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T1 Theorems on Besov and Triebel-Lizorkin Spaces on Spaces of Homogeneous Type and Their Applications

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Abstract. The author first establishes the reduced $T1$ theorems for Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. Using these $T1$ theorems, the author proves that an operator of Bessel potential type can be used as the lifting operator of these spaces.

Keywords: *Spaces of homogeneous type, Besov spaces, Triebel-Lizorkin spaces, $T1$ theorems, Calderón-Zygmund operators, Bessel potentials*

AMS subject classification: 43A85, 42B20, 31B10, 46E35

1. Introduction

Recently, in [15], for some $p_0 \in (0,1)$ the inhomogeneous Besov spaces $B^s_{pq}(X)$ with $p_0 < p \le \infty$ and $0 < q \le \infty$ and the Triebel-Lizorkin spaces $F^s_{pq}(X)$ with $p_0 \le p < \infty$ and $p_0 < q \le \infty$ on spaces of homogeneous type were introduced. Some special cases of these spaces have been introduced in [10, 11] before. Moreover, recently some new characterizations on Besov and Triebel-Lizorkin spaces and their applications were given in [14, 24, 25]. In particular, in [24] it was proved that the Besov spaces on d-sets introduced by Triebel via traces in [21] and, equivalently, via quarkonial decompositions in [22] are the same as those Besov spaces introduced in [10] by regarding d-sets as spaces of homogeneous type. The same is also true for the Besov and Triebel-Lizorkin spaces on Lipschitz manifolds introduced by Triebel in [23] via the localization principle and the real interpolation method and those spaces introduced in [15] via regarding the Lipschitz manifold as a space of homogeneous type.

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The main purpose of this paper is to establish the reduced $T1$ theorems for the Besov spaces $B^s_{pq}(X)$ when $p_0 < p \le \infty$ and $0 < q \le \infty$ and for the Triebel-Lizorkin spaces $F^s_{pq}(X)$ when $p_0 < p < \infty$ and $p_0 < q \le \infty$. To be precise, we will first establish the $T1$ theorem for Triebel-Lizorkin spaces by using the discrete Calderón reproducing formulae in [12] and the Plancherel-Pôlya inequalities in [5]. Then by use of the real interpolation theorems in [25] we will obtain the $T1$ theorem for the Besov spaces. The $T1$ theorems for the homogeneous Besov spaces $\dot{B}^s_{pq}(X)$ and Triebel-Lizorkin spaces $\dot{F}^s_{pq}(X)$ are also stated, parts of which were obtained in [4, 25]. As an application of the $T1$ theorems on $B^s_{pq}(X)$ and $F^s_{pq}(X)$ we will show that an operator of Bessel potential type can be used as the lifting operator of these spaces, which generalizes the corresponding results on these spaces with $p, q > 1$ in [14] to the general cases considered here by a simpler method.

Let us now recall some definitions and notation on spaces of homogeneous type. A *quasi-metric* ρ on a set X is a function $\rho:\, X \times X \to [0, \infty)$ satisfying

$\rho(x, y) = 0$ if and only if $x = y$

$\rho(x, y) = \rho(y, x)$ for all $x, y \in X$

$\rho(x, y) \le A\big[\rho(x, z) + \rho(z, y)\big]$ $(x, y, z \in X)$ for some constant $A \in [1, \infty)$.

Any quasi-metric defines a topology, for which the balls

$$B(x, r) = \big\{y \in X :\, \rho(y, x) < r\big\}$$

for all $x \in X$ and all $r > 0$ form a basis.

In what follows, we set

$$\operatorname{diam} X = \sup\big\{\rho(x, y) :\, x, y \in X\big\}.$$

We also make the following conventions. We denote by $f \sim g$ that there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper we will denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_0, do not change in different occurrences. We denote $\mathbb{N} \cup \{0\}$ simply by $\mathbb{Z}_+$, and for any $q \in [1, \infty]$ we denote by q' its conjugate index, namely $\frac{1}{q} + \frac{1}{q'} = 1$.

Definition 1.1 [14]. Let $d > 0$ and $0 < \theta \le 1$. A *space of homogeneous type* $(X, \rho, \mu)_{d,\theta}$ is a set X together with a quasi-metric ρ and a non-negative Borel regular measure μ on X with $\operatorname{supp}\mu = X$ and there exists a constant $C_0 > 0$ such that, for all $0 < r < \operatorname{diam} X$ and all $x, x', y \in X$,

$$\mu(B(x, r)) \sim r^d \tag{1.1}$$

and

$$\left|\rho(x,y)-\rho(x',y)\right| \le C_0\rho(x,x')^{\theta}\left[\rho(x,y)+\rho(x',y)\right]^{1-\theta}. \qquad (1.2)$$

In particular, when $\operatorname{diam} X < \infty$, spaces of homogeneous type in Definition 1.1 cover the boundaries of bounded Lipschitz domains in $\mathbb{R}^n$, the n-torus in $\mathbb{R}^n$, C^∞-compact Riemannian manifolds, Lipschitz manifolds of compact case in [23], and compact d-sets which include various kinds of fractals (see [19, 21, 22, 24]); while when $\operatorname{diam} X = \infty$, spaces of homogeneous type in Definition 1.1 specifically include Euclidean spaces, the boundaries of unbounded Lipschitz domains in $\mathbb{R}^n$, and Lipschitz manifolds of the non-compact case in [23]. Moreover, the spaces of homogeneous type in Definition 1.1 are just the variants of the spaces of homogeneous type introduced by Coifman and Weiss in [2]. In fact, if we choose $d = 1$, Macias and Segovia in [16] have proved that, in the sense of equivalent topology, $(X,\rho,\mu)_{d,\theta}$ are the spaces of homogeneous type in the sense of Coifman and Weiss, whose definitions only require that ρ is a quasi-metric without property (1.2) and μ satisfies the doubling condition which is weaker than (1.1).

We now recall the definition of the spaces of test functions on X from [13] (see also [8]).

Definition 1.2. Fix $\gamma > 0$ and $\theta \ge \beta > 0$. A function f defined on X is said to be a *test function of type* (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$, if f satisfies the conditions

(i) $|f(x)| \le C\frac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$

(ii) $|f(x) - f(y)| \le C\left(\frac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \frac{r^\gamma}{(r+\rho(x,x_0))^{d+\gamma}}$ for $\rho(x,y) \le \frac{1}{2A}[r + \rho(x,x_0)]$.

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\left\{C : \text{ Properties } (i) \textit{ and } (ii) \text{ hold}\right\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta,\gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta,\gamma)$$

with equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta,\gamma)$. Also, let the dual space $(\mathcal{G}(\beta,\gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a finite constant $C \ge 0$ such that, for all $f \in \mathcal{G}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\mathcal{G}(\beta,\gamma))'$ and $f \in \mathcal{G}(\beta,\gamma)$. It is also easy to see that $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$ if and only if $f \in \mathcal{G}(\beta,\gamma)$. Thus, for all $h \in (\mathcal{G}(\beta,\gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well-known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1,\gamma)$ is not dense in $\mathcal{G}(\beta_2,\gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows we let $\mathring{\mathcal{G}}(\beta,\gamma)$ be the completion of the space $\mathcal{G}(\theta,\theta)$ in $\mathcal{G}(\beta,\gamma)$ when $0 < \beta, \gamma < \theta$.

To state the definition of the inhomogeneous Besov spaces $B^s_{pq}(X)$ and the inhomogeneous Triebel-Lizorkin spaces $F^s_{pq}(X)$ studied in [15] we need the following approximations to the identity which were first introduced in [8].

Definition 1.3. A sequence $\{S_k\}_{k=0}^{\infty}$ of linear operators is said to be an *approximation to the identity of order* $\varepsilon \in (0,\theta]$ if there exist constants $C_1, C_2 > 0$ such that, for all $k \in \mathbb{Z}_+$ and all $x, x', y, y' \in X$, the kernel $S_k(x,y)$ of S_k is a function from $X \times X$ into $\mathbb{C}$ satisfying the following conditions:

(i) $S_k(x,y) = 0$ if $\rho(x,y) \geq C_1 2^{-k}$ and $\|S_k\|_{L^\infty(X\times X)} \leq C_2 2^{dk}$.

