Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 1, 3–16

Bernstein's 'Lethargy' Theorems in SF-Spaces

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Abstract. We prove a version of Bernstein's 'lethargy' theorem for cones in the class of SF-spaces. Moreover, an analogue of the Bernstein theorem for linear projections onto closed subspaces of an SF-space is obtained. This extends results given by G. Lewicki.

Keywords: *Bernstein's 'lethargy' theorem, SF-spaces, Orlicz-Musielak spaces, best approximations, linear projections*

AMS subject classification: Primary 41A65, 46E30, secondary 46A16

1. Introduction

The Bernstein 'lethargy' theorem, one of the most important results in the constructive theory of functions, reads as follows.

Theorem 1.1. *Let X be a Banach space and let $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq X$ be an ascending sequence of distinct finite-dimensional linear subspaces of X. Then for every real sequence $\varepsilon_n \downarrow 0$ there exists $x \in X$ such that $\|x\| = \varepsilon_1$ and* $\operatorname{dist}(x, V_n) = \varepsilon_n$ *for all $n \geq 1$.*

This result was first obtained by Bernstein for $X = C_{\mathbb{R}}[0,1]$ and $V_n = P_n$, where $C_{\mathbb{R}}[0,1]$ denotes the space of all continuous real functions on [0,1] equipped with the supremum norm and P_n is the space of all real polynomials of degree at most n (see [3]). Later, other versions of the 'lethargy' theorem were proved (see, e.g., [2, 5, 9, 11 - 14]). This theorem became a very useful tool in the theory of quasianalytic functions of several complex variables (see [10]).

In [7], Lewicki proved the following, more general version of Bernstein's 'lethargy' theorem in the class of SF-spaces, which includes all F-spaces and Orlicz-Musielak spaces with the condition Δ_2 (basic definitions and notation of SF-spaces are given in Section 2).

Theorem 1.2. *Let (X, N) be an SF-space, let $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq X$ be an ascending sequence of distinct finite-dimensional subspaces of X and let $R_N(\cup_{n=1}^{\infty} V_n) > 0$. Then for every real sequence $\varepsilon_n \downarrow 0$ there exist $n_0 \in \mathbb{N}$ and $x \in X$ such that* $\operatorname{dist}_N(x, V_n) = \varepsilon_n$ *for all $n \geq n_0$.*

In the present note, using a similar technique, we obtain an analogue of the Bernstein theorem (Theorem 3.1), replacing the sequence of finite-dimensional linear subspaces of an SF-space by a sequence of closed cones which generate such subspaces (the

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ISSN 0232-2064 / $ 2.50

case of cones has not been discussed before). We also present a version of Bernstein's 'lethargy' theorem, in which projections onto closed subspaces of an SF-space replace best approximation operators (Theorem 4.3). This generalizes [6: Proposition 3].

2. SF-Spaces: definitions and basic properties

In this section we have compiled some basic facts on SF-spaces, following [7]. We start with

Definition 2.1. Let X be a linear space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) and let $N : X \to [0, \infty)$ be a function satisfying the following conditions:

$$N(x) = 0 \text{ if and only if } x = 0 \tag{2.1}$$

$$N(a_n x_n - ax) \to 0 \text{ for all sequences } \{a_n\} \subset \mathbb{K} \text{ and } \{x_n\} \subset X \text{ such that } a_n \to a \text{ and } N(x_n - x) \to 0 \tag{2.2}$$

$$N(x_n + y_n - x - y) \to 0 \text{ for all sequences } \{x_n\}, \{y_n\} \subset X \text{ such that } N(x_n - x) \to 0 \text{ and } N(y_n - y) \to 0 \tag{2.3}$$

$$N(ax) = N(x) \text{ for all } x \in X \text{ and } a \in \mathbb{K} \text{ such that } |a| = 1 \tag{2.4}$$

$$N(x_n) \to N(x) \text{ for every sequence } \{x_n\} \subset X \text{ such that } N(x_n - x) \to 0. \tag{2.5}$$

Then N is called an SF-*norm*. If, moreover, the space X is complete with respect to the topology induced by the family $\{K(x,r)\}_{x \in X, r>0}$, where $K(x,r) = \{y \in X : N(y-x) < r\}$, then the pair (X, N) is said to be an SF-*space*. If $N(a_1 x) \geq N(a_2 x)$ when $x \in X$ and $|a_1| \geq |a_2|$, then N is a *non-decreasing SF-norm*.

Remark 2.2. Every SF-space is a complete metrizable topological linear space (as a Hausdorff space with a countable basis of neighbourhoods of 0; see, e.g., [4]). Consequently, all finite-dimensional subspaces of an SF-space are closed.

It is obvious that all F-spaces (in particular, all Banach spaces) are SF-spaces.

Example 2.3. Let (Ω, Σ, μ) be a measurable space and let $f : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a φ-function with a parameter, i.e. f satisfies the following properties:

(φ1) For every $t \in \Omega$, $f(t, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing, continuous function such that $f(t, 0) = 0$ and $f(t, x) > 0$ for $x > 0$.

(φ2) For every $x \in \mathbb{R}_+$, $f(\cdot, x) : \Omega \to \mathbb{R}_+$ is a Σ-measurable function.

We denote by M the set of all $\mathbb{K}$-valued Σ-measurable functions defined on Ω, with equality μ-almost everywhere. Set

$$\rho_f(x) = \int_\Omega f(t, |x(t)|)\, d\mu(t) \qquad (x \in M).$$

Then ρ_f is the Orlicz-Musielak modular given by f, and the corresponding modular space X_{ρ_f} will be called Orlicz-Musielak space. In [7] Lewicki showed that an Orlicz-Musielak space $(X_{\varrho_f}, \varrho_f)$ is an SF-space with respect to the modular ϱ_f if the generating function f is locally integrable and satisfies the condition Δ_2 (for definitions and basic properties of modular spaces we refer the reader to [8]).

Now we present a sequence of lemmas concerning SF-spaces which shall be needed in the following sections.

