 3.2, $A, L \in \mathcal{B}(H)$ are self-adjoint, $\langle Lu, u\rangle > 0$ for all $u \in H \setminus \{0\}$ and $0 \notin \sigma(A)$. In Subsection 5.1 we have seen that $-\infty < a < 0 < b < \infty$, thus by definition of a and b, $V \neq \{0\}$ and $W \neq \{0\}$. Then $\langle Au, u\rangle \geq \beta\|u\|^2$ for all $u \in V$ and $\langle Au, u\rangle \leq -\alpha\|u\|^2$ for all $u \in W$, with $\alpha, \beta > 0$ (see [13: p. 7] or [7: Section 2]). Hence $\sup_{\|u\|=1}\langle Au, u\rangle > 0$ and $\inf_{\|u\|=1}\langle Au, u\rangle < 0$. Since $A \in \mathcal{B}(H)$ is self-adjoint, $\sup_{\|u\|=1}\langle Au, u\rangle \in \sigma(A)$ and $\inf_{\|u\|=1}\langle Au, u\rangle \in \sigma(A)$ (see [11: p. 148/Theorem 4]). We have thus $\sigma(A) \cap \mathbb{R}^+ \neq \emptyset$ and $\sigma(A) \cap \mathbb{R}^- \neq \emptyset$.

Let us show now that condition (A2) is true. By Proposition 1, $\varphi \in C^2(H, \mathbb{R})$. We set thus $\phi = \varphi|_{B_{\varepsilon_0}}$ for $\varepsilon_0 = 1$ so that $\phi \in C^2(B_{\varepsilon_0}, \mathbb{R})$. Since $F \geq 0$ a.e., $\phi \geq 0$. Again by Proposition 1, $\lim_{\|u\|\to 0} \frac{\phi(u)}{\|u\|^2} = 0$.

By Lemma 11, hypothesis (A3) is true.

Let us show that condition $T(\delta)_{GJ}$ is satisfied for a $\delta \geq 1$. We use Lemma 5.

Case (1): By the lemma, condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{\tilde{p}}$. Since $\frac{4}{\tilde{p}} > 1$, $T(\delta)$ is satisfied for a $\delta \geq 1$

Case (2): By the lemma, condition $T(\delta)$ is satisfied for all $\delta < \frac{4}{\tilde{p}-2}$. Since $\frac{4}{\tilde{p}-2} > 1$, $T(\delta)$ is satisfied for a $\delta \geq 1$. Since $\varepsilon_0 = 1$, we can chose $\varepsilon = 1$ so that $T(\delta)_{GJ}$ is satisfied for a $\delta \geq 1$.

By Lemma 11 again, hypothesis (A4) is true.

Let us check hypothesis (A5). In case (1), $N = \nabla\varphi : H \to H$ is compact by Lemma 7, thus $N_{GJ} = \nabla\phi : B_{\varepsilon_0} \to H$ is also compact.

In case (2), Lemma 12 implies that $J(\lambda, \cdot) : H \to \mathbb{R}$ is weakly upper G-compact for all $\lambda \in \mathbb{R}$. Clearly, $J_{GJ}(\lambda, \cdot) : B_{\varepsilon_0} \to \mathbb{R}$ is also weakly upper G-compact for all $\lambda \in \mathbb{R}$.

Now, all the hypotheses of Theorem 5 are checked and Theorem 2 is obtained as a corollary of Theorem 5 and Proposition 3 ∎

7. Proof of bifurcation theorem under local conditions

In this section, we will prove the bifurcation theorem under local conditions (Theorem 3) using Theorem 2. Starting from the function G, we will construct a function $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ satisfying the hypotheses of Theorem 2 such that $F(x, \cdot) = G(x, \cdot)$ for s close to 0.

7.1 Preliminary results. We need a function η to construct the extension F:

Lemma 13. *Let $r_0 > 0$. Then there exists $\eta \in C^2(\mathbb{R}, \mathbb{R})$ with $0 \le \eta \le 1$ such that*

$$\eta(r) = \begin{cases} 1 & \text{if } r \le r_0 \\ 0 & \text{if } r \ge 2r_0, \end{cases}$$

$\eta' \le 0$ and, for some constant $C > 0$, $|\eta'(r)| \le C\eta(r)^{1/2}$ for all r.

Proof. We set

$$\eta(r) = \begin{cases} 1 & \text{if } r \le r_0 \\ (2 - \frac{r}{r_0})^3 \big(6(\frac{r}{r_0})^2 - 9(\frac{r}{r_0}) + 4\big) & \text{if } r_0 < r < 2r_0 \\ 0 & \text{if } r \ge 2r_0. \end{cases}$$

It is not dificult to check the conclusions of the lemma (see [9: Lemma 37]) ∎

In the next lemma we state that G is equal to a function F for s close to 0 such that F satisfies the hypotheses of Theorem 2. We use the function η of Lemma 13.