(ii) $\big|S_k(x,y) - S_k(x',y)\big| \leq C_2 2^{k(d+\varepsilon)} \rho(x,x')^{\varepsilon}$.

(iii) $\big|S_k(x,y) - S_k(x,y')\big| \leq C_2 2^{k(d+\varepsilon)} \rho(y,y')^{\varepsilon}$.

(iv) $\big|\big[S_k(x,y) - S_k(x,y')\big] - \big[S_k(x',y) - S_k(x',y')\big]\big| \leq C_2 2^{k(d+2\varepsilon)} \rho(x,x')^{\varepsilon} \rho(y,y')^{\varepsilon}$.

(v) $\int_X S_k(x,y)\, d\mu(y) = 1$.

(vi) $\int_X S_k(x,y)\, d\mu(x) = 1$.

Remark 1.1. By a construction similar to Coifman's one in [3] one can construct an approximation to the identity of order θ with compact supports as in Definition 1.3 for the spaces of homogeneous type from Definition 1.1.

We also need the following construction of Christ in [1], which provides an analogue of the grid of Euclidean dyadic cubes on a space of homogeneous type.

Lemma 1.1. *Let X be a space of homogeneous type. Then there exist a collection $\{Q^k_\alpha \subset X : k \in \mathbb{Z}_+, \alpha \in I_k\}$ of open subsets, where I_k is some (possibly finite) index set, and constants $\delta \in (0,1)$ and $C_3, C_4 > 0$ such that:*

(i) *$\mu(X \setminus \cup_\alpha Q^k_\alpha) = 0$ for each fixed k and $Q^k_\alpha \cap Q^k_\beta = \emptyset$ if $\alpha \neq \beta$.*

(ii) *For any α, β, k, l with $l \geq k$, either $Q^l_\beta \subset Q^k_\alpha$ or $Q^l_\beta \cap Q^k_\alpha = \emptyset$.*

(iii) *For each (k,α) and each $l < k$ there is a unique β such that $Q^k_\alpha \subset Q^l_\beta$.*

(iv) $\operatorname{diam}(Q^k_\alpha) \leq C_3 \delta^k$.

(v) *Each Q^k_α contains some ball $B(z^k_\alpha, C_4\delta^k)$, where $z^k_\alpha \in X$.*

In fact, we can think of Q^k_α as being essentially a cube of diameter rough δ^k with center z^k_α. In what follows we always suppose $\delta = \frac{1}{2}$ (see [13] for how to remove this restriction). Also, we will denote by $Q_\tau^{k,\nu}$ $\big(\nu = 1, 2, \ldots, N(k,\tau)\big)$ the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$, where j is a fixed large positive integer. Denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$. For any dyadic cube Q and any $f \in L^1_{loc}(X)$ we set

$$m_Q(f) = \tfrac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x)$$

and we also let $a_+ = \max(a, 0)$.

Definition 1.4. Let $s \in (-\theta, \theta)$, $\{S_k\}_{k=0}^\infty$ be as in Definition 1.3 with order θ, $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. Suppose β and γ satisfying

$$\max\big(0, -s + d(\tfrac{1}{p} - 1)_+\big) < \beta < \theta \qquad \text{and} \qquad 0 < \gamma < \theta. \tag{1.3}$$

Let $j \in \mathbb{N}$ be fixed and large enough and $\{Q_\tau^{0,\nu} : \tau \in I_0, \nu = 1, \ldots, N(0,\tau)\}$ be as above. The *inhomogeneous Besov space* $B^s_{pq}(X)$ for $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < p \le \infty$ and $0 < q \le \infty$ is the collection of all $f \in \big(\mathring{\mathcal{G}}(\beta,\gamma)\big)'$ such that

$$\|f\|_{B^s_{pq}(X)} = \Big\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \big[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\big]^p \Big\}^{\frac{1}{p}} + \Big\{ \sum_{k=1}^\infty \big[2^{ks} \|D_k(f)\|_{L^p(X)}\big]^q \Big\}^{\frac{1}{q}} < \infty.$$

The *inhomogeneous Triebel-Lizorkin space* $F^s_{pq}(X)$ for $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < p < \infty$ and $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < q \le \infty$ is the collection of all $f \in \big(\mathring{\mathcal{G}}(\beta,\gamma)\big)'$ such that

$$\|f\|_{F^s_{pq}(X)} = \Big\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \big[m_{Q_\tau^{0,\nu}}(|D_0(f)|)\big]^p \Big\}^{\frac{1}{p}} + \Big\| \Big\{ \sum_{k=1}^\infty \big[2^{ks} |D_k(f)|\big]^q \Big\}^{\frac{1}{q}} \Big\|_{L^p(X)} < \infty.$$

Here, for $k \in \mathbb{Z}_+$ and a suitable f,

$$D_k(f)(x) = \int_X D_k(x,y) f(y)\, d\mu(y).$$

It was proved in [15] that Definition 1.4 is independent of the choices of large positive integers j, approximations to the identity and the pairs (β,γ) as in (1.3).

2. T1 theorems

In what follows, for $\eta \in (0,\theta]$ we let $C_0^\eta(X)$ be the set of all functions having compact support such that

$$\|f\|_{C_0^\eta(X)} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{\rho(x,y)^\eta} < \infty.$$

Endow $C_0^\eta(X)$ with the natural topology and let $(C_0^\eta(X))'$ be its dual space.

Definition 2.1. A continuous complex-valued function K on

$$\Omega = \big\{(x,y) \in X \times X : x \neq y\big\}$$

is called an *inhomogeneous Calderón-Zygmund kernel of type* (ε,σ) if there exist $\varepsilon \in (0,\theta], \sigma > 0$ and $C_5 > 0$ such that

$$|K(x,y)| \le C_5 \rho(x,y)^{-d} \qquad \text{for } \rho(x,y) \neq 0 \tag{2.1}$$

$$|K(x,y)| \le C_5 \rho(x,y)^{-d-\sigma} \qquad \text{for } \rho(x,y) \ge 1 \tag{2.2}$$

$$|K(x,y) - K(x',y)| \le C_5 \rho(x,x')^\varepsilon \rho(x,y)^{-d-\varepsilon} \qquad \text{for } \rho(x,x') \le \tfrac{\rho(x,y)}{2A} \tag{2.3}$$

$$|K(x,y) - K(x,y')| \le C_5 \rho(y,y')^\varepsilon \rho(x,y)^{-d-\varepsilon} \qquad \text{for } \rho(y,y') \le \tfrac{\rho(x,y)}{2A}. \tag{2.4}$$

We remark that (2.2) is natural when one considers the boundedness of Calderón-Zygmund operators on inhomogeneous function spaces, which was pointed by Meyer in [17: Chapter 10/Theorem 2].

Definition 2.2. A continuous linear operator $T : C_0^\eta(X) \to (C_0^\eta(X))'$ is an *inhomogeneous Calderón-Zygmund singular integral operator of type* (ε,σ) if there is an inhomogeneous standard kernel K of the type (ε,σ) such that

$$\langle Tf, g\rangle = \int_X \int_X K(x,y) f(y) g(x)\, d\mu(x) d\mu(y)$$

for all $f, g \in C_0^\eta(X)$ with disjoint supports.

We also need the following weak boundedness property.

Definition 2.3. A Calderón-Zygmund singular integral operator T is said to have the *weak boundedness property*, if there are $\eta \in (0,\theta]$ and $C_6 > 0$ such that

$$|\langle Tf, g\rangle| \le C_6 r^{d+2\eta} \|f\|_{C_0^\eta(X)} \|g\|_{C_0^\eta(X)}$$

for all $f, g \in C_0^\eta(X)$ with $\operatorname{diam}(\operatorname{supp} f) \le r$ and $\operatorname{diam}(\operatorname{supp} g) \le r$, and we denote this by $T \in WBP$.