Lemma 2.4 (see [7: Lemma 3.3]). *Let (X, N) be an SF-space and put*

$$N_1(x) = \sup \{N(tx) : t \in [0,1]\} \qquad (x \in X).$$

Then (X, N_1) is an SF-space, N_1 is non-decreasing, and for every sequence $\{x_n\} \subset X$ we have

$$N_1(x_n) \to 0 \text{ if and only if } N(x_n) \to 0. \tag{2.6}$$

Definition 2.5. Let (X, N) be an SF-space. If $Y \subset X$ and $Y \setminus \{0\} \neq \emptyset$, then we define the *radius* of the set Y by

$$R_N(Y) = \inf \Big\{ \sup\{N(ty) : t \geq 0\} : y \in Y \setminus \{0\} \Big\} \in [0, +\infty].$$

In the general case it may occur that $R_N(Y) = 0$ (see [7: Examples 3.6 and 4.4]).

Lemma 2.6 (see [7: Corollary 3.8]). *Let (X, N) be an SF-space. Assume that $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq X$ is an ascending sequence of distinct finite-dimensional subspaces of X and $R_N(\cup_{n=1}^{\infty} V_n) > 0$. Then there exists $d > 0$ such that $\overline{K}(0, d) \cap V_n$ is a compact set for all $n \in \mathbb{N}$, where $\overline{K}(x, r) = \{y \in X : N(y - x) \leq r\}$.*

If (X, N) is an SF-space, $\emptyset \neq Y \subset X$ and $x \in X$, then, as in metric spaces, we may define

$$\operatorname{dist}_N(x, Y) = \inf \{N(x - y) : y \in Y\}$$

and

$$P_Y(x) = \{y \in Y : N(x - y) = \operatorname{dist}_N(x, Y)\}.$$

If $y \in P_Y(x)$, then we call y the *best approximation* to x in Y.

Remark 2.7. If $Z \subset Y \subset X$ and $x \in X$, then

$$\operatorname{dist}_N(x, Y) \leq \operatorname{dist}_N(x, Z). \tag{2.7}$$

If V is a linear subspace of X, $x \in X$ and $v \in V$, then

$$\operatorname{dist}_N(x + v, V) = \operatorname{dist}_N(x, V). \tag{2.8}$$

If V is a linear subspace of X, $x \in X$, $t_1, t_2 \in \mathbb{K}$ such that $|t_1| \leq |t_2|$ and N is non-decreasing, then

$$\operatorname{dist}_N(t_1 x, V) \leq \operatorname{dist}_N(t_2 x, V). \tag{2.9}$$

Let us recall that a subset K of a linear space V is a (convex) cone if $K + K \subset K$, $aK \subset K$ for all $a \geq 0$ and $K \cap (-K) = \{0\}$. If K is a cone, then $\operatorname{Span} K = K - K$.

A slight modification of [7: Proposition 3.4] gives

Lemma 2.8. *Suppose that (X, N) is an SF-space and $K_1 \subsetneq K_2 \subsetneq \ldots \subsetneq X$ is an ascending sequence of distinct closed cones satisfying $V_n \subsetneq V_{n+1}$ and $\dim V_n < \infty$ for all $n \in \mathbb{N}$, where $V_n = K_n - K_n$. Furthermore, assume that there exists $d > 0$ such that $\overline{K}(0, d) \cap V_n$ is a compact set for every $n \in \mathbb{N}$. Define*

$$a_n = \sup\Big\{\operatorname{dist}_N(k, V_n) : k \in K_{n+1}\Big\} \tag{2.10}$$

$$b_n = \inf\Big\{\sup\{\operatorname{dist}_N(v + k, V_n) : k \in K_{n+1}\} : v \in \cup_{i=n+2}^{\infty} V_i \setminus V_{n+1}\Big\}. \tag{2.11}$$

Then $a_n, b_n \geq d$ for every $n \in \mathbb{N}$.

Proof. Choose any $n \in \mathbb{N}$ and $v \in K_{n+1} \setminus V_n$. Let us prove that

$$\sup\big\{\operatorname{dist}_N(tv, V_n) : t \geq 0\big\} \geq d.$$

Suppose this is not true. Then for every $k \in \mathbb{N}$ there exists $v_k \in V_n$ with $N(kv - v_k) \leq d$. Since $kv - v_k \in \overline{K}(0, d) \cap V_{n+1}$ for all k, by argument of compactness we may assume that $N(kv - v_k - z) \to 0$ for some $z \in V_{n+1}$. But $kv - v_k \in V_n \oplus [v]$, which (by Remark 2.2) is a closed subspace of X. Hence $z = tv + w$, where $t \in \mathbb{K}$ and $w \in V_n$. Since $N(kv - v_k - z) \to 0$, from (2.6) we get $N_1\big((k - t)v - (v_k + w)\big) \to 0$, therefore $\operatorname{dist}_{N_1}((k - t)v, V_n) \to 0$. Now fix $k_0 \in \mathbb{N}$ such that $k_0 > |t|$. As V_n is a closed set and $(k_0 - t)v \notin V_n$, by (2.9) and Lemma 2.4 we have

$$\operatorname{dist}_{N_1}\big((k - t)v, V_n\big) \geq \operatorname{dist}_{N_1}((k_0 - t)v, V_n) > 0 \qquad (k \geq k_0).$$

This contradicts the fact that $\operatorname{dist}_{N_1}((k - t)v, V_n) \to 0$ if $k \to \infty$. Consequently, $a_n \geq \sup\{\operatorname{dist}_N(tv, V_n) : t \geq 0\} \geq d$. The proof of the second inequality $b_n \geq d$ is completely similar to that of [7: Proposition 3.4]. So we omit the details ∎

Lemma 2.9. *Suppose that (X, N) is an SF-space and $K_1 \subset K_2 \subset \ldots \subset X$ is an ascending sequence of closed cones which generate finite-dimensional subspaces $V_n = K_n - K_n$ of X. Furthermore, assume that there exists $d > 0$ such that $\overline{K}(0, d) \cap V_n$ is a compact set for all $n \in \mathbb{N}$. Then, for $i > j$ and $x \in K_i$:*

1. *If $\operatorname{dist}_N(x, K_j) < d$, then $P_{K_j}(x) \neq \emptyset$.*
2. *If $\operatorname{dist}_N(x, V_j) < d$, then $P_{V_j}(x) \neq \emptyset$.*

