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An Extended Cauchy-Kovalevskaya Problem and its Solution in Associated Spaces

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Abstract. The classical Cauchy-Kovalevskaya problem with holomorphic intial functions is uniquely solvable provided the right-hand sides of the differential equations are holomorphic in their variables, i.e., they transform holomorphic functions into holomorphic functions. Moreover, the solutions depend holomorphically on the space-like variables. A far-reaching generalization of the Cauchy-Kovalevskaya Theorem is its abstract version which considers an abstract operator equation in a scale of Banach spaces where the behaviour of complex derivatives at the boundary is expressed by a certain mapping property of the operator under consideration in the underlying scale. Another generalization of the Cauchy-Kovalevskaya Theorem replaces the space of holomorphic functions by another so-called associated space which is defined by an elliptic operator. Making use of this second approach, the present short note solves an extended Cauchy-Kovalevskaya problem in which an initial value problem is combined with an implicit equation.

Keywords: *Equivalent integro-differential equations, weighted Banach spaces, interior estimates*

AMS subject classification: 35A10, 35B45, 46E15

M. Nagumo's functional-analytic approach [4] to the Cauchy-Kovalevskaya problem

$$\left.\begin{aligned} \partial_t u &= \mathcal{F}(t,x,u,\partial_{x_j} u) \\ u(0,x) &= \varphi(x) \end{aligned}\right\}$$

is based on the equivalent integro-differential equation

$$u(t,x) = \varphi(x) + \int_0^t \mathcal{F}(\tau,x,u,\partial_{x_j} u)\,d\tau. \tag{1}$$

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A highlight of this development led to abstract versions of the Cauchy-Kovelevskaya Theorem (cf. F. Treves [6] and L. Nirenberg [5]). Generalizing the abstract non-linear Cauchy-Kovalevskaya Theorem, in the paper [1] the extended equation

$$u(t) = F(t, u(\cdot)) + \int_0^t E(t,s)G(s,u(\cdot))\,ds \qquad (0 \le t \le T) \tag{2}$$

is solved in a scale of Banach spaces using the contraction-mapping principle.

On the other hand, using interior estimates for solutions of an associated equation, integro-differential equations of type (1) can be solved [9] in a suitably defined Banach space whose elements depend on the space-like variable x and the time t as well. This is another functional-analytic approach to initial value problems of Cauchy-Kovalevskaya type which generalizes W. Walter's elementary approach [10] to the classical Cauchy-Kovalevskaya Theorem. Although associated spaces can also be used to define suitable scales of Banach spaces (e.g., this has be done for initial value problems with generalized analytic initial functions, cf. [7]), the present paper does not construct the corresponding scales of Banach spaces (which would also be possible), but it constructs a single Banach spaces whose elements depend on both the space-like variables and the time t as well.

Combining equations (1) and (2) with each other, the present paper solves equations of the form

$$u(t,x) = \mathcal{F}_1(t,x,u) + \int_0^t \mathcal{F}_2(\tau, x, u, \partial_{x_j}u)\,d\tau. \tag{3}$$

Both operators $\mathcal{F}_j$ are supposed to be associated to an (elliptic) operator $\mathcal{G}$ with time-independent coefficients, i.e., $\mathcal{G}u = 0$ implies $\mathcal{G}(\mathcal{F}_j u) = 0$ for each fixed t. Further, both operators have to satisfy global Lipschitz conditions with respect to a suitable norm (such as the Hölder norm):

$$\begin{aligned} \big\|\mathcal{F}_1(t,x,u) - \mathcal{F}_1(t,x,v)\big\| &\le l_0\|u-v\| \\ \big\|\mathcal{F}_2(t,x,u,\partial_j u) - \mathcal{F}_2(t,x,v,\partial_j v)\big\| &\le L_0\|u-v\| + \sum_j L_j\|\partial_j u - \partial_j v\|. \end{aligned}$$

Solutions of the associated equation $\mathcal{G}u = 0$ have (for fixed t) to satisfy a first order interior estimate, i.e., if Ω' is a subdomain of Ω'', then

$$\|\partial_j u\|_{\Omega'} \le \frac{c_1}{\operatorname{dist}(\Omega', \partial\Omega'')}\,\|u\|_{\Omega''} \tag{4}$$

with a universal constant c_1.

