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Necessary Conditions for Local and Global Solvability of Non-Diagonal Degenerate Systems

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Abstract. Cross-diffusion has been widely considered either in the mechanical description of diffusion or in the stochastic point process description of interacting populations, in the mathematical modelling of spatially structured epidemic or ecological systems and for the geographical diffusion of innovation. In this paper, specific attention is devoted to blowing-up solutions of some systems which may reflect either failures in the modelling or genuine phenomena like aggregation of populations. Furthemore, necessary conditions for local and global existence of solutions to the considered systems are presented.

Keywords: *Nonlinear degenerate reaction-diffusion systems, porous media, solvability conditions, blowing-up*

AMS subject classification: 35K45, 35K55, 35K65

1. Introduction

The role of spatial heterogeneities and dispersal for chemical reacting species or biological interacting populations in the linear or nonlinear regime has been the subject of a sizeable literature (see, e.g., the authorative books of Aris [1] and Cussler [10]). In particular, cross-diffusion in modelling interactions among different species has attracted special attentions. Apart the above quoted books, one can cite [16, 18, 32 - 34] in physical chemistry, [7, 23] in epidemics, [19, 29] in ecology and population dynamics, [21] in biology and very recently [8] in economics. The recent papers [20, 23, 30, 31] on reaction-diffusion systems with "non-diagonal" diffusion matrices are devoted to global existence and large time behaviour.

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In the present article we consider the system

$$\left.\begin{aligned} u_t(x,t) &= \Delta(|u|^{n-1}u) + \alpha\Delta(|v|^{m-1}v) + f(x,t)|v|^p + w_1(x,t) \\ v_t(x,t) &= \Delta(|v|^{l-1}v) + g(x,t)|u|^q + w_2(x,t) \end{aligned}\right\}$$

for $(x,t) \in \mathbb{R}^N \times \mathbb{R}^+$, subject to the initial distributions

$$\big(u(x,0), v(x,0)\big) = \big(u_0(x), v_0(x)\big) \qquad (x \in \mathbb{R}^N),$$

the constant α may be positive or negative, f and g are given non-negative functions, and the functions w_1 and w_2 may represent some "noises". The cross-term $\alpha\Delta(|v|^{m-1}v)$ gives a measure of the flux of one component engendered by the concentration gradient of the second component.

Before announcing our main results, let us dwell for a while on the modelling part [12]. Consider, for example, two substances (species, chemicals, etc.) that are activating or inhibating each other according to some law of reaction and diffusing in a spatial domain by Fick's law, but the diffusion of one of the substances is influenced also by the other one and vice versa. The density of the two substances at time t and place x are denoted by $u(x,t)$ and $v(x,t)$, respectively. On the one hand, the substance u flows from places where its density is high towards places where the density is low. On the other hand, v has an attracting or repelling effect on u, so that u flows towards high, respectively low density places of v. In this situation the flow vector of u is given by

$$J_u = -d_{11}(u,v)\nabla u - d_{12}(u,v)\nabla v$$

where $d_{11}(u,v) > 0$ and $d_{12}(u,v) \leq 0$, resp. ≥ 0 according as v attracts, resp. repels u. Similarly, the flow of v is given by

$$J_v = -d_{21}(u,v)\nabla u - d_{22}(u,v)\nabla v$$

where $d_{22}(u,v) > 0$ and $d_{21}(u,v) \leq 0$, resp. ≥ 0 according as u attracts, resp. repels v. Then we obtain the reaction-diffusion system

$$\left.\begin{aligned} u_t &= \nabla(d_{11}(u,v)\nabla u + d_{12}(u,v)\nabla v) + U(u,v) \\ v_t &= \nabla(d_{21}(u,v)\nabla u + d_{22}(u,v)\nabla v) + V(u,v) \end{aligned}\right\} \tag{1}$$

where $U(u,v)$ and $V(u,v)$ are the reaction terms. In the particular case

$$d_{11} = nu^{n-1}, \quad d_{22} = lv^{l-1}, \quad d_{12} = d_{21} = 0, \quad U = v^p, \quad V = u^q$$

we obtain the system

$$\left.\begin{aligned} u_t &= \Delta(u^n) + v^p \\ v_t &= \Delta(v^l) + u^q \end{aligned}\right\}. \tag{2}$$