In what follows we let T^* be the dual operator of T. The following theorem is the main theorem in this section.

Theorem 2.1. *Let $\varepsilon \in (0,\theta]$ and $|s| < \varepsilon$ be such that $\max(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}) < p < \infty$ and $\max(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}) < q \le \infty$. Suppose $T \in WBP$ is an inhomogeneous Calderón-Zygmund singular integral operator of type (ε,σ) with $\sigma > d(\frac{1}{p}-1)_+$ and $T1 = 0 = T^*1$, and its kernel K satisfies* (2.1) - (2.4)*. Then T is bounded on $F^s_{pq}(X)$ with an operator norm not larger than $C\max(C_5, C_6)$.*

To prove Theorem 2.1 we need to use the discrete Calderón reproducing formulas in [12] and the Plancherel-Pôlya inequality in [5].

Lemma 2.1 [12]. *Suppose $\{D_k\}_{k=0}^{\infty}$ is the same as in Definition* 1.4. *Then there exist functions $\widetilde{D}_{Q^{0,\nu}_\tau}$ with $\tau \in I_0$ and $\nu = 1,\dots,N(0,\tau)$ and $\widetilde{D}_k$ with $k \in \mathbb{N}$ such that, for any fixed $y^{k,\nu}_\tau \in Q^{k,\nu}_\tau$ with $k \in \mathbb{N}, \tau \in I_k$ and $\nu \in \{1,\dots,N(k,\tau)\}$ and all $f \in (\mathcal{G}(\beta_1,\gamma_1))'$ with $0 < \beta_1 < \theta$ and $0 < \gamma_1 < \theta$,*

$$f(x) = \sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau)\widetilde{D}_{Q^{0,\nu}_\tau}(f)\, m_{Q^{0,\nu}_\tau}(D_0(x,\cdot)) + \sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)\widetilde{D}_k(f)(y^{k,\nu}_\tau)D_k(x,y^{k,\nu}_\tau)$$

where the series converge in $(\mathcal{G}(\beta_1',\gamma_1'))'$ for $\beta_1 < \beta_1' < \theta$ and $\gamma_1 < \gamma_1' < \theta$. Here $\widetilde{D}_k$ with $k \in \mathbb{N}$ satisfies for any given $\varepsilon \in (0,\theta)$ the following conditions:

- **(i)** $|\widetilde{D}_k(x,y)| \le C\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{d+\varepsilon}}$.
- **(ii)** $\big|\widetilde{D}_k(x,y) - \widetilde{D}_k(x,y')\big| \le C\big(\frac{\rho(y,y')}{2^{-k}+\rho(x,y)}\big)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k}+\rho(x,y))^{d+\varepsilon}}$ *for* $\rho(y,y') \le \frac{1}{2A}\big(2^{-k}+\rho(x,y)\big)$.
- **(iii)** $\int_X \widetilde{D}_k(x,y)\,d\mu(x) = \int_X \widetilde{D}_k(x,y)\,d\mu(y) = 0$ *and* $\operatorname{diam}(Q^{0,\nu}_\tau) \sim 2^{-j}$ *for* $\tau \in I_0$ *and* $\nu = 1,\dots,N(0,\tau)$ *with some* $j \in \mathbb{N}$.

Further, $\widetilde{D}_{Q^{0,\nu}_\tau}$ for $\tau \in I_0$ and $\nu = 1,\dots,N(0,\tau)$ satisfies the following conditions:

- **(iv)** $\int_X \widetilde{D}_{Q^{0,\nu}_\tau}(x)\,d\mu(x) = 1$.
- **(v)** $|\widetilde{D}_{Q^{0,\nu}_\tau}(x)| \le C\frac{1}{(1+\rho(x,y^{0,\nu}_\tau))^{d+\varepsilon}}$.
- **(vi)** $\big|\widetilde{D}_{Q^{0,\nu}_\tau}(x) - \widetilde{D}_{Q^{0,\nu}_\tau}(y)\big| \le C\big(\frac{\rho(x,y)}{1+\rho(x,y^{0,\nu}_\tau)}\big)^{\varepsilon}\frac{1}{(1+\rho(x,y^{0,\nu}_\tau))^{d+\varepsilon}}$ *for* $\rho(x,y) \le \frac{1}{2A}(1+\rho(x,y^{0,\nu}_\tau))$.
- **(vii)** $\widetilde{D}_{Q^{0,\nu}_\tau}(f) = \int_X \widetilde{D}_{Q^{0,\nu}_\tau}(y)f(y)\,d\mu(y)$.

Moreover, j can be any fixed large positive integer and the constant C in properties (v) *and* (vi) *is independent of j.*

The following Plancherel-Pôlya inequality was given in the proof of the main theorem in [5] (see also [15]).

Lemma 2.2. *With the notation as in Lemma* 2.1,

$$\left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\widetilde{D}_{Q_\tau^{0,\nu}}(f)\right|^p\right\}^{\frac1p} + \left\{\sum_{k=1}^\infty 2^{ksq}\left(\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\left[\sup_{z\in Q_\tau^{k,\nu}}\left|\widetilde{D}_k(f)(z)\right|\right]^p\right)^{\frac qp}\right\}^{\frac1q} \le C\|f\|_{B^s_{pq}(X)}$$

when $\max\left(\frac{d}{d+\theta},\frac{d}{d+\theta+s}\right)<p\le\infty$ *and* $0<q\le\infty$, *and*

$$\left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\mu(Q_\tau^{0,\nu})\left|\widetilde{D}_{Q_\tau^{0,\nu}}(f)\right|^p\right\}^{\frac1p} + \left\|\left\{\sum_{k=1}^\infty\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\left[2^{ks}\sup_{z\in Q_\tau^{k,\nu}}\left|\widetilde{D}_k(f)(z)\right|\chi_{Q_\tau^{k,\nu}}(\cdot)\right]^q\right\}^{\frac1q}\right\|_{L^p(X)} \le C\|f\|_{F^s_{pq}(X)}$$

when $\max\left(\frac{d}{d+\theta},\frac{d}{d+\theta+s}\right)<p<\infty$ *and* $\max\left(\frac{d}{d+\theta},\frac{d}{d+\theta+s}\right)<q\le\infty$.

In what follows we will denote $m_{Q_{\tau'}^{0,\nu'}}(D_0(x,\cdot))$ simply by $D_{Q_{\tau'}^{0,\nu'}}(x)$ for $\tau'\in I_0$ and $\nu'=1,\dots,N(0,\tau')$. The following estimates are basic.

Lemma 2.3. *With the notation as in Lemma* 2.1,

(i) *for* $k\in\mathbb{Z}_+$, $\tau'\in I_0$ *and* $\nu'=1,\dots,N(0,\tau')$,

$$\left|D_kTD_{Q_{\tau'}^{0,\nu'}}(x)\right|\le C2^{-k\varepsilon}(1+k)\frac{1}{\left(1+\rho(x,y_{\tau'}^{0,\nu'})\right)^{d+\sigma_1}} \tag{2.5}$$

where $\sigma_1=\sigma$ *when* $k=0$ *and* $\sigma_1=\varepsilon$ *when* $k\in\mathbb{N}$,

(ii) *for* $k\in\mathbb{Z}_+$, $k'\in\mathbb{N}$, $\tau'\in I_{k'}$ *and* $\nu'=1,\dots,N(k',\tau')$,

$$\left|D_kTD_{k'}(x,y_{\tau'}^{k',\nu'})\right|\le C2^{-|k-k'|\varepsilon}(1+|k-k'|)\frac{2^{-(k\wedge k')\varepsilon}}{\left(2^{-(k\wedge k')}+\rho(x,y_{\tau'}^{k',\nu'})\right)^{d+\varepsilon}}. \tag{2.6}$$

Proof. We first show (2.5) with $k=0$. We consider two cases.

