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Wiener-Type Tauberian Theorems for Fourier Hyperfunctions

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Abstract. Two Wiener-type Tauberian theorems concerning Fourier hyperfunctions are proved and commented. It is shownt that the shift asymptotics (S-asymptotics) of a hyperfunction f is determined by the ordinary asymptotics of $(f * \mathcal{K})(x)$ as $x \to \infty$, where $\mathcal{K}$ is Hörmaner's kernel. Moreover, Wiener-type theorems are used for the asympthotic analysis of solutions to some (pseudo-)differential equations.

Keywords: *Fourier hyperfunction, Tauberian theorem*

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1. Introduction

Recall the celebrated Wiener Tauberian theorem [27: p. 25]: Suppose that $f \in L^\infty(\mathbb{R}), k \in L^1(\mathbb{R})$ and $\mathcal{F}(k)(y) \neq 0$ for $y \in \mathbb{R}$, where $\mathcal{F}(k)$ is the Fourier transform of k. If

$$\lim_{x\to\infty} \int_{\mathbb{R}} f(y)k(x-y)\,dy = A \int_{\mathbb{R}} k(y)\,dy$$

for a constant $A \in \mathbb{C}$, then for every $g \in L^1(\mathbb{R})$

$$\lim_{x\to\infty} \int_{\mathbb{R}} f(y)g(x-y)\,dy = A \int_{\mathbb{R}} g(y)dy.$$

Pitt's form of Wiener's theorem (cf. [21] and [26: V.10]) gives the behaviour of the function f as $x \to \infty$, with some additional assumptions on f.

Many generalizations of these basic results have been proved in the framework of harmonic analysis. The translation invariance and the completeness of translates in various subspaces or subalgebras are essentialy characterized by the Wiener-type Tauberian theorems [1] (cf. [1, 2, 5, 22]). On the other hand, Abelian and Tauberian theorems in the framework of generalized function spaces have been applied in various

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[1] Results in which we pass from a function to its transform are called Abelian; results in the converse – Tauberian.

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fields of analysis, probability and mathematical physics (cf. [7, 17, 25] and references therein). Note that generalized integral transforms have been elaborated for Schwartz generalized function spaces [17, 20, 25] while, at present, they are studied only in some particular spaces of ultradistributions and hyperfunctions.

The general concept of extending the given Wiener theorem is based on the extention of a generalized function space to whom f has to belong and the analysis of additional properties of k which will imply a Wiener-type Tauberian theorem in a new setting (cf. [7, 10, 15]). Essential extensions in this sense were obtained in [6, 18] for distributions and in [19] for ultradistributions.

In the present paper we prove two Wiener-type Tauberian theorems in the space of Fourier hyperfunctions $\mathcal{Q}(\mathbb{D}^n)$. We refer to Sato and a number of his pupils for the hyperfunction theory and the theory of various classes of differential operators in this framework (cf. [9, 23]). In general, it can be said that recently hyperfunctions play an important role in the quantum field theory extending the results of distribution theory realized already by Bogoljubov and colaborators [4], where Tauberian theorems are unavoidable (cf. [24, 28]). Here, as illustration, we mention that any Fourier hyperfunction is of the form $P(D)f$, where f is a slowly increasing continuous function and $P(D)$ is a local operator –

$$\left[e^{-\frac{1}{z}}\right] = \sum_{k=0}^{\infty} \frac{2\pi}{k!\,(k+1)!}\delta^{(k)}$$

(this is an ultradistribution but not distribution)

$$\cos\sqrt{\frac{d}{dx}}\ \delta = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\,\delta^{(k)}$$

(this is a hyperfunction but not ultradistribution)

are examples of Fourier hyperfunctions. Even these simple examples show that our Wiener-type Tauberian theorems, if hold for f, do not imply that a hyperfunction f is a distribution or an ultradistribution (cf. Section 4).

The space of Fourier hyperfunctions contain distributions and ultradistributions fulfilling the assumptions of Wiener-type Tauberian theorems for distributions and ultradistributions. Our results in a natural way extend and unify previous ones. Thus, a Fourier hyperfunction (whatever it is – distribution or ultadistribution or just a Fourier hyperfunction) can be tested through an appropriately chosen function k. It appears that one can take $k = \mathcal{K}$, where $\mathcal{K}$ is a kernel called here Hörmander kernel, introduced and exploited very much in [8: Chapters 8 - 9] in connection with the microlocal properties of ultradistributions and hyperfunctions. Here, we will show that this kernel is useful for the analysis of a hyperfuncion at infinity. Especially, if a local differential operator (this is in fact a pseudodifferential operator) $P(D)$ satisfies an estimate from below, then we can have the asymptotic behaviour of a solution to $P(D)u = f$ from the asymptotic behaviour of a hyperfunction f. At the end, we pose a question related to the heat kernel which would be of interest for further applications.

The analiticity of k as well as the behaviour of a Fourier hyperfunction f at ∞ make our investigations different in relation to methods which we used in our previous investigations.

2. Notation and definitions

We denote by $\mathbb{D}^n$ the directional compactification of $\mathbb{R}^n, \mathbb{D}^n = \mathbb{R}^n \cup S_\infty^{n-1}$ and supply it with the usual topology. The sheaf $\widetilde{\mathcal{O}}^{-\delta}$ $(\delta \geq 0)$ on $\mathbb{D}^n + i\mathbb{R}^n$ is defined as follows [11: p. 375]. For any open set $U \subset \mathbb{D}^n + i\mathbb{R}^n$, $\widetilde{\mathcal{O}}^{-\delta}(U)$ consists of holomorphic functions F on $U \cap \mathbb{C}^n$, i.e. elements F of $\mathcal{O}(U \cap \mathbb{C}^n)$ which satisfy

$$|F(z)| \leq C_{V,\varepsilon} \exp\big(-(\delta - \varepsilon)|\mathrm{Re}\, z|\big) \qquad (z \in V)$$

for any open set $V \subset \mathbb{C}^n$ with $\overline{V} \subset U$ and for every $\varepsilon > 0$. Hence, $\widetilde{\mathcal{O}}^{-\delta}|_{\mathbb{C}^n} = \mathcal{O}$.

Put $\widetilde{\mathcal{O}}^0(U) \equiv \widetilde{\mathcal{O}}(U)$. The derived sheaf $\mathcal{H}^n_{\mathbb{D}^n}(\widetilde{\mathcal{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on $\mathbb{D}^n$. We need only the space of global sections $\mathcal{Q}(\mathbb{D}^n)$.

Let $I_k \ni 0$ $(k = 1, ..., n)$ be open intervals, $I = I_1 \times ... \times I_n$ and $U_j = \big\{(\mathbb{D}^n + iI) \cap \{\mathrm{Im}\, z_j \neq 0\}\big\}$ $(j = 1, ..., n)$. The family $\{\mathbb{D}^n + iI, U_j : j = 1, ..., n\}$ gives a relative Leray covering for the pair $\{\mathbb{D}^n + iI, (\mathbb{D}^n + iI) \setminus \mathbb{D}^n\}$ relative to the sheaf $\widetilde{\mathcal{O}}$. Thus

$$\mathcal{Q}(\mathbb{D}^n) = \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big) \Big/ \sum_{j=1}^n \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#_j\mathbb{D}^n\big)$$

where

$$(\mathbb{D}^n + iI)\#\mathbb{D}^n = U_1 \cap ... \cap U_n$$
$$(\mathbb{D}^n + iI)\#_j\mathbb{D}^n = U_1 \cap ... \cap U_{j-1} \cap U_{j+1} \cap ... \cap U_n.$$

Similarly, $\mathcal{Q}^{-\delta}$ $(\delta > 0)$ is defined using $\widetilde{\mathcal{O}}^{-\delta}$ instead of $\widetilde{\mathcal{O}}$ (cf. [11: Definition 8.2.5]).