Proof. Fix $i > j$ and $x \in K_i$ satisfying $\operatorname{dist}_N(x, K_j) = \varepsilon < d$. Choose $l_0 \in \mathbb{N}$ such that $\varepsilon + \frac{1}{l_0} \leq d$. Then for every $l \geq l_0$ there exists $k_j^l \in K_j$ with $N(x - k_j^l) \leq \varepsilon + \frac{1}{l}$. By compactness of $\overline{K}(0, d) \cap V_i$, without loss of generality we can assume that $N(x - k_j^l - v) = N(k_j^l - (x - v)) \to 0$ for some $v \in V_i$. Since K_j is a closed set, $v = x - k_j$, where $k_j \in K_j$. Using (2.5), we get $N(x - k_j^l) \to N(x - k_j)$, thus $N(x - k_j) \leq \varepsilon$. By definition of ε, $k_j \in P_{K_j}(x)$, and (1) is proved. To show statement 2, we argue as in the previous case, observing that all spaces V_n are closed in X ∎

Lemma 2.10. *Let (X, N) be an SF-space and assume that $V \subset U \subset X$, $(x_n) \subset U$, $x \in X$ and $N(x_n - x) \to 0$. Then*

$$\limsup_{n \to \infty} \operatorname{dist}_N(x_n, V) \leq \operatorname{dist}_N(x, V). \tag{2.12}$$

If, moreover, U is a finite-dimensional linear subspace of X, the set $U \cap \overline{K}(0,r)$ is compact and $\operatorname{dist}_N(x_n, V) < r$ *for some $r > 0$ and almost all $n \in \mathbb{N}$, then*

$$\lim_{n\to\infty} \operatorname{dist}_N(x_n, V) = \operatorname{dist}_N(x, V). \tag{2.13}$$

Proof. Suppose that $N(x_n - x) \to 0$. We may find $v_k \in V$ satisfying

$$N(x - v_k) \le \operatorname{dist}_N(x, V) + \frac{1}{k} \qquad (k \in \mathbb{N}).$$

By (2.5),

$$\operatorname{dist}_N(x_n, V) \le N(x_n - v_k) \to N(x - v_k)$$

for all $k \in \mathbb{N}$, which gives (2.12).

Now let U be a linear subspace of X and choose $(x_n) \subset U$, $x \in X$ and $r > 0$ which have the required properties. In order to prove (2.13), we only need to show that

$$\liminf_{n\to\infty} \operatorname{dist}_N(x_n, V) \ge \operatorname{dist}_N(x, V).$$

Suppose this is false. Then, without loss of generality, we may assume that there exist $\varepsilon > 0$ and $v_n \in V$ with

$$N(x_n - v_n) \le \operatorname{dist}_N(x, V) - \varepsilon \le r.$$

Hence $x_n - v_n \in U \cap \overline{K}(0,r)$ and, consequently, $N(x_n - v_n - u) \to 0$ for some $u \in U$. By (2.5), $N(x_n - v_n) \to N(u)$, thus $N(u) \le \operatorname{dist}_N(x, V) - \varepsilon$. On the other hand, according to (2.3) - (2.5), $N(x - v_n) \to N(u)$, which implies $N(u) \ge \operatorname{dist}_N(x, V)$. This is a contradiction ∎

3. Bernstein's 'lethargy' theorem for cones

In this section we prove a version of the Bernstein theorem, in which a sequence of cones replaces a sequence of finite-dimensional vector subspaces. The proof will be similar in spirit to that of Theorem 1.2, given by Lewicki in [7]. However, since property (2.8), used constantly in that proof, is no longer valid for cones, we must proceed in a slightly different way.

Our result is the following

Theorem 3.1. *Let (X, N) be an SF-space and suppose that $K_1 \subsetneq K_2 \subsetneq \ldots \subsetneq X$ is an ascending sequence of distinct closed cones such that $V_n \subsetneq V_{n+1}$ and* $\dim V_n < \infty$ *for all $n \in \mathbb{N}$, where $V_n = K_n - K_n$. Furthermore, assume that $R_N(\cup_{n=1}^{\infty} V_n) > 0$ and the following condition is satisfied:*

$$\left.\begin{array}{l} \text{For every } n \in \mathbb{N}, x \in K_n \text{ and all } i, j \in \mathbb{N} \text{ with } j \le i \le n: \\ \text{if } y \in P_{K_i}(x) \text{ and } z \in P_{K_j}(x), \text{ then } y - z \in K_i \end{array}\right\}. \tag{3.1}$$

Then for every sequence $\varepsilon_n \downarrow 0$ *there exist* $n_0 \in \mathbb{N}$ *and* $x \in X$ *such that*

$$\operatorname{dist}_N(x, K_n) = \varepsilon_n \qquad (n \geq n_0). \tag{3.2}$$

Proof. According to Lemma 2.6, we may find $d > 0$ such that $\overline{K}(0,d) \cap V_n$ is a compact set for all $n \in \mathbb{N}$. By (2.3), there exists $\delta > 0$ satisfying $N(x+y) \leq d$ if $x, y \in \overline{K}(0,\delta)$. Put $n_0 = \min\{k \in \mathbb{N} : \varepsilon_k < \delta\}$ and define

$$E_n = \Big\{x \in K_{n+1} : \operatorname{dist}_N(x, K_l) = \operatorname{dist}_N(x, V_l) = \varepsilon_l \quad (l = n_0, \ldots, n)\Big\}$$

where $n \geq n_0$. Let us first prove that $E_n \neq \emptyset$ if $n \geq n_0$ and $\varepsilon_n > 0$. For this purpose we choose such $n \geq n_0$ and set

$$F_j = \Big\{x \in K_{n+1} : \operatorname{dist}_N(x, K_l) = \operatorname{dist}_N(x, V_l) = \varepsilon_l \quad (l = n, \ldots, n-j)\Big\}$$

where $j = 0, \ldots, n - n_0$. We will show by induction on j that all the sets F_j are non-empty.