Case 1: $\rho(x,y_{\tau'}^{0,\nu'})\ge 6A^3C_7$, where $C_7=\max(1,C_3)$. In this case we write

$$\begin{aligned}
&\left|D_0TD_{Q_{\tau'}^{0,\nu'}}(x)\right|\\
&\quad=\left|\int_X\int_X D_0(x,u)K(u,v)D_{Q_{\tau'}^{0,\nu'}}(v)\,d\mu(u)d\mu(v)\right|\\
&\quad\le\frac{1}{\mu(Q_{\tau'}^{0,\nu'})}\int_{Q_{\tau'}^{0,\nu'}}\int_X\int_X|D_0(x,u)|\frac{1}{\rho(u,v)^{d+\sigma}}|D_0(v,y)|\,d\mu(u)d\mu(v)d\mu(y)\\
&\quad\le\frac{C}{\rho(x,y_{\tau'}^{0,\nu'})^{d+\sigma}}
\end{aligned}$$

which is a desired estimate.

Case 2: $\rho(x, y_{\tau'}^{0,\nu'}) < 6A^3C_7$. In this case let $\bar{\psi}_0 \in C_0^\infty(\mathbb{R})$ be a radial function with $0 \le \bar{\psi}_0(x) \le 1$, $\bar{\psi}_0(x) = 1$ for $x \in (0, 6)$ and $\bar{\psi}_0(x) = 0$ for $x > 12$. We now define $\psi_0(x) = \bar{\psi}_0\big(\frac{\rho(x, y_{\tau'}^{0,\nu'})}{A^3C_7}\big)$ and $\psi_1(x) = 1 - \psi_0(x)$. We then write

$$\begin{aligned}
D_0TD_{Q_{\tau'}^{0,\nu'}}(x) &= \big\langle D_0(x,\cdot)\psi_0(\cdot), TD_{Q_{\tau'}^{0,\nu'}}\big\rangle + \big\langle D_0(x,\cdot)\psi_1(\cdot), TD_{Q_{\tau'}^{0,\nu'}}\big\rangle \\
&=: E_1 + E_2.
\end{aligned}$$

Since $T \in WBP$, we have $|E_1| \le CC_6$. For any given x, since $\operatorname{supp}\big(D_0(x,\cdot)\psi_1(\cdot)\big) \cap \operatorname{supp} D_{Q_{\tau'}^{0,\nu'}}(\cdot) = \emptyset$, we can estimate $|E_2|$ by

$$|E_2| \le C\frac{1}{\mu(Q_{\tau'}^{0,\nu'})}\int_{Q_{\tau'}^{0,\nu'}}\int_X\int_X \big|D_0(x,u)D_0(v,y)\big|\,d\mu(u)d\mu(v)d\mu(y) \le C.$$

Thus Cases 1 and 2 tell us (2.5) for $k = 0$.

Let us show that (2.5) is also true when $k \in \mathbb{N}$. We still consider two cases.

If $\rho(x, y_{\tau'}^{0,\nu'}) \ge 12A^3C_7$, then by (2.3) we have

$$\begin{aligned}
&\big|D_kTD_{Q_{\tau'}^{0,\nu'}}(x)\big| \\
&\quad= \left|\int_X\int_X D_k(x,u)\big[K(u,v) - K(x,v)\big]D_{Q_{\tau'}^{0,\nu'}}(v)\,d\mu(u)d\mu(v)\right| \\
&\quad\le \frac{C}{\mu(Q_{\tau'}^{0,\nu'})}\int_{Q_{\tau'}^{0,\nu'}}\int_X\int_X |D_k(x,u)|\frac{\rho(x,u)^\varepsilon}{\rho(u,v)^{d+\varepsilon}}|D_0(v,y)|\,d\mu(u)d\mu(v)d\mu(y) \\
&\quad\le C2^{-k\varepsilon}\frac{1}{\rho(x, y_{\tau'}^{0,\nu'})^{d+\varepsilon}}
\end{aligned}$$

which is a desired estimate.

In what follows we let $\bar{\psi}_1(x) = 1 - \bar{\psi}_0(x)$. If $\rho(x, y_{\tau'}^{0,\nu'}) < 12A^3C_7$, then by $T1 = 0$ and $\int_X D_k(x,u)\,d\mu(u) = 0$ we can write

$$\begin{aligned}
&D_kTD_{Q_{\tau'}^{0,\nu'}}(x) \\
&\quad= \int_X\int_X D_k(x,u)K(u,v)\big[D_{Q_{\tau'}^{0,\nu'}}(v) - D_{Q_{\tau'}^{0,\nu'}}(x)\big]\,d\mu(u)d\mu(v) \\
&\quad= \int_X\int_X D_k(x,u)K(u,v)\big[D_{Q_{\tau'}^{0,\nu'}}(v) - D_{Q_{\tau'}^{0,\nu'}}(x)\big]\bar{\psi}_0\Big(\frac{\rho(v,x)}{2A^3C_72^{-k}}\Big)d\mu(u)d\mu(v) \\
&\qquad+ \int_X\int_X D_k(x,u)\big[K(u,v) - K(x,v)\big]\big[D_{Q_{\tau'}^{0,\nu'}}(v) - D_{Q_{\tau'}^{0,\nu'}}(x)\big] \\
&\qquad\times \bar{\psi}_1\Big(\frac{\rho(v,x)}{2A^3C_72^{-k}}\Big)d\mu(u)d\mu(v) \\
&\quad=: G_1 + G_2.
\end{aligned}$$

For G_1, letting $\psi(u) = D_k(x,u)$ and

$$\phi(v) = \big[D_{Q_{\tau'}^{0,\nu'}}(v) - D_{Q_{\tau'}^{0,\nu'}}(x)\big]\bar{\psi}_0\Big(\frac{\rho(v,x)}{2A^3C_7 2^{-k}}\Big),$$

by $T \in WBP$ we obtain

$$\begin{aligned}|G_1| &\le C_6 2^{-k(d+2\eta)}\|\phi\|_{C_0^\eta(X)}\|\psi\|_{C_0^\eta(X)}\\ &\le C2^{-k(d+2\eta)}2^{k(\eta-\varepsilon)}2^{k(d+\eta)}\\ &\le CC_6 2^{-k\varepsilon}\end{aligned}$$

where we choose $\eta \in (0,\varepsilon]$. To estimate G_2 we first point that it is easy to see the estimate

$$\big|D_{Q_{\tau'}^{0,\nu'}}(v) - D_{Q_{\tau'}^{0,\nu'}}(x)\big| \le C\Big(\frac{\rho(v,x)}{1+\rho(v,x)}\Big)^\varepsilon.$$

From this it is easy to deduce

$$\begin{aligned}|G_2| \le C &\int_X \int_{\{v\in X:\, \rho(x,v)>12A^3C_7 2^{-k}\}} |D_k(x,u)|\\ &\times \frac{\rho(x,u)^\varepsilon}{\rho(v,x)^{d+\varepsilon}}\Big(\frac{\rho(v,x)}{1+\rho(v,x)}\Big)^\varepsilon d\mu(v)d\mu(u)\\ \le{}& C2^{-k\varepsilon}\int_{\{v\in X:\, \rho(x,v)>12A^3C_7\}} \frac{1}{\rho(v,x)^{d+\varepsilon}}\, d\mu(v)\\ &+ C2^{-k\varepsilon}\int_{\{v\in X:\, 12A^3C_7 2^{-k}<\rho(x,v)\le 12A^3C_7\}} \frac{1}{\rho(v,x)^d}\, d\mu(v)\\ \le{}& C2^{-k\varepsilon}(1+k).\end{aligned}$$

Thus (2.5) always holds.

We now prove (2.6) in the case of $k' \ge k$. We still need to consider two cases.