We denote by Γ a convex cone in $\mathbb{R}^n$. Open orthants in $\mathbb{R}^n$ are denoted by Γ_σ $(\sigma \in \Lambda)$, where Λ is the set of n-vectors with entries from $\{-1, 1\}$. A global section $f = [F] \in \mathcal{Q}(\mathbb{D}^n)$ is defined by the *defining function* $F \in \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big), F = (F_\sigma)$, where $F_\sigma \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI_\sigma); \mathbb{D}^n + iI_\sigma$ is an infinitesimal wedge of type $\mathbb{R}^n + i\Gamma_\sigma 0$ with $I_\sigma = I \cap \Gamma_\sigma$ $(\sigma \in \Lambda)$.

Recall the topological structure of $\mathcal{Q}(\mathbb{D}^n)$. Let $f = [F] \in \mathcal{Q}(\mathbb{D}^n)$ and $F \in \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$. Then by

$$P_{K,\varepsilon}(F) = \sup_{z \in \mathbb{R}^n + iK} |F(z)| \exp(-\varepsilon|\mathrm{Re}\, z|) \qquad (\varepsilon > 0, K \subset\subset I \setminus \{0\})$$

there is defined a family of semi-norms on $\widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$. The corresponding quotient topology is the topology on $\mathcal{Q}(\mathbb{D}^n)$. Note $\widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$ is a Fréchet and Montel space, hence so is $\mathcal{Q}(\mathbb{D}^n)$.

Let $f = [F] \in \mathcal{Q}(\mathbb{D}^n)$. Then it is convenient to associate to f a formal sum:

$$f(x) \cong \sum_{\sigma \in \Lambda} \mathrm{sgn}(\sigma) F_\sigma(x + i\Gamma_\sigma 0) \qquad \big(F_\sigma \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI_\sigma)\big) \tag{1}$$

(cf. [11: Theorem 8.5.3 and Definition 8.3.1]).

Recall, $\mathcal{P}_* = \operatorname{ind}\lim_{I\ni 0} \operatorname{ind}\lim_{\delta\downarrow 0} \widetilde{\mathcal{O}}^{-\delta}(\mathbb{D}^n + iI)$. (In order to simplify notation, if $\beta \in \mathcal{P}_*$, we also denote by β a representative of the class β.) $\mathcal{P}_*$ and $\mathcal{Q}(\mathbb{D}^n)$ are topologically dual to each other; for $f \in \mathcal{Q}(\mathbb{D}^n)$ and $\varphi \in \mathcal{P}_*$,

$$\langle f, \varphi\rangle = \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = \sum_{\sigma\in\Lambda} \operatorname{sgn}(\sigma) \int_{\mathbb{R}^n} (F_\sigma\varphi)(x + i\Gamma_\sigma 0)\,dx$$

(see [11: Theorem 8.6.2]) where f is of form (1). We will use

$$\mathcal{Q}(\mathbb{D}^n) \ni \mathbf{1} = \sum_{\sigma\in\Lambda} \operatorname{sgn}(\sigma)\mathbf{1}_\sigma \qquad \text{where } \mathbf{1}_\sigma = \frac{\operatorname{sgn}(\sigma)}{2^n}\kappa_{\mathbb{R}^n+i\Gamma_\sigma}, \tag{2}$$

$\mathbf{1}$ equals one on $\mathbb{C}^n$ and $\kappa_{\mathbb{R}^n+i\Gamma_\sigma}$ is the characteristic function of $\mathbb{R}^n + i\Gamma_\sigma$.

3. Wiener-type Tauberian theorems

The next proposition is given in [12] in more general situations. Here we adapt the assertion for our purposes.

Proposition 1. *If $f = [F] \in \mathcal{Q}(\mathbb{D}^n)$ and $\varphi \in \mathcal{P}_*$, then $f * \varphi \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI'')$ for an appropriate interval $I'' \subset \mathbb{R}^n$ with $I'' \ni 0$, i.e. $f * \varphi$ is a slowly increasing real analytic function.*

Proof. Let $f = [F] \in \mathcal{Q}(\mathbb{D}^n), F \in \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$. Recall,

$$(f * \varphi)(x) = \langle f, \check{\varphi}(\cdot - x)\rangle \qquad \big(x \in \mathbb{R}^n, \check{\varphi}(x) = \varphi(-x)\big).$$

We can always suppose that the existence of an $\alpha > 0$ such that $I = (-\alpha, \alpha)^n$ and

$$f \cong \sum_{\sigma\in\Lambda} \operatorname{sgn}(\sigma)F_\sigma(x + i\Gamma_\sigma 0), \quad I_\sigma = I \cap \Gamma_\sigma, \quad F_\sigma \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI_\sigma).$$

Since $\varphi \in \mathcal{P}_*$, there exist $\delta > 0$ and $\alpha' > 0$ such that $\varphi \in \widetilde{\mathcal{O}}^{-\delta}(\mathbb{D}^n + iI')$ with $I' = (-\alpha', \alpha')^n$. By [11: Proposition 8.4.3],

$$\begin{aligned}(f * \varphi)(x) &= \int_{\mathbb{R}^n} f(\xi)\varphi(x - \xi)\,d\xi \\ &= \sum_{\sigma\in\Lambda} \int_{\mathbb{R}^n} \operatorname{sgn}(\sigma)F_\sigma(\xi + i\eta^0_\sigma)\varphi(x - \xi - \eta^0_\sigma)\,d\xi\end{aligned} \qquad (x \in \mathbb{R}^n)$$

where $\eta^0_\sigma \in I_\sigma, \eta^0_\sigma = (\eta^0_{\sigma_1}, ..., \eta^0_{\sigma_n})$ and I'' are determined as follows:

Take a number η^0 such that $0 < \eta^0 = \frac{\alpha^0}{2}$ where $\alpha^0 = \min(\alpha, \alpha')$. We choose $\eta^0_\sigma \in I_\sigma$ such that $\eta^0_{\sigma_k} = \eta^0$ if $\eta^0_{\sigma_k} > 0$ and $\eta^0_{\sigma_i} = -\eta^0$ if $\eta^0_{\sigma_i} < 0$. Clearly, $\eta^0_\sigma \in I_\sigma \cap I'$. Now, we take $I'' = (-\alpha' + \eta^0, \alpha' - \eta^0)^n$. If $z = x + iy \in \mathbb{R}^n + iI''$ and $\zeta = \xi + i\eta^0_\sigma$ $(\xi \in \mathbb{R}^n)$, then $z - \zeta \in \mathbb{R}^n + iI'$, because $-\alpha' < y_k - \eta^0_{\sigma_k} \le 0$ if $\eta^0_{\sigma_k} > 0$ and $0 < y_i - \eta^0_{\sigma_i} < \alpha'$ if $\eta^0_{\sigma_i} < 0$.