Lemma 14. *Let us consider $G : \mathbb{R} \times B(0, R) \to \mathbb{R}$ with hypotheses* (G1), (G3), (G4), (G6), (G8), (G9). *Moreover, let us suppose that*

- **(1.1)** *$|g(x, s)| \le a(x)|s|^{q_1}$ for a.e. $x \in \mathbb{R}$ for all $s \in B(0, R)$ where $q_1 > 1$ and a is measurable with $\lim_{|x|\to\infty} a(x) = 0$.*
- **(1.2)** *$\tilde{p} < 4$*

or

- **(2.1)** *$G(x + 1, s) = G(x, s)$ for a.e. $x \in \mathbb{R}$ for all $s \in B(0, R)$.*
- **(2.2)** *$\tilde{p} < 6$.*

Then there exists $r_0 \in (0, R)$ and $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ such that:

- **(i)** *$F(x, s) = G(x, s)$ for a.e. $x \in \mathbb{R}$ for all $s \in B(0, r_0)$.*
- **(ii)** *F satisfies the hypotheses of Theorem 2.*

Proof. We chose $r_0 > 0$ such that $3r_0 < R$ and $\frac{\sqrt{2N}a}{r_1(r_1+1)}(3r_0)^{r_1+1} \le 1$ where a and r_1 are given by hypothesis (G1). To r_0 we associate the function η given by Lemma 13. We define $F : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ by

$$F(x, s) = \begin{cases} \eta(|s|)G(x, s) & \text{if } 0 \le |s| < 3r_0 \\ 0 & \text{if } |s| \ge 3r_0. \end{cases}$$

Since $\eta(|s|) = 1$ for $0 \le |s| < r_0$, $F(x, s) = G(x, s)$ for a.e. $x \in \mathbb{R}$, for all $s \in B(0, r_0)$. It is not difficult to check that F satisfies the hypotheses of Theorem 2 (see [9: Lemma 40]) ∎

7.2 Proof of Theorem 3. Using Lemma 14, the proof of Theorem 3 is very short:

Proof of Theorem 3. By Lemma 14 and Theorem 2, b is a bifurcation point of $Ju' + Mu - f(x, u) = \lambda u$. We have thus $\lambda_n \to b$ and $\|u_n\|_{H^1} \to 0$. But $\|u_n\|_{L^\infty} \le C\|u_n\|_{H^1}$, thus for n large, $\|u_n\|_{L^\infty} < r_0$. Hence, for n large, $f(x, u_n) = g(x, u_n)$, thus b is also a bifurcation point of $Ju' + Mu - g(x, u) = \lambda u$ ∎

Acknowledgement. I would like to thank

- Prof. C. A. Stuart who gave me this subject, followed my work with enthousiasm, answered to all my questions and read the preliminary versions of this paper.

- Prof. L. Jeanjean, Prof. B. Buffoni and Dr. J. Giacomoni who answered to my questions.

I would also like to thank the Swiss National Fund for Scientific Research and the EPFL who supported this work financially.

References

[1] Adams, R. A.: *Sobolev Spaces.* New York: Acad. press 1975.

[2] Buffoni, B.: *Un problème variationnel fortement indéfini sans compacité.* Thèse de doctorat. Lausanne: EPFL 1992.

[3] Coti-Zelati, V., Ekeland, I. and E. Séré: *A variational approach to homoclinic orbits in Hamiltonian systems.* Math. Ann. 288 (1990), 133 – 160.

[4] Ding, Y. and S. Li: *Homoclinic orbits for first order Hamiltonian systems.* J. Math. Anal. Appl. 189 (1995), 585 – 601.

[5] Ding, Y. and M. Willem: *Homoclinic orbits of a Hamiltonian system.* ZAMP 50 (1999), 759 – 778.

[6] Edmunds, D. E. and W. D. Evans: *Spectral Theory and Differential Equations.* Oxford: Oxford Univ. Press 1987.

[7] Giacomoni, J. and L. Jeanjean: *A variational approach to bifurcation into spectral gaps.* Ann. Scuola Norm. Super. di Pisa, Sci. Fis. et Mat. (Serie IV) 28 (1999)4.

[8] Hofer, H. and K. Wysocki: *First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems.* Math. Ann. 288 (1990), 483 – 503.

[9] Joosten, R.: *Existence and bifurcation of homoclinic solutions for Hamiltonian systems.* PhD thesis. Lausanne: EPFL 2000.

[10] Joosten, R.: *Existence of homoclinic solutions for Hamiltonian systems.* Adv. Diff. Equ. 7 (2002), 1315 – 1342.

[11] Ljusternik, L. A. and V. J. Sobolev: *Elements of Functional Analysis.* New York: Frederick Ungar Publ. Comp. 1961.

[12] Séré, E.: *Existence of infinitely many orbits in Hamiltonian systems.* Math. Z. 209 (1992), 27 – 42.

[13] Stuart, C. A.: *Bifurcation into Spectral Gaps.* Société Math. de Belgique 1995.

[14] Tanaka, K.: *Homoclinic orbits in a first order superquadratic Hamiltonian system: Convergence of subharmonic orbits.* J. Diff. Equ. 94 (1991), 315 – 339.

[15] Vainberg, M. M.: *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations.* New York: Halsted Press 1975.

Received 06.05.2002