Note that this system describes the processes of diffusion of heat and burning in two-component continuous media with nonlinear conductivity and volume energy release. The functions u and v can thus be treated as temperatures of interacting components of a combustible mixture [15]. When the cross-diffusion $d_{12}(u,v)$ obeys to a similar law as d_{11} and d_{22}, say $d_{12}(u,v) = mv^{m-1}$, we obtain the system

$$\left.\begin{aligned} u_t &= \Delta(u^n) + \Delta(v^m) + v^p \\ v_t &= \Delta(v^l) + u^q \end{aligned}\right\} \tag{3}$$

which concerns the present paper. With an aim of giving more general results, we consider the case where the reaction terms also depend on t and x.

Section 2 is motivated by paper [5] in which Baras and Kersner showed that the problem

$$\left.\begin{aligned} u_t &= \Delta u + h(x)u^p \\ u(x,0) &= u_0(x) \geq 0 \end{aligned}\right\}$$

has no non-negative local weak solution if the initial data satisfies

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x) = +\infty,$$

and any possible non-negative local weak solution blows up at a finite time if

$$\lim_{|x|\to+\infty} u_0^{p-1} h(x)|x|^2 = +\infty.$$

We show similar results for a degenerated nonlinear parabolic system with triangular diffusion matrix.

Section 3 deals with Fujita's type results. Its aim is not only to generalize the results in [9] to triangular diffusion matrix systems but also to weaken the assumptions on the data. Indeed, we require non-negative integrability of the initial data and of the non-homogeneous forcing terms while [9] requires their positivity.

2. Necessary conditions for local and global solvability

Consider the system

$$
\textbf{(P)}\begin{cases} u_t(x,t) = \Delta(|u|^{n-1}u) \pm \Delta(|v|^{m-1}v) & \\ \qquad\qquad + f(x,t)|v|^p + w_1(x,t) & \text{in } Q_T = \mathbb{R}^N \times (0,T) \\ v_t(x,t) = \Delta(|v|^{l-1}v) + g(x,t)|u|^q + w_2(x,t) & \text{in } Q_T \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N \end{cases}
$$

where $p, q, m, n, l \geq 1$ and $0 < T \leq +\infty$, with the following hypotheses on the data:

(H1) $(0,0) \leq (f,g) \in L^{p'}_{loc}(Q_T) \times L^{q'}_{loc}(Q_T)$, where $(p', q') = (\frac{p}{p-1}, \frac{q}{q-1})$.

(H2) $w_i \in L^1(Q_T)$ and $\int_{Q_T} w_i dxdt \geq 0 \quad (i = 1,2)$.

(H3) $(u_0, v_0) \in L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N)$, with $\int_{\mathbb{R}^N} u_0 dx \geq 0$ and $\int_{\mathbb{R}^N} v_0 dx \geq 0$.

In the sequel, if $T = +\infty$, the domain Q_T will be denoted by Q.

Definition 1. A pair of functions (u,v) is called a *weak solution* of problem (P) in Q_T if

- **(i)** $u, v : Q_T \to \mathbb{R}$
- **(ii)** $(u,v) \in L^q_{loc}(Q_T) \times L^p_{loc}(Q_T)$
- **(iii)** For any $\varphi \in \mathcal{D}(\mathbb{R}^N \times [0,T])$ vanishing at $t = T$ if $T < +\infty$ or for any $\varphi \in \mathcal{D}(Q)$ if $T = +\infty$ one has

$$
\int_{Q_T} \Big(u\varphi_t + \big(|u|^{n-1}u \pm |v|^{m-1}v\big)\Delta\varphi + (f|v|^p + w_1)\varphi \Big) dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\, dx = 0 \tag{4}
$$

$$
\int_{Q_T} \Big(v\varphi_t + |v|^{l-1}v\Delta\varphi + g|u|^q\varphi + w_2\varphi \Big) dxdt + \int_{\mathbb{R}^N} v_0(x)\varphi(x,0)\, dx = 0. \tag{5}
$$

We attempt to get insight into the relationship between local and global solvability of problem (P) on the one hand, and the behaviour at infinity of the data f, g, w_1, w_2, u_0, v_0 on the other hand.