We can now prove that $(f * \varphi)(z) \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI'')$. For this, let $\varepsilon > 0$ be chosen so that $0 < 2\varepsilon < \delta$ and $K'' \subset\subset I''$. Then $K'' - \eta_\sigma^0 \subset I'$ and using the inequalities $\varepsilon|x| + (\delta - \varepsilon)|\xi - x| \geq \varepsilon|x| + \varepsilon|\xi - x| \geq \varepsilon|\xi|$ we get

$$\begin{aligned} \sup_{z\in\mathbb{R}^n+iK''} & \left| e^{-\varepsilon|x|} \int_{\mathbb{R}^n} F_\sigma(\xi + i\eta_\sigma^0)\varphi(x - \xi + i(y - \eta_\sigma^0))\,d\xi \right| \\ & \leq C_{K''} \int_{\mathbb{R}^n} |F_\sigma(\xi + i\eta_\sigma^0)| e^{-\varepsilon|x|} e^{-(\delta-\varepsilon)|\xi - x|} d\xi \\ & \leq C_{K''} \int_{\mathbb{R}^n} |F_\sigma(\xi + i\eta_\sigma^0)| e^{-\varepsilon|\xi|} d\xi \\ & \leq C_{K''} C_\varepsilon \int_{\mathbb{R}^n} e^{-\varepsilon\frac{|\xi|}{2}} d\xi \\ & < \infty \end{aligned} \tag{3}$$

where $C_\varepsilon = \max_{\xi\in\mathbb{R}^n} |F_\sigma(\xi + i\eta_\sigma^0)| e^{-\varepsilon\frac{|\xi|}{2}}$. Since (3) holds for every $\sigma \in \Lambda$, it follows $(f * \varphi)(z) \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI'')$ ∎

In the sequal, we will use the notation $z^2 = \langle z, z\rangle = z_1 z_1 + \ldots + z_n z_n$ for $z \in \mathbb{C}^n$.

Proposition 2. *Let $\varphi \in \mathcal{P}_*$ and $\mathcal{F}(\varphi) = \psi$. Assume that there exists $\delta > 0$ such that $\frac{1}{\psi}\exp(-\delta\sqrt{\cdot^2 + 1}) \in \mathcal{P}_*$. Let M be the subspace of $\mathcal{P}_*$ consisting of all finite linear combinations of $\varphi(\cdot + x)$ $(x \in \mathbb{R}^n)$. Then M is dense in $\mathcal{P}_*$.*

Proof. Let $f \in \mathcal{Q}(\mathbb{D}^n)$. Note $\mathcal{F}(f)\exp(-\delta\sqrt{\cdot^2 + 1}) \in \mathcal{Q}^{-\delta}(\mathbb{D}^n)$ and $f * \varphi \in \mathcal{Q}(\mathbb{D}^n)$. By Proposition 1, $f * \varphi \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI'')$ with $I'' \ni 0$. We have to prove

$$\text{if } (f * \varphi)(x) = \langle f, \check{\varphi}(\cdot - x)\rangle = 0 \text{ for every } x \in \mathbb{R}^n, \qquad \text{then } f = 0.$$

By [11: Proposition 8.4.3] we have $0 = \mathcal{F}(f * \varphi) = \mathcal{F}(f)\psi$ in $\mathcal{Q}(\mathbb{D}^n)$. Since

$$0 = (\mathcal{F}(f)\psi)\Big(\frac{1}{\psi}\exp(-\delta\sqrt{\cdot^2 + 1})\Big) = \mathcal{F}(f)\Big(\psi\frac{1}{\psi}\Big)\exp(-\delta\sqrt{\cdot^2 + 1}),$$

it follows

$$\mathcal{F}(f)\exp(-\delta\sqrt{\cdot^2 + 1}) = 0 \qquad \text{in } \mathcal{Q}(\mathbb{D}^n).$$

By [11: Theorem 8.4.1] (or [12: Lemma 3.3]), the natural imbeding $\mathcal{Q}^{-\delta}(\mathbb{D}^n)$ into $\mathcal{Q}(\mathbb{D}^n)$ is injective and thus

$$\mathcal{F}(f)\exp(-\delta\sqrt{\cdot^2 + 1}) = 0 \qquad \text{in } \mathcal{Q}^{-\delta}(\mathbb{D}^n).$$

This implies $\mathcal{F}(f) = 0$ in $\mathcal{Q}(\mathbb{D}^n)$. By the properties of Fourier transform it follows that $f = 0$ in $\mathcal{Q}(\mathbb{D}^n)$ ∎

We shall use the following notation:

$h \to \infty$ for $h = (h_1, \ldots, h_n) \in \mathbb{R}^n, h_i \to \infty$ $(i = 1, \ldots, n)$

$\alpha_n \downarrow 0$ for a positive sequence which monotonically tends to zero as $n \to \infty$

$b_n \uparrow b$ for a sequence $b_n < b \le \infty$ which monotonically tends to b as $n \to \infty$. In order to form a basis of neighbourhoods of zero in $\mathcal{P}_*$ we note that

$$\mathcal{P}_* = \operatorname{ind} \lim_{\nu \to \infty} \widetilde{\mathcal{O}}^{-\delta_\nu}(\mathbb{D}^n + iI_\nu)$$

where $\delta_\nu \downarrow 0$ and $I_\nu = (-\alpha_\nu, \alpha_\nu)^n$ with $\alpha_\nu \downarrow 0$. We can take fixed sequences $\{\delta_\nu\}$ and $\{\alpha_\nu\}$. For a fixed ν, a basis of neighbourhoods of zero in $\widetilde{\mathcal{O}}^{-\delta_\nu}(\mathbb{D}^n + iI_\nu)$ is given by

$$V_{\nu,K_j,\varepsilon_k,\eta_p} = \Big\{\beta \in \widetilde{\mathcal{O}}^{-\delta_\nu}(\mathbb{D}^n + iI_\nu) : \sup_{z \in \mathbb{R}^n + iK_j} |\beta(z)| \exp((\delta_\nu - \varepsilon_k)|x|) \le \eta_p\Big\}$$

for $j, k, p \in \mathbb{N}$ where $K_j = [-b_j, b_j]^n, b_j \uparrow \alpha_\nu, \varepsilon_k \downarrow 0, \varepsilon_k < \delta_\nu$ and $\eta_p \downarrow 0$ are given sequences.

Let us choose for every $\nu \in \mathbb{N}$ sequences $b_{j,\nu} \uparrow \alpha_\nu, \varepsilon_{k,\nu} \downarrow 0$ $(\varepsilon_{k,\nu} < \delta_\nu)$ and $\eta_{p,\nu} \downarrow 0$. Fixing $K_{j_\nu,\nu} = [-b_{j_\nu,\nu}, b_{j_\nu,\nu}]^n, \varepsilon_{k_\nu,\nu}, \eta_{p_\nu,\nu}$ $(\nu \ge \nu_0)$ and making the absolutely convex linear hull of $V_{\nu,K_{j_\nu,\nu},\varepsilon_{k_\nu,\nu},\eta_{p_\nu,\nu}}$ $(\nu \ge \nu_0)$, denoted by

$$\Gamma_{\nu,K_{j_\nu,\nu}\varepsilon_{k_\nu,\nu},\eta_{p_\nu,\nu}} \qquad (\nu \ge \nu_0) \tag{4}$$

we obtain a neighbourhood of zero in $\mathcal{P}_*$. In this way we construct a basis of neighbourhoods of zero which will be used in the sequel.