Case 1: $\rho(x, y_{\tau'}^{k',\nu'}) \ge 12A^3C_7 2^{-k}$. In this case, by $\int_X D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(v) = 0$ we have

$$\begin{aligned}&\big|D_kTD_{k'}(x, y_{\tau'}^{k',\nu'})\big|\\ &\quad= \Big|\int_X\int_X D_k(x,u)\big[K(u,v) - K(u, y_{\tau'}^{k',\nu'})\big]D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(u)d\mu(v)\Big|\\ &\quad\le C\int_X\int_X |D_k(x,u)|\frac{\rho(v, y_{\tau'}^{k',\nu'})^\varepsilon}{\rho(u,v)^{d+\varepsilon}}\big|D_{k'}(v, y_{\tau'}^{k',\nu'})\big|\, d\mu(u)d\mu(v)\\ &\quad\le C2^{-(k'-k)\varepsilon}\frac{2^{-k\varepsilon}}{\rho(x, y_{\tau'}^{k',\nu'})^{d+\varepsilon}}\end{aligned}$$

which is a desired estimate.

Case 2: $\rho(x, y_{\tau'}^{k',\nu'}) < 12A^3C_7 2^{-k}$. In this case, by $T^*1 = 0$ we obtain

$$\begin{aligned}
&D_k T D_{k'}(x, y_{\tau'}^{k',\nu'})\\
&\quad= \int_X \int_X \big[D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big] K(u,v) D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(u) d\mu(v)\\
&\quad= \int_X \int_X \big[D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big] \bar{\psi}_0\Big(\frac{\rho(u, y_{\tau'}^{k',\nu'})}{2A^3C_7 2^{-k'}}\Big)\\
&\qquad \times K(u,v) D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(u) d\mu(v)\\
&\qquad + \int_X \int_X \big[D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big] \bar{\psi}_1\Big(\frac{\rho(u, y_{\tau'}^{k',\nu'})}{2A^3C_7 2^{-k'}}\Big)\\
&\qquad \times K(u,v) D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(u) d\mu(v)\\
&\quad =: H_1 + H_2.
\end{aligned}$$

To estimate H_1 let

$$\psi(u) = \big[D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big] \bar{\psi}_0\Big(\frac{\rho(u, y_{\tau'}^{k',\nu'})}{2A^3C_7 2^{-k'}}\Big)$$

and $\phi(v) = D_{k'}(v, y_{\tau'}^{k',\nu'})$. By $T \in WBP$ we have

$$\begin{aligned}
|H_1| &\le C 2^{-(d+2\eta)k'} \|\psi\|_{C_0^\eta(X)} \|\phi\|_{C_0^\eta(X)}\\
&\le C 2^{-(d+2\eta)k'} 2^{kd-(k'-k)\varepsilon + k'\eta} 2^{k'(d+\eta)}\\
&\le C 2^{-(k'-k)\varepsilon} 2^{kd}
\end{aligned}$$

where we choose $\eta \in (0, \varepsilon]$. To estimate H_2 we first note that

$$\big|D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big| \le C 2^{kd} \frac{\rho(u, y_{\tau'}^{k',\nu'})^\varepsilon}{\big(2^{-k} + \rho(u, y_{\tau'}^{k',\nu'})\big)^\varepsilon}.$$

is easy to see. From this and $\int_X D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(v) = 0$ it follows that

$$\begin{aligned}
|H_2| &= \bigg| \int_X \int_X \big[D_k(x,u) - D_k(x, y_{\tau'}^{k',\nu'})\big] \bar{\psi}_1\Big(\frac{\rho(u, y_{\tau'}^{k',\nu'})}{2A^3C_7 2^{-k'}}\Big) \\
&\qquad \times \big[K(u,v) - K(u, y_{\tau'}^{k',\nu'})\big] D_{k'}(v, y_{\tau'}^{k',\nu'})\, d\mu(u) d\mu(v)\bigg| \\
&\le C 2^{kd-k'\varepsilon} \int_X \int_{\{u:\, \rho(u, y_{\tau'}^{k',\nu'}) \ge 12A^3C_7 2^{-k'}\}} \frac{\rho(u, y_{\tau'}^{k',\nu'})^{\varepsilon}}{\big(2^{-k} + \rho(u, y_{\tau'}^{k',\nu'})\big)^{\varepsilon}} \\
&\qquad \times \frac{1}{\rho(u, y_{\tau'}^{k',\nu'})^{d+\varepsilon}} |D_{k'}(v, y_{\tau'}^{k',\nu'})|\, d\mu(u) d\mu(v) \\
&\le C 2^{kd-k'\varepsilon} \int_{\{u:\, \rho(u, y_{\tau'}^{k',\nu'}) \ge 12A^3C_7 2^{-k}\}} \frac{1}{\rho(u, y_{\tau'}^{k',\nu'})^{d+\varepsilon}}\, d\mu(u) \\
&\qquad + C 2^{-(k'-k)\varepsilon + kd} \int_{\{u:\, 12A^3C_7 2^{-k'} \le \rho(u, y_{\tau'}^{k',\nu'}) < 12A^3C_7 2^{-k}\}} \frac{1}{\rho(u, y_{\tau'}^{k',\nu'})^{d}}\, d\mu(u) \\
&\le C(1 + k' - k) 2^{-(k'-k)\varepsilon + kd}
\end{aligned}$$

which is a desired estimate. Thus (2.6) is true when $k' \ge k$. The proof of (2.6) when $k' < k$ is similar. We leave the details to the reader (see also [4]). This finishes the proof of Lemma 2.3 ∎

The following lemma can be found in [13: pp. 93].

Lemma 2.4. *Let $0 < r \le 1$, $k, \eta \in \mathbb{Z}_+$ with $\eta \le k$ and, for any dyadic cube $Q_\tau^{k,\nu}$,*

$$|f_{Q_\tau^{k,\nu}}(x)| \le \big(1 + 2^{\eta} \rho(x, y_\tau^{k,\nu})\big)^{-d-\gamma}$$

where $y_\tau^{k,\nu}$ is any point in $Q_\tau^{k,\nu}$ and $\gamma > d(\frac{1}{r} - 1)$. Then

$$\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}|\, |f_{Q_\tau^{k,\nu}}(x)| \le C 2^{\frac{(k-\eta)d}{r}} \bigg[M\Big(\sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} |\lambda_{Q_\tau^{k,\nu}}|^r \chi_{Q_\tau^{k,\nu}} \Big)(x) \bigg]^{\frac{1}{r}}$$

where C is independent of x, k and η, and M is the Hardy-Littlewood maximal operator on X.

Proof of Theorem 2.1. By Definition 1.4 we can have

$$\begin{aligned}
\|Tf\|_{F_{pq}^s(X)} &\le \bigg\{ \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) \big[m_{Q_\tau^{0,\nu}}(|D_0(Tf)|) \big]^p \bigg\}^{\frac{1}{p}} \\
&\qquad + \bigg\| \bigg\{ \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \sup_{z \in Q_\tau^{k,\nu}} |D_k(Tf)(z)|^q \chi_{Q_\tau^{k,\nu}} \bigg\}^{\frac{1}{q}} \bigg\|_{L^p(X)} \\
&=: J_1 + J_2.
\end{aligned}$$

By noting that $\mu(Q_\tau^{0,\nu}) \sim C$ and Lemma 2.1 we have

$$\begin{aligned}
J_1 \le\ & C\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\Big[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|\sup_{z\in Q_\tau^{0,\nu}}\big|D_0TD_{Q_{\tau'}^{0,\nu'}}(z)\big|\Big]^p\Big\}^{\frac{1}{p}}\\
&+C\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\Big[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|\\
&\times\sup_{z\in Q_\tau^{0,\nu}}\big|D_0TD_{k'}(z,y_{\tau'}^{k',\nu'})\big|\Big]^p\Big\}^{\frac{1}{p}}\\
=:\ & J_1^1+J_1^2.
\end{aligned}$$

When $p \le 1$, by (2.5), the following well-known inequality

$$\Big(\sum_i |a_i|\Big)^p \le \sum_i |a_i|^p \tag{2.7}$$

for $p \le 1$ and $a_i \in \mathbb{C}$ and by Lemma 2.2 we obtain

$$\begin{aligned}
J_1^1 &\le C\Big\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\Big[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\frac{1}{\big(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'})\big)^{(d+\sigma)p}}\Big]\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\int_X\frac{1}{(1+\rho(x,y_{\tau'}^{0,\nu'}))^{(d+\sigma)p}}\,d\mu(x)\Big\}^{\frac{1}{p}}\\
&\le C\Big\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\Big\}^{\frac{1}{p}}\\
&\le C\|f\|_{F_{pq}^s(X)}
\end{aligned}$$

where we used the fact that $1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'}) \sim 1+\rho(x,y_\tau^{0,\nu})$ for all $x \in Q_\tau^{0,\nu}$, $\mu(Q_\tau^{0,\nu}) \sim C$, Lemma 1.1 and $\sigma > d(\frac{1}{p}-1)$.