According to Lemma 2.8, $\varepsilon_n < \delta \leq d \leq a_n$, with a_n defined by (2.10). Therefore $\operatorname{dist}_N(v, V_n) > \varepsilon_n$ for some $v \in K_{n+1}$. Put

$$t_0 = \inf\big\{t \geq 0 : \operatorname{dist}_N(tv, V_n) \geq \varepsilon_n\big\}.$$

By (2.12), $\operatorname{dist}_N(t_0 v, V_n) \geq \varepsilon_n$. Since $\varepsilon_n < d$, (2.13) now implies $\operatorname{dist}_N(x_1, V_n) = \varepsilon_n$, where $x_1 = t_0 v$. By Lemma 2.9, there exist $k_1, k_2 \in K_n$ satisfying $N(x_1 - (k_1 - k_2)) = \varepsilon_n$. Put $x = x_1 + k_2$. Then $x \in K_{n+1}$. From (2.7) and (2.8) we have

$$\begin{aligned}\operatorname{dist}_N(x, V_n) &= \operatorname{dist}_N(x_1, V_n) = N(x_1 + k_2 - k_1)\\ &\geq \operatorname{dist}_N(x, K_n) \geq \operatorname{dist}_N(x, V_n),\end{aligned}$$

hence $x \in F_0$.

Now suppose that $x \in F_j$. By Lemma 2.9, for $l \in \{n-j, \ldots, n\}$,

$$\operatorname{dist}_N(x, V_l) = \operatorname{dist}_N(x, K_l) = N(x - v_l)$$

with some $v_l \in K_l$. According to (3.1), $x_1 = x - v_{n-j} \in K_{n+1}$. Relation (2.8) now implies

$$\operatorname{dist}_N(x_1, V_l) = \operatorname{dist}_N(x, V_l) = N((x - v_{n-j}) - (v_l - v_{n-j})) \geq \operatorname{dist}_N(x_1, K_l)$$

since $v_l - v_{n-j} \in K_l$, which follows from (3.1). Consequently,

$$\operatorname{dist}_N(x_1, K_l) = \operatorname{dist}_N(x_1, V_l) = \varepsilon_l \qquad (l = n, \ldots, n-j)$$

and so $x_1 \in F_j$. Moreover,

$$\operatorname{dist}_N(x_1, V_{n-j}) = N(x_1 - 0) \geq \operatorname{dist}_N(x_1, K_{n-j-1}) \geq \operatorname{dist}_N(x_1, V_{n-j})$$

which means that

$$\operatorname{dist}_N(x_1, K_{n-j-1}) = \operatorname{dist}_N(x_1, V_{n-j-1}) = \varepsilon_{n-j}.$$

If $\varepsilon_{n-j-1} = \varepsilon_{n-j}$, then $x_1 \in F_{j+1}$, so it suffices to consider the case $\varepsilon_{n-j-1} > \varepsilon_{n-j}$. By Lemma 2.8, $\varepsilon_{n-j-1} < \delta \le d \le b_{n-j-1}$, where b_{n-j-1} is given by (2.11). Consequently, there exists $u \in K_{n-j}$ such that $\operatorname{dist}_N(x_1 + u, V_{n-j-1}) > \varepsilon_{n-j-1}$. Set

$$t_0 = \inf\Big\{t \ge 0 : \operatorname{dist}_N(x_1 + tu, V_{n-j-1}) \ge \varepsilon_{n-j-1}\Big\}.$$

As in the previous case, Lemma 2.10 gives $\operatorname{dist}_N(x_1 + t_0 u, V_{n-j-1}) = \varepsilon_{n-j-1}$. According to Lemma 2.9,

$$\operatorname{dist}_N(x_1 + t_0 u, V_{n-j-1}) = N\big(x_1 + t_0 u - (w_1 - w_2)\big)$$

with $w_1, w_2 \in K_{n-j-1}$. Put $z = x_1 + t_0 u + w_2$. Then $z \in K_{n+1}$. By (2.7) and (2.8), we have

$$\begin{aligned}\operatorname{dist}_N(z, V_{n-j-1}) &= \operatorname{dist}_N(x_1 + t_0 u, V_{n-j-1}) = N(z - w_1)\\ &\ge \operatorname{dist}_N(z, K_{n-j-1}) \ge \operatorname{dist}_N(z, V_{n-j-1}).\end{aligned}$$

Hence

$$\operatorname{dist}_N(z, K_{n-j-1}) = \operatorname{dist}_N(z, V_{n-j-1}) = \varepsilon_{n-j-1}.$$

Since $v_l - v_{n-j} \in K_l$, for $l = n, \dots, n-j$ we get

$$\begin{aligned}\operatorname{dist}_N(x_1, V_l) &= N(x_1 - (v_l - v_{n-j})) = N(z - (v_l - v_{n-j} + t_0 u + w_2))\\ &\ge \operatorname{dist}_N(z, K_l) \ge \operatorname{dist}_N(z, V_l) = \operatorname{dist}_N(x_1, V_l).\end{aligned}$$

Therefore $\operatorname{dist}_N(z, K_l) = \operatorname{dist}_N(z, V_l) = \varepsilon_l$, and, in consequence, $z \in F_{j+1}$. Thus we have proved that all the sets F_j are non-empty. As $E_n = F_{n-n_0}$, we conclude that $E_n \ne \emptyset$.

In the case of $\varepsilon_n > 0$ and $\varepsilon_{n+1} = 0$ for some $n \ge n_0$, each element $x \in E_n$ satisfies condition (3.2). So we can assume that $\varepsilon_n > 0$ for all $n \ge n_0$. Our claim is that $E_n \cap \overline{K}(0, \delta) \ne \emptyset$ for every $n \ge n_0$. To prove this, fix $n \ge n_0$ and choose $\widetilde{x_n} \in E_n$. By Lemma 2.9, there exist $v_l \in K_l$ satisfying

$$\operatorname{dist}_N(\widetilde{x_n}, K_l) = \operatorname{dist}_N(\widetilde{x_n}, V_l) = N(\widetilde{x_n} - v_l) \qquad (l = n_0, \dots, n).$$

Define $x_n = \widetilde{x_n} - v_{n_0}$. Then $x_n \in K_{n+1}$, which follows from (3.1). According to (2.7) and (2.8), we have

$$\begin{aligned}\operatorname{dist}_N(x_n, V_l) &= \operatorname{dist}_N(\widetilde{x_n}, V_l) = N(\widetilde{x_n} - v_l) = N(x_n - (v_l - v_{n_0}))\\ &\ge \operatorname{dist}_N(x_n, K_l) \ge \operatorname{dist}_N(x_n, V_l)\end{aligned}$$

hence $\operatorname{dist}_N(x_n, K_l) = \operatorname{dist}_N(x_n, V_l) = \varepsilon_l$. Furthermore,

$$\operatorname{dist}_N(x_n, K_{n_0}) = N(\widetilde{x_n} - v_{n_0}) = N(x_n) = \varepsilon_{n_0} < \delta$$

which means that $x_n \in E_n \cap \overline{K}(0,\delta)$.