Theorem 1. *Let $\varphi \in \mathcal{P}_*, \psi = \mathcal{F}(\varphi), f = [F] \in \mathcal{Q}(\mathbb{D}^n), F \in \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$ with $I = (-\alpha, \alpha)^n$ $(\alpha > 0)$ and let c be a positive and measurable function on $\mathbb{R}^n$. Assume the following:*

1. (i) $\lim_{x\to\infty} \frac{c(x)}{c(x+t)} = 1$ *for all* $t \in \mathbb{R}^n$.

(ii) *For every $\varepsilon > 0$ there exist constants $B > 0$ and $B_1 > 0$ such that $Be^{-\varepsilon|t|} \le \frac{c(x)}{c(x+t)} \le B_1 e^{\varepsilon|t|}$ for all $x, t \in \mathbb{R}^n$.*

2. *There exists $\omega > 0$ such that $\frac{1}{\psi(z)} e^{-\omega\sqrt{z^2+1}} \in \mathcal{P}_*$.*

3. *For every $\sigma \in \Lambda$, every compact set $K_\sigma \subset\subset I_\sigma = I \cap \Gamma_\sigma$ and for every $\eta > 0$, there exists $C > 0$ such that $\left|\frac{F_\sigma(x+h+iy_\sigma)}{c(h)}\right| \le Ce^{\eta|x|}$ for all $x \in \mathbb{R}^n, h \in \mathbb{R}^n_+$ and $y_\sigma \in K_\sigma$.*

4. *There exists $A \in \mathbb{C}$ such that $\lim_{h\to\infty} \frac{(f*\varphi)(x+h)}{c(h)} = \langle A, \varphi\rangle$ in $\mathcal{Q}(\mathbb{D}^n)$.*

Then

$$\lim_{h\to\infty} \frac{(f*\lambda)(x+h)}{c(h)} = \langle A, \lambda\rangle \qquad \text{in } \mathcal{Q}(\mathbb{D}^n), \text{ for every } \lambda \in \mathcal{P}_*. \tag{5}$$

Proof. `Step 1`. We shall prove the assertion with $c = 1$. Thus if $\lim_{h\to\infty}(f * \varphi)(x+h) = \langle A', \varphi\rangle$ in $\mathcal{Q}(\mathbb{D}^n)$ with $A' \in \mathbb{C}$, we have to prove that

$$\lim_{h\to\infty} (f*\lambda)(x+h) = \langle A', \lambda\rangle \qquad \text{in } \mathcal{Q}(\mathbb{D}^n), \text{ for every } \lambda \in \mathcal{P}_*. \tag{6}$$

We use the notation of Proposition 2. Let $\lambda \in \mathcal{P}_*, \lambda \in \widetilde{\mathcal{O}}^{-\delta}(\mathbb{D}^n + iI')$ with $I' = (-\alpha', \alpha')^n$. Assume that $\nu_0 \in \mathbb{N}$ is chosen so that $\delta_{\nu_0} \le \delta$ and $\alpha_{\nu_0} < \alpha'$. (Here we use the notation given before the theorem.) By Proposition 2, for given $\lambda \in \mathcal{P}_*$ there exist $\rho \in M$ and $\gamma \in \mathcal{P}_*$ such that $\lambda = \rho + \gamma$ and $\gamma \in \Gamma_{\nu, K_{j_\nu,\nu}, \varepsilon_{k_\nu,\nu}, \eta_{p_\nu,\nu}}$ (cf. (4)) which implies

$$\sup_{z \in \mathbb{R}^n + iK_{j_\nu,\nu}} |\gamma(z)| \exp\big((\delta_\nu - \varepsilon_{k_\nu,\nu})|x|\big) \le \eta_{p_\nu,\nu} \qquad (\nu \ge \nu_0). \tag{7}$$

By (1) and (2) we have

$$\begin{aligned}&\big((f - A') * (\lambda - \rho)\big)(z + h)\\ &\quad = \sum_{\sigma \in \Lambda} \int_{\mathbb{R}^n} \big(\operatorname{sgn}(\sigma) F_\sigma(\xi + h + i\eta^0_\sigma) - A'_\sigma\big)\gamma(z - \xi - i\eta^0_\sigma)\, d\xi\end{aligned}$$

for $z \in \mathbb{R}^n + iI''$ where I'' and η^0_σ are settled down in Proposition 1 and $A'_\sigma = A\mathbf{1}_\sigma$ $(\sigma \in \Lambda)$.

Assumption 4 and Proposition 2 imply that for the proof of (6) it is enough to show that for every $\sigma \in \Lambda, \mu > 0, \varepsilon > 0$ and $K \subset\subset I''$ there exist $h_0 > 0, \rho \in M, \gamma \in \mathcal{P}_*$ such that $\lambda = \rho + \gamma$ and

$$J(h) = \sup_{z \in \mathbb{R}^n + iK} e^{-\varepsilon|x|}\left| \int_{\mathbb{R}^n} \big(F_\sigma(\xi + h + i\eta^0_\sigma) - A'_\sigma\big)\gamma(z - \xi - i\eta^0_\sigma)\, d\xi \right| < \mu$$

for $h_i \ge h_0$ $(i = 1, ..., n)$. Let $\sigma \in \Lambda, \mu > 0, \varepsilon > 0$ and $K \subset\subset I''$ be fixed. Choose $\nu_0 > 0$ and $\eta > 0$, a compact set $K_{j_{\nu_0},\nu_0}$ and $\varepsilon_{k_{\nu_0},\nu_0} > 0$ such that

$$K_{j_{\nu_0},\nu_0} \supset K - \eta^0_\sigma, \quad \varepsilon_{k_{\nu_0},\nu_0} + \eta < \varepsilon, \quad 2\varepsilon_{k_{\nu_0},\nu_0} + \eta < \delta_{\nu_0}.$$

Then choose $\eta_{p_{\nu_0},\nu_0} > 0$ such that $\eta_{p_{\nu_0},\nu_0} < \frac{\mu}{DD_1}$ where

$$D = \sup_{h \in \mathbb{R}^n_+, \sigma \in \Lambda} e^{-\eta|\xi|}\big(|F_\sigma(\xi + h + i\eta^0_\sigma)| + |A'_\sigma|\big), \qquad D_1 = \int_{\mathbb{R}^n} e^{-\varepsilon_{k_{\nu_0},\nu_0}|\xi|}\, d\xi.$$

Note that $D < \infty$ (this follows from condition 3). We have

$$\begin{aligned}\varepsilon|x| + (\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|x - \xi| &\ge (\eta + \varepsilon_{k_{\nu_0},\nu_0})(|x| + |x - \xi|)\\ &\ge (\eta + \varepsilon_{k_{\nu_0},\nu_0})|\xi|.\end{aligned} \tag{8}$$

Take $\rho \in M$ such that

$$\sup_{z \in \mathbb{R}^n + iK_{j_{\nu_0},\nu_0}} \big|\gamma(z) \exp\big((\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|x|\big)\big| \le \eta_{p_{\nu_0},\nu_0}.$$