In this section we will confine ourselves to the case

$$
\begin{aligned} f(x,t) &= t^\alpha F(x) \\ g(x,t) &= t^\beta G(x) \end{aligned} \qquad \text{and} \qquad \begin{aligned} w_1(x,t) &= t^{\gamma_1} W_1(x) \\ w_2(x,t) &= t^{\gamma_2} W_2(x) \end{aligned}
$$

where F and G are positive and continuous functions, u_0 and v_0 are non-negative and integrable functions. We add the following assumption:

$$\textbf{(H4)}\begin{cases} p > \max\{l, m, l+\alpha, m+\alpha\} \\ q > \max\{n, n+\beta\} \\ \min\{m, n, l\} > 1. \end{cases}$$

2.1 Necessary conditions for local solvability. Before stating our first result, we need to assume that, for R sufficiently large, the estimates

$$\int_{|x|\le 2R} G(x)^{-\frac{1}{q-n}}\,dx = o\big(R^{\frac{2q}{q-n}}\big) \tag{6}$$

$$\int_{|x|\le 2R} F(x)^{-\frac{1}{p-l}}\,dx = o\big(R^{\frac{2p}{p-l}}\big) \tag{7}$$

$$\int_{|x|\le 2R} F(x)^{-\frac{1}{p-m}}\,dx = o\big(R^{\frac{2p}{p-m}}\big) \tag{8}$$

hold.

Theorem 1. *If problem* (P) *has a non-negative local solution defined in* Q_T *with* $T < +\infty$, *then the estimates*

$$\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \le \frac{K_1}{T^\delta} \qquad \liminf_{|x|\to+\infty} v_0(x)\mathcal{F}(x) \le \frac{K_1}{T^\delta} \tag{9}$$

$$\liminf_{|x|\to+\infty} W_1(x)\mathcal{F}(x) \le \frac{K_2}{T^{1+\gamma_1+\delta}} \qquad \liminf_{|x|\to+\infty} W_2(x)\mathcal{F}(x) \le \frac{K_3}{T^{1+\gamma_2+\delta}} \tag{10}$$

hold, where $\mathcal{F}(x) = \frac{F^{\frac{p'}{p}}(x)\,G^{\frac{q'}{q}}(x)}{F^{\frac{p'}{p}}(x)+G^{\frac{q'}{q}}(x)}$ *and* $T^\delta = \min\big(T^{\frac{\alpha+1}{p-1}}, T^{\frac{\beta+1}{q-1}}\big)$ *and the constants* K_1, K_2, K_3 *will be specified in the proof.*

Proof. Let (u, v) be a non-negative weak solution of problem (P) in Q_T. For any $\varphi \in \mathcal{D}([0,T]\times\mathbb{R}^N)$ with $\varphi \ge 0$ and $\varphi(\cdot, T) \equiv 0$ one has

$$\int_{Q_T}\Big(u\varphi_t + \big(u^n \pm v^m\big)\Delta\varphi + fv^p\varphi + w_1\varphi\Big)dxdt + \int_{\mathbb{R}^N} u_0(x)\varphi(x,0)\,dx = 0$$

$$\int_{Q_T}\Big(v\varphi_t + v^l\Delta\varphi + gu^q\varphi + w_2\varphi\Big)dxdt + \int_{\mathbb{R}^N} v_0(x)\varphi(x,0)\,dx = 0.$$

Then

$$\int_{\mathbb{R}^N} u_0\varphi(0) + \int_{Q_T} fv^p\varphi + \int_{Q_T} w_1\varphi \le \int_{Q_T}\big(u|\varphi_t| + (u^n+v^m)(-\Delta\varphi)_+\big)$$

$$\int_{\mathbb{R}^N} v_0\varphi(0) + \int_{Q_T} gu^q\varphi + \int_{Q_T} w_2\varphi \le \int_{Q_T}\big(v|\varphi_t| + v^l(-\Delta\varphi)_+\big)$$

where $\varphi(0) \equiv \varphi(\cdot,0)$ and $(-\Delta\varphi)_+ = \max(0,-\Delta\varphi)$. Furthermore, Young's inequality gives

$$\int_{Q_T} u|\varphi_t| \le \tfrac{1}{2}\int_{Q_T} u^q(g\varphi) + c_1\int_{Q_T} |\varphi_t|^{\frac{q}{q-1}}(g\varphi)^{-\frac{1}{q-1}}$$
$$\int_{Q_T} u^n(-\Delta\varphi)_+ \le \tfrac{1}{2}\int_{Q_T} u^q(g\varphi) + c_2\int_{Q_T} (-\Delta\varphi)_+^{\frac{q}{q-n}}(g\varphi)^{-\frac{1}{q-n}}.$$