Now, choose any $x_n \in E_n \cap \overline{K}(0,\delta)$, where $n \geq n_0$. According to Lemma 2.9, for every $l \geq n_0$ and $n \geq l$ we may find $v_n^l \in P_{K_l}(x_n)$. Since $N(x_n - v_n^l) = \varepsilon_l < \delta$ and $N(x_n) \leq \delta$, by (2.4) and the choice of δ we have $N(v_n^l) \leq d$. Therefore $v_n^l \in K_l \cap \overline{K}(0,d)$, which is a compact subset of X (as a closed subset of a compact set). Consequently, for each $l \geq n_0$ there exist a subsequence $(k_n) \subset \mathbb{N}$ and $v_l \in K_l$ such that $N(v_{k_n}^l - v_l) \to 0$. Applying the diagonal argument, we may assume that $N(v_{k_n}^l - v_l) \to 0$ for all $l \geq n_0$. Let us fix $\varepsilon > 0$. By (2.3), we may find $r > 0$ such that $N(y_1 + y_2 + y_3 + y_4) \leq \varepsilon$, when $y_1, y_2, y_3, y_4 \in \overline{K}(0,r)$. Choose any $l_0 \geq n_0$ satisfying $\varepsilon_{l_0} \leq r$. Then there exists $k(l_0) \in \mathbb{N}$ such that

$$\left.\begin{array}{l} N(x_{k_n} - v_{k_n}^{l_0}) \\ N(v_{k_n}^{l_0} - v_{l_0}) \\ N(v_{k_m}^{l_0} - v_{l_0}) \\ N(x_{k_m} - v_{k_m}^{l_0}) \end{array}\right\} \leq r$$

for all $n, m \geq k(l_0)$. By (2.4) and the choice of r, $N(x_{k_n} - x_{k_m}) \leq \varepsilon$ if $n, m \geq k(l_0)$. Thus we have proved that (x_{k_n}) is a Cauchy sequence in X. Since the space X is complete, $N(x_{k_n} - x) \to 0$, with some $x \in X$.

It remains to show that $\operatorname{dist}_N(x, K_l) = \varepsilon_l$ for $l \geq n_0$. Fixing such l, by (2.3) and (2.5) we get $N((x_{k_n} - v_{k_n}^l) - (x - v_l)) \to 0$, hence $N(x_{k_n} - v_{k_n}^l) \to N(x - v_l)$, and, finally, $N(x - v_l) = \varepsilon_l$. Moreover,

$$N(x_{k_n} - v_{k_n}^l) \leq N(x_{k_n} - v) \to N(x - v)$$

for every $v \in K_l$ and $k_n \geq l$. Therefore $N(x - v) \geq \varepsilon_l$ and, consequently, $v_l \in P_{K_l}(x)$, which completes the proof ∎

As a corollary, we obtain the Bernstein 'lethargy' theorem for cones in Banach spaces (another version is given in Section 4 – see Theorem 4.5).

Corollary 3.2. *Let X be a Banach space and suppose that $K_1 \subsetneq K_2 \subsetneq \ldots \subsetneq X$ is an ascending sequence of distinct closed cones such that $K_n - K_n \subsetneq K_{n+1} - K_{n+1}$ and $\dim(K_n - K_n) < \infty$ for all $n \in \mathbb{N}$. Furthermore, assume that condition (3.1) is satisfied. Then for every sequence $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq 0 = \lim \varepsilon_n$ there exists $x \in X$ such that $\operatorname{dist}(x, K_n) = \varepsilon_n$ for $n \geq 1$.*

Let us observe that the 'lethargy' theorem for cones may not be true if we drop the assumptions of Theorem 3.1, because of

Example 3.3.

1. Let X be the space $\mathbb{R}^2$ with the Euclidean norm and define

$$K_n = \Big\{(x,y) \in X : 0 \leq x < \infty \text{ and } 0 \leq y \leq nx\Big\} \qquad (n \in \mathbb{N}).$$

Then $K_n - K_n = X$ for all $n \in \mathbb{N}$ and property (3.1) is not valid. Observe that the condition $\operatorname{dist}(z, K_n) = \delta_n \downarrow 0$ implies $z = (0,y)$ for some $y > 0$. Since for such z and

$t > 0$ we have $\operatorname{dist}(tz, K_n) = t \operatorname{dist}(z, K_n) = t\delta_n$, in this case we cannot obtain (3.2) if $\varepsilon_n \downarrow 0$ and $\frac{\varepsilon_n}{\delta_n}$ is not constant.

2. Let X be the space of all real (or complex) sequences $(x_n)_{n=1}^{\infty}$ equipped with the F-norm $|(x_n)| = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1+|x_i|}$. Set

$$K_n = \Big\{(x_n) \in X : x_i \geq 0 \text{ for } i \leq n \text{ and } x_i = 0 \text{ for } i > n\Big\}$$

and $V_n = K_n - K_n$. Then $R_{|\cdot|}(V_n) = \frac{1}{2^n}$ and, consequently, $R_{|\cdot|}(\cup_{n=1}^{\infty} V_n) = 0$ (compare [7: Example 3.6]).

Choose any sequence $\varepsilon_n \downarrow 0$ such that $(\varepsilon_k - \varepsilon_{k+1})2^{k+1} \geq 1$ for infinitely many $k \in \mathbb{N}$ and assume that (3.2) holds for some $x \in X$. Then $x_i \geq 0 \quad (i \geq 1)$. Since $\operatorname{dist}_{|\cdot|}(x, K_n) = \sum_{i=n+1}^{\infty} \frac{1}{2^i} \frac{|x_i|}{1+|x_i|}$, from (3.2) we get $\frac{1}{2^{n+1}} \frac{|x_{n+1}|}{1+|x_{n+1}|} = \varepsilon_n - \varepsilon_{n+1}$, hence $(\varepsilon_n - \varepsilon_{n+1})2^{n+1} < 1$ for almost all natural n, which is a contradiction.