Obviously, solutions $u = u(t,x)$ of equation (3) are fixed points of the operator defined by the right-hand side of (3). Starting from a Banach space $\mathcal{B}$ (e.g., the space of Hölder continuous functions) and an exhaustion Ω_s $(0 < s < s_0)$ of a given (bounded) domain $\Omega \subset \mathbb{R}^n$, one considers the Banach spaces $\mathcal{B}_s = \mathcal{B}(\Omega_s)$. Introduce the conical domain

$$M(\eta) = \Big\{(t,z) : \; z \in \Omega \;\text{ and }\; 0 \le t < \eta(s_0 - s(x))\Big\}$$

where $s(x)$ is the (uniquely determined) index such that x belongs to the boundary of $\Omega_{s(x)}$. Then

$$d(t,x) = s_0 - s(x) - \frac{t}{\eta}$$

is a pseudo-distance measuring the distance of a point (t,x) of $M(\eta)$ from the lateral surface of $M(\eta)$. Define $\mathcal{B}_*(M(\eta))$ as the Banach space of all (continuous) functions $u = u(t,x)$ whose $*$-norm

$$\|u\|_* = \sup_{M(\eta)} \|u\|_{s(x)} d(t,x)$$

is finite. This definition implies

$$\|u\|_{s(x)} \leq \frac{\|u\|_*}{d(t,x)}. \tag{5}$$

Now consider the subspace $\mathcal{B}_*^{\mathcal{G}}(M(\eta))$ of those elements of $\mathcal{B}_*(M(\eta))$ which satisfy the associated (elliptic) equation $\mathcal{G}u = 0$ for each fixed t.

To be short, denote $\mathcal{F}_1(t,x,u)$ by $\mathcal{F}_1 u$. The Lipschitz condition for $\mathcal{F}_1$ implies

$$\|\mathcal{F}_1 u - \mathcal{F}_1 v\|_{s(x)} \leq l_0 \|u - v\|_{s(x)} \leq l_0 \, \frac{\|u - v\|_*}{d(t,x)}$$

and thus

$$\|\mathcal{F}_1 u - \mathcal{F}_1 v\|_* \leq l_0 \|u - v\|_*. \tag{6}$$

Since $\mathcal{F}_2$ depends on the derivatives $\partial_j u$, an analogous estimate of the integral in (3) requires an interior estimate which holds by hypothesis. Applying the interior estimate (4) to the exhaustion of Ω where $\operatorname{dist}(\Omega_{s'}, \partial\Omega_{s''}) \geq \operatorname{const}(s'' - s')$, we obtain

$$\|\partial_j u\|_{s'} \leq \frac{c_2}{s'' - s'} \, \|u\|_{s''}. \tag{7}$$

In order to estimate the $s(x)$-norm of the first order derivatives, consider $\tilde{s} = s(x) + \frac{1}{2}\, d(t,x)$. Let $\tilde{x}$ be any point with $s(\tilde{x}) = \tilde{s}$. Then

$$d(t,\tilde{x}) = s_0 - s(\tilde{x}) - \frac{t}{\eta} = \frac{1}{2}\, d(t,x)$$

and, consequently, (5) yields

$$\|u(t,\cdot)\|_{\tilde{s}} \leq \frac{\|u\|_*}{d(t,\tilde{x})} = \frac{2\|u\|_*}{d(t,x)}.$$

Applying the interior estimate (7) with the pair $(s(x), \tilde{s})$, the latter estimate gives

$$\|\partial_j u\|_{s(x)} \leq \frac{4c_2}{d^2(t,x)} \, \|u\|_*.$$

Analogously to the abbreviation $\mathcal{F}_1$, denote $\mathcal{F}_2(t, x, u, \partial_j u)$ by $\mathcal{F}_2 u$. Combining the above estimates, and making use of the estimate $d(t, x) \le s_0$, i.e. $1 \le \frac{s_0}{d(t,x)}$, it follows

$$\|\mathcal{F}_2 u - \mathcal{F}_2 v\|_{s(x)} \le \frac{c_3}{d^2(t,x)} \, \|u - v\|_*$$

where $c_3 = L_0 s_0 + 4c_2 \sum_j L_j$. Since $\int_0^t \frac{1}{d^2(\tau,x)} \, d\tau < \frac{\eta}{d(t,x)}$ one has

$$\left\| \int_0^t (\mathcal{F}_2 u - \mathcal{F}_2 v) \, d\tau \right\|_{s(x)} \le \frac{\eta c_3}{d(t,x)} \, \|u - v\|_*$$

and, therefore,

$$\left\| \int_0^t (\mathcal{F}_2 u - \mathcal{F}_2 v) \, d\tau \right\|_* \le \eta c_3 \|u - v\|_*.$$

Together with (6) we obtain for the images

$$U = \mathcal{F}_1 u + \int_0^t \mathcal{F}_2 u \, d\tau \qquad \text{and} \qquad V = \mathcal{F}_1 v + \int_0^t \mathcal{F}_2 v \, d\tau$$

of u and v, respectively, the estimate

$$\|U - V\|_* \le (l_0 + \eta c_3) \, \|u - v\|_*$$

and thus the following lemma has been proved:

Lemma. *The operator defined by the right-hand side of* (3) *is contractive in* $\mathcal{B}_*^{\mathcal{G}}(M(\eta))$ *provided* $l_0 < 1$ *and* $\eta < \frac{1-l_0}{c_3}$.