Hence

$$\int_{\mathbb{R}^N} u_0\varphi(0) + \int_{Q_T}(fv^p\varphi + w_1\varphi) \le \int_{Q_T}\big(v|\varphi_t| + (v^l+v^m)(-\Delta\varphi)_+\big) + A(\varphi,g,T)$$

where

$$A(\varphi,g,T) = c_1X_1(\varphi,g,T) + c_2X_2(\varphi,g,T)$$

with

$$X_1(\varphi,g,T) = \int_{Q_T} |\varphi_t|^{\frac{q}{q-1}}(g\varphi)^{-\frac{1}{q-1}}$$
$$X_2(\varphi,g,T) = \int_{Q_T} (-\Delta\varphi)_+^{\frac{q}{q-n}}(g\varphi)^{-\frac{1}{q-n}}.$$

Similarly, Young's inequality allows us to obtain

$$\int_{Q_T}\big(v|\varphi_t| + (v^l+v^m)(-\Delta\varphi)_+\big) \le \int_{Q_T} fv^p\varphi + B(\varphi,f,T)$$

where

$$B(\varphi,f,T) = c_3X_3(\varphi,f,T) + c_4X_4(\varphi,f,T) + c_5X_5(\varphi,f,T)$$

with

$$X_3(\varphi,f,T) = \int_{Q_T} |\varphi_t|^{\frac{p}{p-1}}(f\varphi)^{-\frac{1}{p-1}}$$
$$X_4(\varphi,f,T) = \int_{Q_T} (-\Delta\varphi)_+^{\frac{p}{p-l}}(f\varphi)^{-\frac{l}{p-l}}$$
$$X_5(\varphi,f,T) = \int_{Q_T} (-\Delta\varphi)_+^{\frac{p}{p-m}}(f\varphi)^{-\frac{m}{p-m}}.$$

Finally, we have

$$\int_{\mathbb{R}^N} u_0\varphi(0) + \int_{Q_T} w_1\varphi \le \sum_{i=1}^{5} c_iX_i. \tag{11}$$

Following the same idea, we show that there are constants $d_i > 0 \ \ (i = 1, ..., 5)$ so that

$$\int_{\mathbb{R}^N} v_0\varphi(0) + \int_{Q_T} w_2\varphi \le \sum_{i=1}^{5} d_i X_i. \tag{12}$$

At this stage, the test function φ is chosen as

$$\varphi(x,t) = \Big[\eta\Big(\frac{t}{T}\Big)\Big]^s \Phi\Big(\frac{x}{R}\Big)$$

where

i) $\Phi \in \mathcal{D}(\mathbb{R}^N)$, $0 \le \Phi \le 1$, $\operatorname{supp}\Phi \subset \{1 < |x| < 2\}$ and $-\Delta\Phi \le \Phi$

ii) $\eta \in \mathcal{D}(\mathbb{R}^+)$, $0 \le \eta \le 1$, and $\eta(r) = \begin{cases} 1 & \text{if } r \le \frac{1}{2} \\ 0 & \text{if } r \ge 1 \end{cases}$

iii) $s = \max(p', q')$ and $R > 0$.

The choice of this test function is inspired by the paper of P. Baras and R. Kersner [5]. It allows us to obtain interesting estimations connecting the initial data and the reaction terms. Consequently, the integrals $X_i \ \ (i = 1, ..., 5)$ are convergent. More precisely,

$$\begin{aligned} X_1 &\le \frac{s^{q'}\|\eta'\|_\infty^{q'}}{T^{q'}} \frac{T^{1-\frac{\beta}{q-1}}}{1 - \frac{\beta}{q-1}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) G(x)^{-\frac{q'}{q}}\, dx \\ &= \frac{(q-1)s^{q'}\|\eta'\|_\infty^{q'}}{q-(\beta+1)} T^{-\frac{\beta+1}{q-1}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) G(x)^{-\frac{q'}{q}}\, dx \end{aligned}$$