However, in some situations these assumptions may be weakened, which is shown by

Example 3.4. Suppose that $(V_n)_{n=1}^{\infty}$ is a sequence of distinct finite-dimensional subspaces of the space $X = L^{\infty}[a, b]$ with the supremum norm and $1 \in V_1$. Set $K_n = \{v \in V_n : v \geq 0\} \quad (n \geq 1)$. Here condition (3.1) may not be satisfied, but the Bernstein theorem is still true.

To prove this, fix a sequence $\varepsilon_n \downarrow 0$. By Theorem 1.1, there exists $y \in X$ such that $\operatorname{dist}(y, V_n) = \varepsilon_n$ for $n \in \mathbb{N}$ and $\|y\| = \varepsilon_1$. Obviously, if $w_n \in P_{V_n}(y)$, then $\|w_n\| \leq 2\varepsilon_1$. Put $x = y + 2\varepsilon_1$. By (2.8), $\operatorname{dist}(x, V_n) = \operatorname{dist}(y, V_n) = \varepsilon_n$ for all $n \in \mathbb{N}$. Moreover, if $w_n \in P_{V_n}(y)$, then $v_n = w_n + 2\varepsilon_1 \in P_{V_n}(x)$ and $v_n \in K_n$, thus $\operatorname{dist}(x, K_n) = \varepsilon_n$.

For example, the 'lethargy' theorem holds for the cones $K_n = \{p \in P_n[a, b] : p \geq 0\}$, where $P_n[a, b]$ denotes the space of all polynomials of degree at most n, restricted to $[a, b]$.

Example 3.5. Let S be the space of all real (or complex) sequences $(x_n)_{n=1}^{\infty}$ such that $x_n = 0$ for almost all $n \in \mathbb{N}$, equipped with a norm $\|\cdot\|$ satisfying $\|(x_n)\| \leq \|(y_n)\|$ if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$. Let X be the completion of the space $(S, \|\cdot\|)$ and set K_n as in Example 3.3/2. If $i \geq j$ and $(x_n) \in K_i$, then $P_{K_j}((x_n)) = (x_1, \ldots, x_j, 0, 0, \ldots)$ and, in consequence, the assumptions of Corollary 3.2 are fulfilled.

In particular, if $X = l^p \quad (1 \leq p < \infty)$, then $\operatorname{dist}(x, K_n) = \varepsilon_n$ for $n \geq 1$, when $x_1 \geq 0$ and $x_k = (\varepsilon_{k-1}^p - \varepsilon_k^p)^{1/p}$ for $k \geq 2$.

To end this section, we note that the assumption $R_N(\cup_{n=1}^{\infty} V_n) > 0$ in Theorem 3.1 is not restrictive in many cases as the following examples show.

Remark 3.6.

1. If $(X, |\cdot|)$ is an F-space with an s-homogeneous norm (see Definition 4.5), then $R_{|\cdot|}(X) = \infty$.

2. If (X_ϱ, ϱ) is a modular space and ϱ is s-convex with some $s \in (0, 1]$, i.e. $\varrho(ax + by) \leq a^s \varrho(x) + b^s \varrho(y)$ for all $x, y \in X_\varrho$ and $a, b \geq 0$ with $a^s + b^s = 1$, then $R_\varrho(X_\varrho) = \infty$.

3. If an SF-space (X, N) is locally bounded, then $R_N(X) > 0$.

4. (See [7: Proposition 4.5].) If $(X_{\varrho_f}, \varrho_f)$ is an Orlicz-Musielak space defined as in Example 2.3 and $Y \subset X_{\varrho_f}$ with $Y \setminus \{0\} \neq \emptyset$, then

$$R_{\varrho_f}(Y) = \inf \left\{ \int_{A_y} f^\infty(t)\, d\mu(t) : \, y \in Y \setminus \{0\} \right\} \tag{3.3}$$

where $A_y = \{t \in \Omega : \, y(t) \neq 0\}$ and $f^\infty(t) = \lim_{s \to \infty} f(t,s)$ for $t \in \Omega$.

4. Bernstein's 'lethargy' theorem for linear projections

Let us now start with

Definition 4.1. Let V be a linear subspace of an SF-space (X, N). Then a linear operator $P : \, X \to V$ is called a *projection* if it is continuous and $Px = x$ for all $x \in V$. We will denote by $P(X, V)$ the set of all projections from X onto V.

We shall prove a version of the Bernstein theorem for projections onto closed subspaces of an SF-space. This is a generalization of [6: Proposition 3], which deals only with the case of a Banach space and projections onto its finite-dimensional subspaces.

The following lemma will be useful for our purposes.

Lemma 4.2. *Let (X, N) be an SF-space, let V be one of its finite-dimensional linear subspaces and assume that $\overline{K}(0,d) \cap V$ is a compact set. If $\varepsilon \leq d$, $x \in V$, $v \in V \setminus \{0\}$ and $N(x) \leq \varepsilon$, then there exists $t_0 \geq 0$ with $N(x + t_0 v) = \varepsilon$.*

Proof. By (2.2) and (2.5), the mapping $t \mapsto N(x+tv)$ is continuous on $\mathbb{R}$. Therefore e only need to show that $N(x+tv) > d$ for some $t > 0$. On the contrary, suppose this false. Then for every $k \in \mathbb{N}$ we have $N(x+kv) \leq d$, hence $x + kv \in \overline{K}(0,d) \cap V$. By argument of compactness, there exists $z \in V$ satisfying

$$N(x + kv - z) = N\big(kv - (z - x)\big) \to 0.$$

Since the one-dimensional space $[v]$ is closed in X, $z - x = t_1 v$, where $t_1 \in \mathbb{K}$. According to (2.6), we have $N_1((k - t_1)v) \to 0$, with N_1 defined as in Lemma 2.4. On the other hand, for $k > 2|t_1| + 1$,

$$N_1\big((k - t_1)v\big) \geq N_1\big((|t_1| + 1)v\big) > 0$$

which is a contradiction ∎

Our theorem reads as follows.