When $1 < p \le \infty$, from (2.5), the Hölder inequality and Lemma 2.2 it follows that

$$
\begin{aligned}
J_1^1 &\le C\bigg\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\bigg[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\frac{1}{\big(1+\rho(y_\tau^{0,\nu},y_{\tau'}^{0,\nu'})\big)^{d+\sigma}}\bigg]\\
&\quad\times\bigg[\int_X\frac{1}{\big(1+\rho(y_\tau^{0,\nu},y)\big)^{d+\sigma}}\,d\mu(y)\bigg]^{\frac{p}{p'}}\bigg\}^{\frac1p}\\
&\le C\bigg\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\int_X\frac{1}{\big(1+\rho(x,y_{\tau'}^{0,\nu'})\big)^{d+\sigma}}\,d\mu(x)\bigg\}^{\frac1p}\\
&\le C\bigg\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\bigg\}^{\frac1p}\\
&\le C\|f\|_{F_{pq}^s(X)}.
\end{aligned}
$$

By (2.6) - (2.7), the Hölder inequality, the Fefferman-Stein vector-valued inequality in [6], Lemma 2.4, the arbitrariness of $y_{\tau'}^{k',\nu'}$ and Lemma 2.2 we obtain

$$
\begin{aligned}
J_1^2 &\le C\bigg\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau)}\int_X\chi_{Q_\tau^{0,\nu}}(x)\bigg[\sum_{k'=1}^{\infty}(1+k')2^{-k'(\varepsilon+d+s)}\\
&\quad\times\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k's}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|\frac{1}{\big(1+\rho(x,y_{\tau'}^{k',\nu'})\big)^{d+\varepsilon}}\bigg]^p d\mu(x)\bigg\}^{\frac1p}\\
&\le C\bigg\{\int_X\bigg(\sum_{k'=1}^{\infty}(1+k')2^{-k'(\varepsilon+d+s-\frac{d}{r})}\\
&\quad\times\bigg[M\bigg(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\bigg)\bigg]^{\frac1r}\bigg)^p d\mu(x)\bigg\}^{\frac1p}\\
&\le C\bigg\|\bigg\{\sum_{k'=1}^{\infty}\bigg[M\bigg(\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\bigg)\bigg]^{\frac{q}{r}}\bigg\}^{\frac1q}\bigg\|_{L^p(X)}\\
&\le C\bigg\|\bigg\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^q\chi_{Q_{\tau'}^{k',\nu'}}\bigg\}^{\frac1q}\bigg\|_{L^p(X)}\\
&\le C\|f\|_{F_{pq}^s(X)}
\end{aligned}
$$

where we choose $\max\big(\frac{d}{d+s+\varepsilon},\frac{d}{d+\varepsilon}\big)<r<\min(1,p,q)$. So far we have obtained a desired estimate for J_1.

Let us now estimate J_2 by writing

$$\begin{aligned}
J_2 &\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\Bigg[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big| \\
&\quad\times \sup_{z\in Q_\tau^{k,\nu}}\big|D_k T D_{Q_{\tau'}^{0,\nu'}}(z)\big|\Bigg]^q \chi_{Q_\tau^{k,\nu}}\Bigg\}^{\frac{1}{q}}\Bigg\|_{L^p(X)} \\
&\quad + C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\Bigg[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big| \\
&\quad\times \sup_{z\in Q_\tau^{k,\nu}}\big|D_k T D_{k'}(z, y_{\tau'}^{k',\nu'})\big|\Bigg]^q \chi_{Q_\tau^{k,\nu}}\Bigg\}^{\frac{1}{q}}\Bigg\|_{L^p(X)} \\
&=: J_2^1 + J_2^2.
\end{aligned}$$

Estimate (2.5), Lemma 2.4, the Fefferman-Stein vector-valued inequality in [6] and Lemma 2.2 tell us that

$$\begin{aligned}
J_2^1 &\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} k^q 2^{k(s-\varepsilon)q} \\
&\quad\times\Bigg[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|\frac{1}{\big(1+\rho(y_\tau^{k,\nu}, y_{\tau'}^{0,\nu'})\big)^{d+\varepsilon}}\Bigg]^q \chi_{Q_\tau^{k,\nu}}\Bigg\}^{\frac{1}{q}}\Bigg\|_{L^p(X)} \\
&\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty} k^q 2^{k(s-\varepsilon)q}\Bigg\}^{\frac{1}{q}}\Bigg\{M\Bigg(\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^r \chi_{Q_{\tau'}^{0,\nu'}}\Bigg)\Bigg\}^{\frac{1}{r}}\Bigg\|_{L^p(X)} \\
&\le C\Bigg\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau')}\mu(Q_{\tau'}^{0,\nu'})\big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big|^p\Bigg\}^{\frac{1}{p}} \\
&\le C\|f\|_{F_{pq}^s(X)}
\end{aligned}$$

where we choose $r \in \big(\frac{d}{d+\varepsilon}, \min(p,1)\big)$. From (2.6), Lemma 2.4, (2.7), the Fefferman-Stein vector-valued inequality in [6], the arbitrariness of $y_{\tau'}^{k',\nu'}$ and

Lemma 2.2 it follows that

$$\begin{aligned}
J_2^2 &\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\Bigg[\sum_{k'=1}^{\infty}\big(1+|k-k'|\big)\,2^{(k\wedge k')d-|k-k'|\varepsilon-k'd-k's}\\
&\quad\times\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k's}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|\frac{1}{\big(1+2^{k\wedge k'}\rho(\cdot,y_{\tau'}^{k',\nu'})\big)^{d+\varepsilon}}\Bigg]^q\chi_{Q_\tau^{k,\nu}}\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\Bigg[\sum_{k'=1}^{\infty}\big(1+|k-k'|\big)\,2^{(k\wedge k')d-|k-k'|\varepsilon-k'd+(k-k')s+[k'-(k\wedge k')]\frac{d}{r}}\\
&\quad\times\Bigg(M\Bigg[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\Bigg]\Bigg)^{\frac1r}\Bigg]^q\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\le C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\Bigg[\sum_{k'=1}^{k}\big(1+k-k'\big)\,2^{(k-k')(s-\varepsilon)}\\
&\quad\times\Bigg(M\Bigg[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\Bigg]\Bigg)^{\frac1r}\Bigg]^q\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\quad+C\Bigg\|\Bigg\{\sum_{k=1}^{\infty}\Bigg[\sum_{k'=k+1}^{\infty}\big(1+k'-k\big)\,2^{(k-k')(d+s+\varepsilon-\frac{d}{r})}\\
&\quad\times\Bigg(M\Bigg[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\Bigg]\Bigg)^{\frac1r}\Bigg]^q\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\le C\Bigg\|\Bigg\{\sum_{k'=1}^{\infty}\Bigg(M\Bigg[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sr}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^r\chi_{Q_{\tau'}^{k',\nu'}}\Bigg]\Bigg)^{\frac qr}\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\le C\Bigg\|\Bigg\{\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq}\big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big|^q\chi_{Q_{\tau'}^{k',\nu'}}\Bigg\}^{\frac1q}\Bigg\|_{L^p(X)}\\
&\le C\|f\|_{F_{pq}^s(X)}
\end{aligned}$$

where we choose $\max(\frac{d}{d+s+\varepsilon},\frac{d}{d+\varepsilon})<r<\min(1,p,q)$. This finishes the proof of Theorem 2.1 ∎

To establish a similar theorem for the Besov spaces $B_{pq}^s(X)$ we need the following real interpolation theorems from [25].