Then, by (7) and (8), we get $(z \in \mathbb{R}^n + iK)$

$$\begin{aligned}&e^{-\varepsilon|x|}\left| \int_{\mathbb{R}^n} \big(F_\sigma(\xi + h + i\eta^0_\sigma) - A'_\sigma\big)\gamma(x + iy - \xi - i\eta^0_\sigma)\, d\xi \right|\\ &\quad \le \eta_{p_{\nu_0},\nu_0} \int_{\mathbb{R}^n} \big(|F_\sigma(\xi + h + i\eta^0_\sigma)| + |A'_\sigma|\big) \exp\big(-\varepsilon|x| - (\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|x - \xi|\big) d\xi\\ &\quad \le \eta_{p_{\nu_0},\nu_0} \int_{\mathbb{R}^n} \big(|F_\sigma(\xi + h + i\eta^0_\sigma)| + |A'_\sigma|\big) \exp\big(-(\eta + \varepsilon_{k_{\nu_0},\nu_0})|\xi|\big) d\xi\\ &\quad \le \eta_{p_{\nu_0},\nu_0} DD_1\\ &\quad < \mu.\end{aligned}$$

`Step 2.` We introduce

$$e(z) = \int_{\mathbb{R}^n} c(t) e^{-(t-z)^2} dt \qquad (z \in \mathbb{C}^n).$$

This is an entire function, and by assumption 1/(ii) there exists a strip $I_{c_0} = \{z \in \mathbb{C}^n : |y| \le c_0\}$ $(c_0 > 0)$ so that $e(z) \ne 0$ for $z \in I_{c_0}$. Let us prove this assertion. For this we will use

$$\cos\left(\sum_{i=1}^n 2t_i y_i\right) = \prod_{i=1}^n \cos(2|t_i y_i|) + \sum_{j=1}^n M_j \sin(2t_j y_j)$$

where M_j are linear combinations of products of $\cos(2t_i y_i)$ or $\sin(2t_i y_i)$ $(i \ne j, j = 1, ..., n)$ and that for every $p \in (0,1)$ there exists $q > 0$ such that

$$\begin{aligned}
&\sum_{j=1}^n \left| \int_{\mathbb{R}^n} e^{-t^2+y^2+|t|} M_j \sin(2y_j t_j)\, dt \right| \\
&\quad \le 2^{n-1} \sum_{j=1}^n \int_{\mathbb{R}^n} e^{-t^2+y^2+|t|} |\sin(2y_j t_j)|\, dt \qquad (|y| \le q). \\
&\quad \le p
\end{aligned}$$

Let $|y| \le q$. We get

$$\begin{aligned}
|e(z)| &\ge |Re(e(z))| \\
&= \left| \int_{\mathbb{R}^n} c(x+t) e^{-t^2+y^2} \cos\left(\sum_{i=1}^n 2t_i y_i\right) dt \right| \\
&\ge \int_{\mathbb{R}^n} c(x+t) e^{-t^2+y^2} \cos(2|y_1|t_1) \cdots \cos(2|y_n|t_n)\, dt \\
&\quad - \sum_{j=1}^n \int_{\mathbb{R}^n} c(x+t) e^{-t^2+y^2} |M_j \sin(2t_j y_j)|\, dt \\
&\ge \frac{1}{B_1} c(x) \int_{t\in\Pi} e^{-t^2+y^2-|t|} dt - \frac{1}{B} c(x) \int_{t\in\mathbb{R}^n\setminus\Pi} e^{-t^2+y^2+|t|} dt \\
&\quad - \frac{2^{n-1}}{B} c(x) \sum_{j=1}^n \int_{\mathbb{R}^n} e^{-t^2+y^2+|t|} |\sin(2t_j y_j)|\, dt \\
&= c(x) \Big\{ A(|y|) - A_1(|y|) \frac{-2^{n-1} p}{B} \Big\}
\end{aligned}$$

where

$$\Pi = \Big\{ t : |t_i| \le \frac{\pi}{8|y_i|}\ (i = 1, ..., n) \Big\},$$

and the observation that $A(|y|) > 0$, $A(|y|)$ grows as $|y| \to 0$, while $A_1(|y|) \to 0$ as $|y| \to 0$. Thus, by choosing p sufficiently small and then by taking sufficiently small $c_0 \le q$, it follows that $\frac{1}{e}$ is an analitic function in the strip I_{c_0} and

$$|e(x+iy)| \ge C c(x) \qquad (x \in \mathbb{R}^n, |y| < c_0)$$

where $C = A(c_0) - A_1(c_0) - \frac{2^{n-1}p}{B}$. Shrincking I if necessary we can hereafter assume that c_0 is chosen so that $I \subset I_{c_0}$. Notice that condition 1/(ii) implies for every $\varepsilon > 0$ the existence of constants $B_2 > 0$ and $B_3 > 0$ so that

$$B_2 e^{-\varepsilon|x|} \le e(x+iy) \le B_3 e^{\varepsilon|x|} \qquad (x \in \mathbb{R}^n, y \in I).$$

Thus $e, \frac{1}{e} \in \widetilde{\mathcal{O}}(\mathbb{D}^n + iI)$. Moreover, the given estimates for e imply for every $\varepsilon > 0$ the existence of a constant $C > 0$ such that

$$\Big|\frac{c(h)}{e(x+h+iy)}\Big| \le C e^{\varepsilon|x|} \qquad (x \in \mathbb{R}^n, h \in \mathbb{R}^n_+, y \in I). \tag{9}$$

We need also

$$\lim_{x\to\infty} \frac{e(x+iy)}{c(x)} = \int_{\mathbb{R}^n} \exp(-(t-iy)^2)\, dt = \pi^{\frac{n}{2}} \qquad (y \in I). \tag{10}$$

Note, the Cauchy formula implies that the integral in (10) does not depend on $y \in I$. Let us prove (10). Indeed, letting $y \in I$ for this, we have

$$\begin{aligned}
\lim_{x\to\infty} \frac{e(x+iy)}{c(x)} &= \lim_{x\to\infty} \int_{\mathbb{R}^n} \exp(-(t-x-iy)^2)\frac{c(t)}{c(x)}\, dt \\
&= \lim_{x\to\infty} \int_{\mathbb{R}^n} \exp(-(t-iy)^2)\frac{c(x+t)}{c(x)}\, dt \\
&= \int_{\mathbb{R}^n} \exp(-(t-iy)^2) \lim_{x\to\infty} \frac{c(x+t)}{c(x)}\, dt \\
&= \int_{\mathbb{R}^n} \exp(-(t-iy)^2)\, dt \\
&= \int_{\mathbb{R}^n} e^{-t^2} dt \\
&= \pi^{\frac{n}{2}}.
\end{aligned}$$

`Step 3`. Let $y^0_\sigma \in I_\sigma$ and $f_1 = \pi^{\frac{n}{2}} f$. Note that $\frac{f_1}{e} \in \mathcal{Q}(\mathbb{D}^n)$. We will prove that, for every $\gamma \in \mathcal{P}_*$,

$$\lim_{h\to\infty} \frac{(f*\gamma)(x+h)}{c(h)} = \lim_{h\to\infty} \Big(\frac{f_1}{e} * \gamma\Big)(x+h) \tag{11}$$

$$\lim_{h\to\infty} \Big(\frac{f_1}{e} * \gamma\Big)(x+h) = \langle A, \gamma\rangle \tag{12}$$

in $\mathcal{Q}(\mathbb{D}^n)$. This will complete the proof of Theorem 1. For this rewrite (11) as

$$\lim_{h\to\infty} \Big(\frac{f(\cdot+h)}{e(\cdot+h)}\Big(\frac{e(\cdot+h)}{c(h)} - \pi^{\frac{n}{2}}\Big) * \gamma\Big)(x) = 0 \qquad \text{in } \mathcal{Q}(\mathbb{D}^n). \tag{13}$$