Theorem 4.3. *Suppose that (X, N) is an SF-space and let $V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq X$ be an ascending sequence of distinct closed subspaces of X. Furthermore, suppose that $P_n \in P(X, V_n)$ and for every $n \in \mathbb{N}$ we may find $v_n \in V_{n+1} \setminus V_n$ such that the conditions*

$$P_j v_n = 0 \;\; \textit{if}\; j = 1, \ldots, n \tag{4.1}$$

$$R_N\big(\cup_{n=1}^\infty \operatorname{Span}\{v_1, \ldots, v_n\}\big) > 0 \tag{4.2}$$

are fulfilled. Then for every sequence $\varepsilon_n \downarrow 0$ *there exist* $n_0 \in \mathbb{N}$ *and* $x \in X$ *such that*

$$N(x - P_n x) = \varepsilon_n \qquad (n \geq n_0). \tag{4.3}$$

Proof. Choose any sequence (v_n) of vectors which have the properties listed above and put $W_j = \operatorname{Span}\{v_1, \ldots, v_j\}$ $(j \geq 1)$. By (4.2) and Lemma 2.6, $\overline{K}(0,d) \cap W_n$ is a compact set for some $d > 0$ and all $n \in \mathbb{N}$. As in the proof of Theorem 3.1, we may find $\delta > 0$ such that $N(x+y) \leq d$ if $N(x) \leq \delta$ and $N(y) \leq \delta$. Let $n_0 = \min\{n \in \mathbb{N} : \varepsilon_n \leq \delta\}$ and choose $n \geq n_0$ satisfying $\varepsilon_n > 0$. For $j = 0, \ldots, n - n_0$ define

$$U_j^n = \operatorname{Span}\{v_{n-j}, \ldots, v_n\}$$

and

$$F_j = \Big\{x \in U_j^n : N(x) = \varepsilon_{n-j}, N(x - P_l x) = \varepsilon_l \text{ for } l = n, \ldots, n-j\Big\}.$$

We claim that all the sets F_j are non-empty. The proof goes by induction on j.

We first show that $F_0 \neq \emptyset$. To do this, choose $t_0 > 0$ with $N(t_0 v_n) = \varepsilon_n$ (by Lemma 4.2 such t_0 exists). We have

$$N\big(t_0 v_n - P_n(t_0 v_n)\big) = N(t_0 v_n) = \varepsilon_n,$$

hence $x_0 = t_0 v_n \in F_0$. Now, suppose that $x_j \in F_j$. According to Lemma 4.2, we may find $t_0 \geq 0$ satisfying $N(x_j + t_0 v_{n-j-1}) = \varepsilon_{n-j-1}$. Setting $x_{j+1} = x_j + t_0 v_{n-j-1}$, we have $x_{j+1} \in U_{j+1}^n$. Furthermore, by (4.1) we get

$$\begin{aligned} &N(x_{j+1} - P_{n-j-1}x_{j+1}) \\ &\quad = N(x_j + t_0 v_{n-j-1} - P_{n-j-1}x_j - t_0 P_{n-j-1} v_{n-j-1}) \\ &\quad = N(x_{j+1}) \\ &\quad = \varepsilon_{n-j-1} \end{aligned}$$

and

$$\begin{aligned} &N(x_{j+1} - P_l x_{j+1}) \\ &\quad = N(x_j + t_0 v_{n-j-1} - P_l x_j - t_0 P_l v_{n-j-1}) \\ &\quad = N(x_j - P_l x_j) \\ &\quad = \varepsilon_l \end{aligned}$$

when $l = n, \ldots, n-j$. Consequently, $x_{j+1} \in F_{j+1}$. Thus we have showed that every set F_j is non-empty. In particular, $E_n \neq \emptyset$, where $E_n = F_{n-n_0}$, i.e.

$$E_n = \Big\{x \in U_{n-n_0}^n : N(x) = \varepsilon_{n_0}, N(x - P_l x) = \varepsilon_l \text{ for } l = n_0, \ldots, n\Big\}.$$

Let us observe that each vector $x \in E_n$ satisfies condition (4.3) if $\varepsilon_n > 0$ and $\varepsilon_{n+1} = 0$. Therefore it is sufficient to consider the case of $\varepsilon_n > 0$ for all $n \in \mathbb{N}$. Then, given any $n \geq n_0$, we may choose $x_n \in E_n$. Since $x_n \in W_n$, from (4.1) we get $P_l x_n \in W_l$, with $l \geq n_0$. Moreover, $N(x_n) = \varepsilon_{n_0} \leq \delta$ and $N(x_n - P_l x_n) \leq \varepsilon_l \leq \delta$, thus $N(P_l x_n) \leq d$. We have showed that, for $n, l \geq n_0$, $P_l x_n \in \overline{K}(0,d) \cap W_l$, which is compact in X. Applying the diagonal argument, we may find a subsequence $(k_n) \subset \mathbb{N}$ and vectors $v_l \in X$ such that $N(P_l x_{k_n} - v_l) \to 0$ for $l \geq n_0$.

Fix $\varepsilon > 0$ and choose $r > 0$ satisfying $N(x+y+z+w) \leq \varepsilon$ if $x, y, z, w \in \overline{K}(0,r)$. We may find $l_0 \geq n_0$ with $\varepsilon_{l_0} \leq r$ and $k(l_0)$ such that $N(x_{k_n} - P_{l_0} x_{k_n}) \leq r$ and $N(P_{l_0} x_{k_n} - v_{l_0}) \leq r$, when $n \geq k(l_0)$. By (2.4) and the choice of r, we get

$$\begin{aligned} N(x_{k_n} - x_{k_m}) &= N\Big((x_{k_n} - P_{l_0} x_{k_n}) + (P_{l_0} x_{k_n} - v_{l_0}) \\ &\qquad + (v_{l_0} - P_{l_0} x_{k_m}) + (P_{l_0} x_{k_m} - x_{k_m})\Big) \\ &\leq \varepsilon \end{aligned}$$

for $n, m \geq k(l_0)$. From this it follows that (x_{k_n}) is a Cauchy sequence in X, hence $N(x_{k_n} - x) \to 0$, with some $x \in X$. Let us choose $l \geq n_0$. Since P_l is continuous, by (2.3) - (2.5) we have

$$N\big((x_{k_n} - P_l x_{k_n}) - (x - P_l x)\big) \to 0$$

and, consequently,

$$N(x_{k_n} - P_l x_{k_n}) \to N(x - P_l x).$$

Therefore $N(x - P_l x) = \varepsilon_l$ for $l \geq n_0$ and the proof is complete ∎

Let us emphasize that in Theorem 4.3, contrary to the classical 'lethargy' theorem, subspaces V_n do not need to be finite-dimensional.