Lemma 2.5. *Let $\kappa\in(0,1)$, $s_0,s_1\in(-\theta,\theta)$ with $s_0\ne s_1$ and $s=(1-\kappa)s_0+\kappa s_1$.*

(i) *If* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+s_0+\theta}, \frac{d}{d+s_1+\theta}\right) < p \le \infty$ *and* $0 < q_0, q_1, q \le \infty$, *then*

$$\left(B^{s_0}_{p,q_0}(X), B^{s_1}_{p,q_1}(X)\right)_{\kappa,q} = B^s_{pq}(X).$$

(ii) *If* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+s_0+\theta}, \frac{d}{d+s_1+\theta}\right) < p < \infty$ *and* $\max\left(\frac{d}{d+\theta}, \frac{d}{d+s_i+\theta}\right) < q_i \le \infty$ *for* $i = 0, 1$ *and* $0 < q \le \infty$, *then*

$$\left(F^{s_0}_{p,q_0}(X), F^{s_1}_{p,q_1}(X)\right)_{\kappa,q} = B^s_{pq}(X).$$

The following is the $T1$ theorem for the Besov spaces $B^s_{pq}(X)$.

Theorem 2.2. *Let* $\varepsilon \in (0, \theta], |s| < \varepsilon, \max\left(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}\right) < p \le \infty$ *and* $0 < q \le \infty$. *Suppose* $T \in WBP$, $T1 = 0 = T^*1$, *is an inhomogeneous Calderón-Zygmund singular integral operator of type* (ε, σ) *with* $\sigma > d(\frac{1}{p} - 1)_+$ *and its kernel* K *satisfies* (2.1) – (2.4). *Then* T *is bounded on* $B^s_{pq}(X)$ *with an operator norm not larger than* $C \max(C_5, C_6)$.

Proof. The case $p < \infty$ is a simple corollary of Theorem 2.1 and Lemma 2.5. To show the case $p = \infty$, by Lemma 2.5 we only need to show that T is bounded on $B^s_{\infty\infty}(X)$ for $|s| < \varepsilon$. To see this we write

$$\begin{aligned} \|Tf\|_{B^s_{\infty\infty}(X)} &\le C \sup_{\substack{\tau \in I_0 \\ \nu = 1, \ldots, N(0,\tau)}} m_{Q^{0,\nu}_\tau}(|D_0(Tf)|) \\ &\quad + C \sup_{k \in \mathbb{N}} 2^{ks} \sup_{\substack{\tau \in I_k \\ \nu = 1, \ldots, N(k,\tau)}} \sup_{z \in Q^{k,\nu}_\tau} |D_k(Tf)(z)| \\ &=: H_1 + H_2. \end{aligned}$$

By Lemma 2.1, (2.5) and (2.6), the arbitrariness of $y^{k',\nu'}_{\tau'}$ and Lemma 2.2 we obtain

$$
\begin{aligned}
H_1 &\le C \sup_{\substack{\tau'\in I_0\\ \nu=1,\dots,N(0,\tau')}} \left|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\right| \sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau')} \frac{1}{\left(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{0,\nu'})\right)^{d+\sigma}} \\
&\quad + C \sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} \sum_{k'=1}^{\infty} (1+k')\, 2^{-k'\varepsilon} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) \left|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\right| \\
&\quad \times \frac{1}{\left(1+\rho(y_\tau^{0,\nu}, y_{\tau'}^{k',\nu'})\right)^{d+\varepsilon}} \\
&\le C \sup_{\substack{\tau'\in I_0\\ \nu=1,\dots,N(0,\tau')}} \left|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\right| + C \sup_{k'\in\mathbb{N}} 2^{k's} \sup_{\substack{\tau'\in I_{k'}\\ \nu'=1,\dots,N(k',\tau')}} \left|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\right| \\
&\quad \times \sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} \sum_{k'=1}^{\infty} (1+k')\, 2^{-k'(s+\varepsilon)} \int_X \frac{1}{\left(1+\rho(y_\tau^{0,\nu}, y)\right)^{d+\varepsilon}}\, d\mu(y) \\
&\le C \sup_{\substack{\tau'\in I_0\\ \nu=1,\dots,N(0,\tau')}} \left|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\right| + C \sup_{k'\in\mathbb{N}} 2^{k's} \sup_{\substack{\tau'\in I_{k'}\\ \nu'=1,\dots,N(k',\tau')}} \left|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\right| \\
&\le C\|f\|_{B^s_{\infty\infty}(X)}.
\end{aligned}
$$

From Lemma 2.1, (2.5) and (2.6), the arbitrariness of $y_{\tau'}^{k',\nu'}$ and Lemma 2.2

it also follows that

$$
\begin{aligned}
H_2 &\le C \sup_{k\in\mathbb{N}} k2^{k(s-\varepsilon)} \sup_{\substack{\tau\in I_k\\ \nu=1,\dots,N(k,\tau)}} \sum_{\tau'\in I_0} \sum_{\nu'=1}^{N(0,\tau')} \big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big| \frac{1}{\big(1+\rho(y_\tau^{k,\nu}, y_{\tau'}^{0,\nu'})\big)^{d+\varepsilon}} \\
&\quad + C \sup_{k\in\mathbb{N}} 2^{ks} \sup_{\substack{\tau\in I_k\\ \nu=1,\dots,N(k,\tau)}} \sum_{k'=1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) \big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big| \\
&\quad \times \big(1+|k-k'|\big) 2^{-|k-k'|\varepsilon} \frac{2^{-(k\wedge k')\varepsilon}}{\big(2^{-(k\wedge k')}+\rho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'})\big)^{d+\varepsilon}} \\
&\le C \sup_{\substack{\tau'\in I_0\\ \nu'=1,\dots,N(0,\tau')}} \big|\widetilde{D}_{Q_{\tau'}^{0,\nu'}}(f)\big| \sup_{k\in\mathbb{N}} k\, 2^{k(s-\varepsilon)} \\
&\quad \times \sup_{\substack{\tau\in I_k\\ \nu=1,\dots,N(k,\tau)}} \int_X \frac{1}{\big(1+\rho(y_\tau^{k,\nu}, y)\big)^{d+\varepsilon}}\, d\mu(y) \\
&\quad + \sup_{k'\in\mathbb{N}} 2^{k's} \sup_{\substack{\tau'\in I_{k'}\\ \nu'=1,\dots,N(k',\tau')}} \big|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})\big| \sup_{k\in\mathbb{N}} \sum_{k'=1}^{\infty} \big(1+|k-k'|\big) 2^{(k-k')s-|k-k'|\varepsilon} \\
&\quad \times \int_X \frac{2^{-(k\wedge k')\varepsilon}}{\big(2^{-(k\wedge k')}+\rho(y_\tau^{k,\nu}, y)\big)^{d+\varepsilon}}\, d\mu(y) \\
&\le C\|f\|_{B^s_{\infty\infty}(X)}.
\end{aligned}
$$

This proves Theorem 2.2 ∎

Now we assume $\mu(X)=\infty$. The homogeneous Besov spaces $\dot{B}^s_{pq}(X)$ for $s\in(-\theta,\theta)$, $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < p \le \infty$ and $0<q\le\infty$ and the Triebel-Lizorkin spaces $\dot{F}^s_{pq}(X)$ for $s\in(-\theta,\theta)$, $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < p < \infty$ and $\max\big(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}\big) < q \le \infty$ have been introduced by Han in [7]. By using the homogeneous discrete Calderón reproducing formulas in [9] and some arguments similar to those for Theorems 2.1 and 2.2 we can show the following $T1$ theorems for the homogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type. We omit the details.