Notice that similar estimates show that $\|u\|_* < +\infty$ implies $\|U\|_* < +\infty$.

Theorem. *Suppose* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are Lipschitz-continuous operators associated to an* (*elliptic*) *first order operator* $\mathcal{G}$ *which satisfies a first order interior estimate. Provided the Lipschitz constant* l_0 *of* $\mathcal{F}_1$ *is smaller than* 1, *the operator equation* (3) *is uniquely solvable in* $\mathcal{B}_*^{\mathcal{G}}(M(\eta))$ *where the height of the conical domain* $M(\eta)$ *is sufficiently small. The solution* $u = u(t, x)$ *of equation* (3) *satisfies the side condition* $\mathcal{G}u = 0$ *for each* t.

An easy example is given by the Laplace equation $\mathcal{G}w \equiv \partial_z \partial_{\overline{z}} w = 0$ for complex-valued functions in the complex z-plane (cf. [3]). Then $\mathcal{F}_1 w \equiv \varphi_1 + \overline{\varphi_2} + A_1 w + B_1 \overline{w}$ is associated to $\mathcal{G}$ provided φ_1 and φ_2 are holomorphic in the domain under consideration, while A_1 and B_1 are arbitrary (continuous) functions depending on the time t. If $\sup|A_1| + \sup|B_1| < 1$, then one has $l_0 < 1$ for the corresponding Lipschitz constant. [1)] Moreover,

$$\mathcal{F}_2 w \equiv C_1 \partial_z w + C_2 \partial_{\overline{z}} w + C_3 \overline{\partial_z w} + C_4 \overline{\partial_{\overline{z}} w} + A_2 w + B_2 \overline{w}$$

[1)] Of course, an equation of the form $w = A_1 w + B_1 \overline{w} + C$ can be solved for w in the case $|A_1 - 1| \ne |B_1|$. However, if $|A_1 - 1| - |B_1|$ is small, then solving for w can lead to a stronger restriction of η.

is also associated to the Laplace operator provided C_1 and C_4 are holomorphic in z, whereas C_2 and C_3 are supposed to be anti-holomorphic in z (note that all coefficients may depend on t, too). Then the above theorem shows that in sufficiently small conical domains the equation $w = \mathcal{F}_1 w + \int_0^t \mathcal{F}_2 w \, d\tau$ can be solved by a function $w = w(t,z)$ which satisfies the Laplace equation for each t.

Similarly, if $\mathcal{G}$ is the Laplace operator in a bounded domain in $\mathbb{R}^n$, then the theorem can be applied to desired complex-valued functions $u = u(t,x)$ in the case $\mathcal{F}_1 = \varphi + Au + B\overline{u}$ and $\mathcal{F}_2 = \sum_j C_j \partial_j u$ where A, B and C_j depend on t, $\sup|A| + \sup|B| < 1$ and $\Delta\varphi = 0$.

Concluding Remarks.

1. Concerning the compactness of the integro-differential operator on the right-hand side of (3) see the paper [8]. While the operator is compact in some Frechét spaces, this is not the case in the Banach space being in use here.

2. In order to exclude the existence of further solutions of equation (3) not belonging to $\mathcal{B}_*^{\mathcal{G}}(M(\eta))$ one needs a Holmgren-type theorem which is not provived in the present paper.

3. The initial state $u(0,x)$ satisfies the equation

$$u(0,x) = \mathcal{F}_1(0, x, u(0,x)).$$

Its existence and uniqueness in the space of all solutions of the (time-independent) equation $\mathcal{G}u = 0$ is contained in the above theorem.

4. As mentioned above, the present paper does not make use of an abstract Cauchy-Kovalevskaya theorem because the contraction-mapping principle is applied to a single Banach space. Of course, the solution of equation (3) under the side condition $\mathcal{G}u = 0$ for each t can also be constructed using the scale method. Then the Banach spaces $\mathcal{B}_s$ would form a suitable scale.

5. If $\mathcal{F}_1 \equiv \varphi$ where $\mathcal{G}\varphi = 0$, the above theorem solves the initial value problem $\partial_t u = \mathcal{F}_2(t, x, u, \partial_j u)$ with the initial function φ. Then the theorem includes a conservation law because the side condition $\mathcal{G}u = 0$ is satisfied for each t.

6. If $\mathcal{F}_2 \equiv 0$, the above theorem shows also that the implicit equation $u = \mathcal{F}_1(t, x, u)$ has a uniquely determined solution in $\mathcal{B}_*^{\mathcal{G}}(M(\eta))$ provided $l_0 < 1$.

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