Example 4.4. The following sequences of projections have the properties required in Theorem 4.3:

1. Let X be a Hilbert space and suppose that $(V_n)_{n=1}^{\infty}$ is an ascending sequence of its distinct closed subspaces. Let P_n be the operator of the best approximation in V_n, i.e. $P_n x \in P_{V_n}(x)$ for $x \in X$ and $n \in \mathbb{N}$. Then $P_n \in P(X, V_n)$ and there exists $v_n \in V_{n+1} \setminus V_n$ such that $P_n v_n = 0$ and, in consequence, $P_j v_n = 0$ for $j < n$. In this case Theorem 4.3 is a generalization of the classical Bernstein theorem (Theorem 1.1) for Hilbert spaces.

2. Let X be a Banach space with a Schauder basis $(v_j)_{j=1}^{\infty}$. Define $V_n = \operatorname{Span}\{v_1, \ldots, v_n\}$ and set $P_n(\sum_{j=1}^{\infty} \alpha_j v_j) = \sum_{j=1}^{n} \alpha_j v_j$. Then $P_n \in P(X, V_n)$ and $P_j(v_{n+1}) = 0$ for $j = 1, \ldots, n$.

3. Let X_{ϱ_f} be an Orlicz-Musielak space (Example 2.3) generated by a function f satisfying $\lim_{s\to\infty} f(t,s) = \infty$ for almost all $t \in \Omega$. Moreover, assume that $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \in \Sigma$, $\Omega_n \subset \Omega_{n+1}$ and $0 < \mu(\Omega_{n+1} \setminus \Omega_n) < \infty$. Put $V_n = \{x \in X_{\varrho_f} : x|_{\Omega \setminus \Omega_n} = 0\}$ and $P_n x = x \cdot I_{\Omega_n}$, where I_A denotes the characteristic function of the set A. Then $P_n \in P(X_{\varrho_f}, V_n)$ and $P_j v_n = 0$ for $v_n = I_{\Omega_{n+1} \setminus \Omega_n} \in V_{n+1} \setminus V_n$ and $j \leq n$. Additionally, by (3.3) we have

$$\begin{aligned} R_{\varrho_f}\Big(\bigcup_{n=1}^{\infty} \operatorname{Span}\{v_1, \ldots, v_n\}\Big) &= \inf\Big\{\int_{\Omega_{n+1} \setminus \Omega_n} f^{\infty}(t)\, d\mu(t) : n \in \mathbb{N}\Big\} \\ &= \infty. \end{aligned}$$

Theorem 4.3 implies the following theorem of Bernstein type for cones in Banach spaces.

Theorem 4.5. *Let X be a Banach space and let $K_1 \subsetneq K_2 \subsetneq \ldots \subsetneq X$ be an ascending sequence of distinct closed cones, generating distinct closed subspaces $V_n = K_n - K_n$ of X. Assume that $P_n \in P(X, V_n)$, there exist $v_n \in K_{n+1} \setminus K_n$ satisfying condition* (4.1) *and $M > 0$ such that $\|Id - P_n\| \leq M \quad (n \in \mathbb{N})$. Then for every sequence $\varepsilon_n \downarrow 0$ we may find $x \in X$ such that*

$$\operatorname{dist}(x, K_n) \leq \varepsilon_n \leq M \operatorname{dist}(x, K_n)$$

for $n \geq 1$.

Proof. Reasoning as in the proof of Theorem 4.3, we show the existence of an element $x \in X$ such that $\|x - P_n x\| = \varepsilon_n$ for all natural n. Furthermore, since the vectors $x_n \in E_n$, used in the construction of x, are of the form $x_n = \sum_{k=1}^{n} t_k v_k$ with $t_k \geq 0$, by (4.1) and the choice of v_n we get $P_j x_n \in K_j$ and, in consequence, $P_j x \in K_j$ for all $j \geq 1$. Therefore

$$\operatorname{dist}(x, K_n) \leq \|x - P_n x\| = \varepsilon_n \leq \|Id - P_n\| \operatorname{dist}(x, K_n) \leq M \operatorname{dist}(x, K_n)$$

which is the desired conclusion ∎

Example 4.6. Suppose that X is a Hilbert space and K_n, V_n, ε_n are as in Theorem 4.5. Let P_n denote the operator of the best approximation in V_n (compare Example 4.4.1). In this case $\|Id - P_n\| = 1$. By Theorem 4.5, if for every n there exists $v_n \in K_{n+1} \setminus K_n$ with $P_n v_n = 0$, then $\operatorname{dist}(x, K_n) = \varepsilon_n$ for some $x \in X$ and all $n \in \mathbb{N}$.

Finally, we formulate Theorem 4.3 for the space $L(Y, X)$ of all linear, continuous operators from Y into X, where X and Y are F-spaces with s-homogeneous norms. This property may be applied to the theory of approximation numbers and Bernstein pairs (see [1] for more details).

Definition 4.7. Let $(X, |\cdot|)$ be an F-space over $\mathbb{K}$ and $s \in (0, 1]$. We say that $|\cdot|$ is an *s-homogeneous norm* if $|\alpha x| = |\alpha|^s |x|$ for all $\alpha \in \mathbb{K}$ and $x \in X$.

It is easy to check that $L(Y, X)$ is an F-space with the standard F-norm $|L| = \sup\{|L(y)| : |y| \leq 1\}$ if X and Y are F-spaces with s-homogeneous norms. Moreover, $R_{|\cdot|}(L(Y, X)) = \infty$.

Corollary 4.8. *Assume that X is an F-space with an s-homogeneous norm and let V_n, P_n, ε_n be such as in Theorem* 4.3. *Then for any $s_1 \in (0, 1]$ and F-space Y with an s_1-homogeneous norm and $Y^* \neq \{0\}$ there exist $L \in L(Y, X)$ and $n_0 \in \mathbb{N}$ satisfying $|L - W_n L| = \varepsilon_n \quad (n \geq n_0)$ where $W_n \in P\big(L(Y, X), L(Y, V_n)\big)$ is defined by $W_n T = P_n \circ T$ for $T \in L(Y, X)$.*

The proof of this corollary is the same as that of [1: Proposition 2.3], where only the case of 1-homogeneous norm was considered.

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Received 23.10.2001