Theorem 2.3. *Let $\varepsilon\in(0,\theta]$ and $|s|<\varepsilon$. Suppose $T\in WBP$, $T1 = 0 = T^*1$, is a standard Calderón-Zygmund operator of type ε and its kernel K satisfies* $(2.1), (2.3)-(2.4)$*. Then:*

(i) *T is bounded on $\dot{B}^s_{pq}(X)$ with an operator norm not larger than $C\max(C_5, C_6)$ if $\max\big(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}\big) < p \le \infty$ and $0<q\le\infty$.*

(ii) *T is bounded on $\dot{F}^s_{pq}(X)$ with an operator norm not larger than $C\max(C_5, C_6)$ if $\max\big(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}\big) < p < \infty$ and $\max\big(\frac{d}{d+\varepsilon}, \frac{d}{d+s+\varepsilon}\big) < q \le \infty$.*

Here we say a kernel $K(x,y)$ is a *standard Calderón-Zygmund kernel of type* ε if it satisfies (2.1), (2.3) and (2.4). Moreover, we say an operator T is a *standard Calderón-Zygmund singular integral operator of type* ε if it corresponds to a standard Calderón-Zygmund kernel of type ε as in Definition 2.2. We point that, differently from the cases $B^s_{pq}(X)$ and $F^s_{pq}(X)$, we do not need the kernel K to satisfy (2.2) in Theorem 2.3. We should also remark that if $0 < s < \varepsilon$, $T \in WBP$ with $T1 = 0$ and its kernel K only satisfies (2.1) and (2.3), then T is also bounded on $\dot{B}^s_{pq}(X)$ and $\dot{F}^s_{pq}(X)$ for p and q as in Theorem 2.3. This was proved by Deng and Han in [4]. There they also gave a direct proof of the case $\dot{B}^s_{pq}(X)$ instead of using real interpolation.

3. An application

In this section, we will consider the boundedness of the following operator of Bessel potential type I_α on Besov and Triebel-Lizorkin spaces.

Definition 3.1. Let $\{D_l\}_{l=0}^\infty$ be as in Definition 1.4 and $\alpha \in \mathbb{R}$. Then the *operator* I_α for $f \in \mathcal{G}(\beta,\gamma)$ with $0 < \beta \le \theta$ and $0 < \gamma$ is defined by

$$I_\alpha(f)(x) = \sum_{l=0}^\infty 2^{-l\alpha} D_l(f)(x)$$

where $x \in X$.

Operators of this type were first studied by Nahmod in [18]. Definition 3.1 was given in [14]. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge \frac{3}{2}$ and let

$$S_k(x,y) = [\varphi(2^{-k}\cdot)]^\vee(x-y)$$

for $k \in \mathbb{Z}_+$. Then $\{S_k\}_{k=0}^\infty$ is an approximation to the identity on $\mathbb{R}^n$ without compact support (see [15]). Let $S_{-1} = 0$. In this case we have

$$\begin{aligned} I_\alpha(f)^\wedge(\xi) &= \sum_{l=0}^\infty 2^{-l\alpha}[S_l - S_{l-1}]^\wedge(\xi)\hat{f}(\xi) \\ &= \sum_{l=0}^\infty 2^{-l\alpha}[\varphi(2^{-l}\xi) - \varphi(2^{-l+1}\xi)]\hat{f}(\xi) \\ &\sim (1+|\xi|^2)^{-\frac{\alpha}{2}}\hat{f}(\xi). \end{aligned}$$

Thus I_α is equivalent to the Bessel potential operator in the sense of Fourier transforms.

Theorem 3.1. *Let $|\alpha| < \theta$, $|s| < \theta$ and $|s+\alpha| < \theta$.*

(i) *If $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < p \le \infty$, $0 < q \le \infty$ and $d(\frac{1}{p} - 1)_+ < \theta - \alpha$, then I_α is bounded from $B^s_{pq}(X)$ into $B^{s+\alpha}_{pq}(X)$.*

(ii) *If $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < p < \infty$, $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < q \le \infty$ and $d(\frac{1}{\min(p,q)} - 1)_+ < \theta - \alpha$, then I_α is bounded from $F^s_{pq}(X)$ into $F^{s+\alpha}_{pq}(X)$.*

Theorem 3.1 when $p, q > 1$ was obtained in [14] by using the atom and molecule characterizations of these spaces. Moreover, by using Theorems 2.1 and 2.2 we can establish the converse of Theorem 3.1.

Theorem 3.2. *Let $|\alpha| < \theta$, $|s| < \theta$ and $|s+\alpha| < \theta$.*

(i) *If $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < p \le \infty$, $0 < q \le \infty$ and $d(\frac{1}{p} - 1)_+ < \theta + \alpha$, then there are $\alpha_0(s) \in (0, \theta)$ and a constant $C > 0$ such that if $|\alpha| < \alpha_0(s)$, then*

$$\|f\|_{B^s_{pq}(X)} \le C\|I_\alpha(f)\|_{B^{s+\alpha}_{pq}(X)}$$

for all $f \in B^s_{pq}(X)$.

(ii) *If $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < p < \infty$, $\max\left(\frac{d}{d+\theta}, \frac{d}{d+\theta+s}, \frac{d}{d+\theta+s+\alpha}\right) < q \le \infty$ and $d(\frac{1}{\min(p,q)} - 1)_+ < \theta + \alpha$, then there are $\alpha_0(s) \in (0, \theta)$ and a constant $C > 0$ such that, if $|\alpha| < \alpha_0(s)$, then*

$$\|f\|_{F^s_{pq}(X)} \le C\|I_\alpha(f)\|_{F^{s+\alpha}_{pq}(X)}$$

for all $f \in F^s_{pq}(X)$.

Proof. Let $T = I - I_{-\alpha}I_\alpha$ and $K(x,y)$ its kernel. In [14] it was proved that there are $\alpha_1, \delta, \delta_1 \in (0, \theta)$ and constants $C_8, C_9 > 0$ such that if $|\alpha| < \alpha_1$, then K is an inhomogeneous kernel of type (ε, σ) in terms of Definition 2.1 with

$$C_5 \le C_8 2^{-\delta N} + C_9 \sum_{|l| \le N} |1 - 2^{l\alpha}| 2^{-|l|\delta_1}$$

for any $N \in \mathbb{N}$ where $\varepsilon \in (0, \theta)$ and $\sigma > 0$ can be any numbers, the constants C_8 and C_9 are independent of N and α, but C_8 may depend on α_1 and δ. Also, α_1 and δ can be any positive number less than θ. Moreover, $T \in WBP$ with

$$C_6 \le C_8 2^{-\delta N} + C_9 \sum_{|l| \le N} |1 - 2^{l\alpha}| 2^{-|l|\delta_1}.$$

Thus by Theorems 2.1 and 2.3 we know that T is bounded on $B^s_{pq}(X)$ and $F^s_{pq}(X)$ with an operator norm not larger than $C_{10} = C\max(C_5, C_6)$. Now if we choose α_1 small enough, then we can have $C_{10} < 1$. This just means that if α_1 is small enough, then $I_{-\alpha}I_\alpha$ is an invertible operator on $B^s_{pq}(X)$ and $F^s_{pq}(X)$. Then, by Theorem 3.1,

$$\|f\|_{B^s_{pq}(X)} = \big\|(I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha\big\|_{B^s_{pq}(X)} \leq C\|I_{-\alpha}I_\alpha\|_{B^s_{pq}(X)} \leq C\|I_\alpha(f)\|_{B^{s+\alpha}_{pq}(X)}$$

for all $f \in B^s_{pq}(X)$ and

$$\|f\|_{F^s_{pq}(X)} = \big\|(I_{-\alpha}I_\alpha)^{-1}I_{-\alpha}I_\alpha\big\|_{F^s_{pq}(X)} \leq C\|I_{-\alpha}I_\alpha\|_{F^s_{pq}(X)} \leq C\|I_\alpha(f)\|_{F^{s+\alpha}_{pq}(X)}$$

for all $f \in F^s_{pq}(X)$. This proves Theorem 3.2 ∎

From Theorems 3.1 and 3.3 we see that I_α can be used as a lifting operator for the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ (see also [20] for the $\mathbb{R}^n$ case). Finally, we point that Theorem 3.2 for $p, q > 1$ was obtained in [14] by using the atom and molecule characterizations of these spaces, which however is more complicated than the proof given here.

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