We will prove (13) similarly as (6). It is sufficient to prove that, for every $\sigma \in \Lambda$ and $\varepsilon > 0$, $\lim_{h\to\infty} H_{\sigma\gamma}(h) = 0$ $(\gamma \in \mathcal{P}_*)$ where

$$H_{\sigma\gamma}(h) = \sup_{z\in\mathbb{R}^n+iK} e^{-\varepsilon|x|} \int_{\mathbb{R}^n} F_\sigma(\xi+h+i\eta_\sigma^0)\Big(\frac{1}{c(h)} - \frac{\pi^{\frac{n}{2}}}{e(\xi+h+i\eta_\sigma^0)}\Big)\gamma(z-\xi-i\eta_\sigma^0)\,d\xi.$$

We have

$$\begin{aligned} H_{\sigma\gamma}(h) &= \sup_{z\in\mathbb{R}^n+iK} e^{-\varepsilon|x|}\int_{\mathbb{R}^n} \frac{F_\sigma(\xi+h+i\eta_\sigma^0)}{c(h)}\frac{c(h)}{e(\xi+h+iy_\sigma^0)} \\ &\quad\times\Big(\int_{\mathbb{R}^n} e^{-(t-i\eta_\sigma^0)^2}\Big(\frac{c(t+\xi+h)}{c(h)}-1\Big)dt\Big)\gamma(z-\xi-i\eta_\sigma^0)\,d\xi \\ &= \sup_{z\in\mathbb{R}^n+iK} e^{-\varepsilon|x|}\int_{\mathbb{R}^n} e^{-(t-i\eta_\sigma^0)^2} \\ &\quad\times\Big(\int_{\mathbb{R}^n}\frac{F_\sigma(\xi+h+i\eta_\sigma^0)}{c(h)}\frac{c(h)}{e(\xi+h+iy_\sigma^0)}\Big(\frac{c(t+\xi+h)}{c(h)}-1\Big) \\ &\quad\times\gamma(z-\xi-i\eta_\sigma^0)\,d\xi\Big)dt. \end{aligned}$$

The order of integration is changed on the basis of estimates which are to follow. For these estimates we use assumption 1/(ii) (with $\varepsilon_{k_{\nu_0},\nu_0}$ instead of ε in the exponent), (9) (again with $\varepsilon_{k_{\nu_0},\nu_0}$ instead of ε in the exponent) and the assumptions on δ_{ν_0} and $\varepsilon_{k_{\nu_0},\nu_0}$

$$\varepsilon \geq \eta + 2\varepsilon_{k_{\nu_0},\nu_0}, \qquad \delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0} \geq \eta + 2\varepsilon_{k_{\nu_0},\nu_0}.$$

We get

$$\begin{aligned} &\Big|\exp\big(-\varepsilon|x| - (t-i\eta_\sigma^0)^2\big)\frac{F_\sigma(\xi+h+i\eta_\sigma^0)}{c(h)}\Big| \\ &\quad\times\Big|\frac{c(h)}{e(\xi+h+iy_\sigma^0)}\Big(\frac{c(t+\xi+h)}{c(h)}-1\Big)\gamma(z-\xi-i\eta_\sigma^0)\Big| \\ &\leq E\exp\Big(-\varepsilon|x| - t^2 + \eta|\xi| + \varepsilon_{k_{\nu_0},\nu_0}(|t|+|\xi|) - (\delta_{\nu_0}-\varepsilon_{k_{\nu_0},\nu_0})|x-\xi|\Big) \\ &\leq \ldots \text{ and by (8)} \\ &\leq E\exp\big(-t^2 + \varepsilon_{k_{\nu_0},\nu_0}|t| - \varepsilon_{k_{\nu_0},\nu_0}|\xi|\big) \end{aligned}$$

where E is a suitable positive constant. Then the Lebuesgues theorem implies that $\lim_{h\to\infty} H_{\sigma\gamma}(h) = 0$.

Now we prove (12). We will show that, for every $\sigma \in \Lambda$, $\widetilde{F}_\sigma = \frac{F_\sigma}{e}$ satisfies assumption 3 with $c = 1$. Let $K_\sigma \subset\subset I_\sigma$ and $y_\sigma \in K_\sigma$ $(\sigma \in \Lambda)$. Then, for $h \in \mathbb{R}^n_+$,

$$\begin{aligned} |\widetilde{F}_\sigma(x+h+iy_\sigma)| &= \Big|\frac{F_\sigma}{e}(x+h+iy_\sigma)\Big| \\ &\leq \Big|\frac{\widetilde{F}_\sigma(x+h+iy_\sigma)}{c(h)}\Big|\Big|\frac{c(h)}{e(x+h+iy_\sigma)}\Big| \end{aligned} \qquad (x \in \mathbb{R}^n).$$

By assumption 3 of the theorem and by (9), we obtain

$$|\widetilde{F}_\sigma(x+h+iy_\sigma)| \le C' e^{(\eta+\varepsilon)|x|} \qquad (x \in \mathbb{R}^n)$$

for every $\eta > 0$ and $\varepsilon > 0$ and a suitable constant $C' > 0$. This completes the proof of Theorem 1 ∎

In the next theorem we will suppose a stronger assumption than assumption 3 and that the limit in assumption 4 exists in $\mathbb{C}$.

Theorem 2. *Let $\varphi \in \mathcal{P}_*, \psi = \mathcal{F}(\varphi)$ and $f = [F] \in \mathcal{Q}$, $F \in \widetilde{\mathcal{O}}\big((\mathbb{D}^n + iI)\#\mathbb{D}^n\big)$ with $I = (-\alpha,\alpha)^n$ for $\alpha > 0$. Let c and ψ satisfy assumptions* 1 *and* 2 *of Theorem* 1. *Further, assume the following:*

3. *For every $K_\sigma \subset\subset I_\sigma$ $(\sigma \in \Lambda)$ there exists $N > 0$ such that, for every $y_\sigma \in K_\sigma$, $|\frac{F_\sigma(x+h+iy_\sigma)}{c(h)}| \le N$ for all $x \in \mathbb{R}^n$ and $h \in \mathbb{R}^n_+$.*

4. *There exists $A \in \mathbb{C}$ such that $\lim_{x\to\infty} \frac{(f*\varphi)(x)}{c(x)} = \langle A, \varphi\rangle$ in $\mathbb{C}$.*

Then

$$\lim_{x\to\infty} \frac{(f*\lambda)(x)}{c(x)} = \langle A,\lambda\rangle \qquad \textit{in } \mathbb{C},\ \textit{for every } \lambda \in \mathcal{P}_*. \tag{14}$$

Proof. The structure of the proof of Theorem 2 is the same as for Theorem 1. We have only to take care that the limits are not in $\mathcal{Q}$ but in $\mathbb{C}$. In the first step we take $c = 1$. We use the fact that

$$\lim_{x\to\infty} (f*\lambda)(x) - \langle A',\lambda\rangle = 0 \qquad \text{in } \mathbb{C},\ \text{for every } \lambda \in \mathcal{P}_*$$

is equivalent to

$$\lim_{x\to\infty} \big((f-A')(x) * (\lambda-\rho)\big)(x) = 0 \qquad \text{in } \mathbb{C},\ \text{for every } \rho \in M \text{ and } \lambda \in \mathcal{P}_*.$$

We have

$$(f-A')*(\lambda-\rho) = \sum_{\sigma\in\Lambda} \int_{\mathbb{R}^n} \big(\operatorname{sgn}(\sigma)F_\sigma(\xi+i\eta^0_\sigma) - A'_\sigma\big)\gamma(x-\xi-i\eta^0_\sigma)\,d\xi$$

for every $x \in \mathbb{R}^n$, where $\gamma = \lambda - \rho$. (A'_σ is the same as in Theorem 1.) Proposition 1 implies that the functions

$$g_\sigma(\gamma,x) = \int_{\mathbb{R}^n} \big(\operatorname{sgn}(\sigma)F_\sigma(\xi+i\eta^0_\sigma) - A'_\sigma\big)\gamma(x-\xi-i\eta^0_\sigma)\,d\xi \qquad (x\in\mathbb{R}^n, \sigma\in\Lambda)$$

are slowly increasing. By Carleman's theorem (cf. [11: Lemma 8.4.7]),

$$\big((f-A')*(\lambda-\rho)\big)(x) = \sum_{\sigma\in\Lambda} g_\sigma(\gamma,x) \qquad (x\in\mathbb{R}^n).$$

We shall show that for every $\mu > 0$ and $\sigma \in \Lambda$ there exist $x_0 > 0, \rho \in M$ and $\gamma \in \mathcal{P}_*$, $\lambda = \rho + \gamma$ such that

$$|g_\sigma(\gamma, x)| \le \mu \qquad (x_i \ge x_0, i = 1, ..., n). \tag{15}$$

Take $\rho \in M$ such that γ satisfies (7) with the same notation and assumptions on $K_{j_{\nu_0},\nu_0}$ and $\varepsilon_{k_{\nu_0},\nu_0} > 0$. Now chose $\eta_{p_{\nu_0},\nu_0} < \frac{\mu}{D'}$, where

$$D' = (N + |A'_\sigma|) \int_{\mathbb{R}^n} \exp\big(-(\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|t|\big) dt.$$

The proof of (15) is to follow. For every $\sigma \in \Lambda$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\big(-(\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|t|\big)\big(|F_\sigma(x - t + i\eta^0_\sigma)| + |A'_\sigma|\big) \\ &\qquad \times \sup_{t\in\mathbb{R}^n} \big|\exp\big((\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|t|\big)\gamma(t - iy^0_\sigma)\big| dt \\ &\le \eta_{p,n_m}(N + |A'_\sigma|) \int_{\mathbb{R}^n} \exp\big(-(\delta_{\nu_0} - \varepsilon_{k_{\nu_0},\nu_0})|t|\big) dt \\ &< \mu \end{aligned}$$

for $x_i \ge x_0 \quad (i = 1, ..., n)$. Let e be the function defined in Step 2 of the proof of Theorem 1 and $f_1 = \pi^{\frac{n}{2}} f$. We shall prove that, for every $\gamma \in \mathcal{P}_*$,

$$\lim_{x\to\infty} \frac{(f * \gamma)(x)}{c(x)} = \lim_{x\to\infty} \Big(\frac{f_1}{e} * \gamma\Big)(x) \tag{16}$$

$$\lim_{x\to\infty} \Big(\frac{f_1}{e} * \gamma\Big)(x) = \langle A, \gamma\rangle \ \text{ in } \mathbb{C}. \tag{17}$$

First, we prove (16). The properties of e imply that (16) can be rewritten as

$$\lim_{x\to\infty} \Big(f(\cdot + x)\Big(\frac{1}{c(x)} - \frac{\pi^{\frac{n}{2}}}{e(\cdot + x)}\Big) * \gamma\Big)(0) = 0. \tag{18}$$

Let $\gamma \in \widetilde{\mathcal{O}}^{-\delta}(\mathbb{D}^n + iI')$. In order to prove (18), we shall prove that, for every $\eta^0_\sigma \in I_\sigma \quad (\sigma \in \Lambda)$,

$$\begin{aligned} \lim_{x\to\infty} G_\sigma(x) &= \lim_{x\to\infty} \int_{\mathbb{R}^n} F_\sigma(x + \xi + i\eta^0_\sigma)\Big(\frac{1}{c(x)} - \frac{\pi^{\frac{n}{2}}}{e(x + \xi + i\eta^0_\sigma)}\Big)\gamma(-\xi - i\eta^0_\sigma)\, d\xi \\ &= 0. \end{aligned}$$

We have

$$\begin{aligned} G_\sigma(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-(t - i\eta^0_\sigma)^2} \frac{F_\sigma(x + \xi + i\eta^0_\sigma)}{c(x)} \frac{c(x)}{e(x + \xi + i\eta^0_\sigma)} \\ \times \Big(\frac{c(x + \xi + t)}{c(x)} - 1\Big)\gamma(-\xi - i\eta^0_\sigma)\, d\xi dt. \end{aligned}$$

Now, as in the previous theorem, using assumptions 1 and 3 and (9), one can show that $\lim_{x\to\infty} G_\sigma(x) = 0$ for $\sigma \in \Lambda$.

Let us prove (17) by showing that $\widetilde{F} = \frac{F}{e}$ satisfies assumption 3 with $c = 1$. With this, the first part of the proof implies (17). We have

$$\begin{aligned} |\widetilde{F}(x+iy_\sigma)| &= \left|\frac{F_\sigma(x+iy_\sigma)}{e(x+iy_\sigma)}\right| \\ &= \left|\int_{\mathbb{R}^n} e^{-(t-x-iy_\sigma)^2}\frac{F_\sigma(x+iy_\sigma)}{c(t)}\,dt\right| \qquad (y_\sigma \in K_\sigma). \\ &\le C_{K_\sigma}\int_{\mathbb{R}^n} e^{-(t-iy_\sigma)^2}\frac{F_\sigma(x+iy_\sigma)}{c(x)}\frac{c(x)}{c(t+x)}\,dt. \end{aligned}$$

Assumptions 1 and 3 imply the boundedness of the last integral. Now (14) follows from (16) and (17) ∎

4. Applications

1. Recall [3], a function L is called slowly varying if it is a positive measurable function on (t_0,∞) $(t_0 \ge 0)$ such that $\lim_{x\to\infty}\frac{L(xt)}{L(x)} = 1$ $(t > t_0)$. If $n = 1$, then assumption 1/(ii) follows from assumption 1/(i). This follows from [3: Theorems 1.4.1 and 1.5.6] (with the change of variables $x = \ln u, u > 0$). In fact, in this case $c(x) = e^{\alpha x}L(e^x)$ $(x > x_0)$ (cf. [17]). The function

$$c(x) = (x_1^2+1)^{p_1}L_1(e^{x_1})\cdots(x_n^2+1)^{p_n}L_n(e^{x_n})$$

for $x = (x_1,...,x_n) \in \mathbb{R}^n$, where L_i $(i = 1,...,n)$ are slowly varying functions and $p = (p_1,...,p_n) \in \mathbb{R}^n$, satisfies assumption 1 of Theorems 1 and 2.

2. Let $\varphi_\delta = \mathcal{F}^{-1}(\exp(-\delta\sqrt{\cdot^2+1}))$. Since $\exp(-\delta\sqrt{\cdot^2+1}) \in \mathcal{P}_*$ $(\delta > 0)$, it follows that $\varphi_\delta \in \mathcal{P}_*$. One can simply show that $\psi = \varphi_\delta$ satisfies assumption 2 of Theorems 1 and 2. Another important function which satisfies assumption 2 in Theorems 1 and 2 is the Fourier transformation of the function $\mathcal{K}$ introduced by Hörmander (cf. [8: Section 8.4]):

$$\mathcal{K}(z) = \tfrac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{e^{i\langle z,\xi\rangle}}{I(\xi)}\,d\xi \qquad \big(z \in \Omega = \{z \in \mathbb{C}^n : |\mathrm{Im}\, z| < 1\}\big)$$

where $I(\xi) = \int_{|\omega|=1} e^{-\langle\omega,\xi\rangle}d\omega$. Recall, $I(\xi) = I_0(\langle\xi,\xi\rangle^{1/2})$ $(\xi \in \mathbb{R}^n)$ where

$$I_0(\rho) = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_{-1}^{1}(1-t^2)^{\frac{n-1}{2}-1}e^{-t\rho}dt \qquad (\rho \in \mathbb{C})$$

is an entire function which satisfies the estimate

$$|I_0(\rho)| \le C(1+|\rho|)^{-\frac{n-1}{2}}e^{|\mathrm{Re}\,\rho|} \qquad (\rho \in \mathbb{C}) \tag{19}$$

(cf. [8: Lemma 8.4.9]).

We shall prove that $\mathcal{K} \in \mathcal{P}_*$. By [8: Lemma 8.4.10], $\mathcal{K}$ is analytic in every open connected set Ω satisfying $\Omega \subset \widetilde{\Omega} = \{z \in \mathbb{C}^n : \langle z, z\rangle \notin (-\infty, -1]\}$. One can simply prove that the strip $\Omega = \{z \in \mathbb{C}^n : |y_k| < \frac{1}{2\sqrt{n}} \;\; (k = 1, ..., n)\}$ is a subset of $\widetilde{\Omega}$. Let Γ be a closed cone such that if $z \in \Gamma \setminus \{0\}$, then $|x_k| > |y_k| \;\; (k = 1, ..., n)$. If $z \in \Gamma \setminus \{0\}$, then $\langle z, z\rangle \notin (-\infty, 0]$. By [7: Lemma 8.4.10], there exists $c > 0$ such that $\mathcal{K}(z) = O(e^{-c|z|}) \;\; (z \in \Gamma, |z| \to \infty)$. Hence,

$$|\mathcal{K}(z)| \le C_{K,\varepsilon} e^{-(c-\varepsilon)|\mathrm{Re}\, z|} \qquad (z \in \mathbb{R}^n + iK)$$

for every compact set $K \subset \{y \in \mathbb{R}^n : |y_k| < \frac{1}{2\sqrt{n}} \;\; (k = 1, ..., n)\}$ and every $\varepsilon > 0$. Consequently, $\mathcal{K} \in \mathcal{P}_*$.

We denote by ψ the Fourier transform of $\mathcal{K}$, i.e. $\psi = \mathcal{F}(\mathcal{K})$, and let $\omega > 1$. Then $\frac{1}{\psi(\zeta)} = I(\zeta) \;\; (\zeta \in \mathbb{C})$ is an entire function. Let $\zeta = \xi + i\eta$ with $|\eta| < 1$. Then

$$|I(\zeta)| = \left| \int_{|\omega|=1} e^{-\langle \omega, \xi + i\eta\rangle} d\xi \right| = I(\xi) = I_0(\xi).$$

Now, by (19),

$$|I(\zeta)| \le |I(\xi)| \le |I_0(|\zeta|)| \le C(1 + |\zeta|)^{-\frac{n-1}{2}} e^{|\xi|+1} \qquad (\xi \in \mathbb{R}^n, |\eta| < 1).$$

This implies $I(\zeta)\exp(-\omega\sqrt{\zeta^2 + 1}) \in \mathcal{P}_*$ for $w > 1$.

3. The S-asymptotics of Fourier hyperfunctions can be defined in the following way: Suppose that c is a positive function defined on $\mathbb{R}^n$ and $f \in \mathcal{Q}(\mathbb{D}^n)$. It is said that f has the S-asymptotics related to c with the limit $u \in \mathcal{Q}(\mathbb{D}^n)$ if

$$\lim_{x\to\infty} \left\langle \frac{f(t+x)}{c(x)}, \varphi(t) \right\rangle = \langle u, \varphi\rangle \qquad \text{in } \mathbb{C}, \text{ for every } \varphi \in \mathcal{P}_*.$$

Theorem 2 asserts that if $f, \check{\varphi}$ and c satisfy the assumptions of Theorem 2 and if

$$\lim_{x\to\infty} \left\langle \frac{f(\cdot + x)}{c(x)}, \check{\varphi} \right\rangle = \langle A, \check{\varphi}\rangle \qquad \text{in } \mathbb{C},$$

then f has the S-asymptotics related to c with the limit A. Thus if $c = 1$ and $f \in \mathcal{Q}$ satisfies $(f * \mathcal{K})(x) \to A$ as $x \to \infty$, then f has the S-asymptotics at ∞ (equals to A).

By Theorem 1 one can easily show that hyperfunctions quoted in the introduction have the S-asymptotics zero with respect to $c = 1$. This simply shows that a hyperfunction with the S-asymptotics behaviour with respect to $c = 1$ is not a distribution or an ultradistribution, in general.

4. Let $P(D) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha \frac{\partial^\alpha}{\partial x^\alpha}$ be a microlocal operator with constant coefficients, which means $\lim_{|\alpha|\to\infty} \sqrt{|a_\alpha|\alpha!} = 0$, acting as a sheaf homomorphism on the sheaf of Fourier hyperfunctions (cf. [11: Proposition 8.4.8]). The question is whether the S-asymptotic behaviour of $P(D)f$ determines the S-asymptotics of f. Since

$$P(D)f * \mathcal{K} = f * (P(D)\mathcal{K}),$$

we have the following assertion:

Assume that

$$|P(\zeta)| \geq C \exp(-\delta\sqrt{\zeta^2+1}) \qquad \text{for some } C > 0 \text{ and } \delta > 0$$

in some strip $\mathbb{R}^n + i(-\varepsilon,\varepsilon)\times\ldots\times(\varepsilon,\varepsilon)$ $(\varepsilon > 0)$. *Then* $P(D)\mathcal{K}$ *satisfies the assymption of Theorem* 1 *and the behaviour of* $P(D)f * \mathcal{K}$ *determines the behaviour of* f *in the sense of* S*-asymptotics with* $c = 1$.

5. It would be interesting to answer the following problem. Whether the Weierstrass kernel $\sqrt[n]{4\pi t}\exp(-\frac{x^2}{4t})$ can be used as an appropriate element of $\mathcal{P}_*$ since it does not satisfy assumption 2 in both theorems. This theoretical problem is of interest for the analysis of the heat equation.

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