and similarly

$$\begin{aligned} X_2 &\le \frac{q-n}{q-(\beta+n)} T^{\frac{q-(\beta+n)}{q-n}} R^{-\frac{2q}{q-n}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) G(x)^{-\frac{1}{q-n}}\, dx \\ X_3 &\le \frac{(p-1)s^{p'}\|\eta'\|_\infty^{p'}}{p-(\alpha+1)} T^{-\frac{\alpha+1}{p-1}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) F(x)^{-\frac{p'}{p}}\, dx \\ X_4 &\le \frac{p-l}{p-(\alpha+l)} T^{\frac{p-(\alpha+l)}{p-l}} R^{-\frac{2p}{p-l}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) F(x)^{-\frac{1}{p-l}}\, dx \\ X_5 &\le \frac{p-m}{p-(\alpha+m)} T^{\frac{p-(\alpha+m)}{p-m}} R^{-\frac{2p}{p-m}} \int_{\mathbb{R}^N} \Phi\Big(\frac{x}{R}\Big) F(x)^{-\frac{1}{p-m}}\, dx. \end{aligned}$$

Let

$$K_1 = \max\Big\{ c_1 \frac{(q-1)s^{q'}\|\eta'\|_\infty^{q'}}{q-(\beta+1)},\ c_2 \frac{(p-1)s^{p'}\|\eta'\|_\infty^{p'}}{p-(\alpha+1)} \Big\}.$$

In view of (11) and the above estimates we see that

$$\int_{\mathbb{R}^N} u_0(x)\Phi\Big(\frac{x}{R}\Big) \leq K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\Big(\frac{x}{R}\Big) + X_2 + X_4 + X_5 \quad (13)$$

$$\frac{(\frac{T}{2})^{1+\gamma_1}}{1+\gamma_1} \int_{\mathbb{R}^N} W_1(x)\Phi\Big(\frac{x}{R}\Big) \leq K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\Big(\frac{x}{R}\Big) + X_2 + X_4 + X_5 \quad (14)$$

where

$$\mathcal{G}(x,T) = F(x)^{-\frac{p'}{p}} T^{-\frac{\alpha+1}{p-1}} + G(x)^{-\frac{q'}{q}} T^{-\frac{\beta+1}{q-1}}.$$

Hence,

$$\Big\{\inf_{|x|>R} \frac{u_0(x)}{\mathcal{G}(x,T)}\Big\} \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\Big(\frac{x}{R}\Big) \leq K_1 \int_{\mathbb{R}^N} \mathcal{G}(x,T)\Phi\Big(\frac{x}{R}\Big) + X_2 + X_4 + X_5.$$

Finally, using estimates (6) - (8) we have $\lim_{R\to+\infty} \inf_{|x|>R} \frac{u_0(x)}{\mathcal{G}(x,T)} \leq C$ or

$$\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \leq \frac{K_1}{\min\big(T^{\frac{\alpha+1}{p-1}}, T^{\frac{\beta+1}{q-1}}\big)}. \quad (15)$$

The second estimate in (9) and the estimates in (10) can be obtained in the same manner by setting $K_2 = (1+\gamma_1)K_1 2^{1+\gamma_1}$ and $K_3 = (1+\gamma_2)K_1 2^{1+\gamma_2}$. This completes the proof ∎

Consequences.

1. *If* $\max\{\alpha,\beta\} < -1$ *and* $\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) \neq 0$, *then problem* (P) *has no non-negative local solution.*

2. *If problem* (P) *has a non-negative global solution, then we have the implications*

$$\min\{\alpha,\beta\} > -1 \implies \liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) = \liminf_{|x|\to+\infty} v_0(x)\mathcal{F}(x) = 0$$

$$\min\{\alpha,\beta,\gamma_1,\gamma_2\} > -1 \implies \liminf_{|x|\to+\infty} W_1(x)\mathcal{F}(x) = \liminf_{|x|\to+\infty} W_2(x)\mathcal{F}(x) = 0.$$

3. *In the limit case* $\min\{\alpha,\beta\} = -1$, *if* $\liminf_{|x|\to+\infty} u_0(x)\mathcal{F}(x) > K_1$, *there is no non-negative global solution.*

Proof. Assertions 1 and 2 can be seen from (15) by letting $T \to 0$ and $T \to +\infty$, respectively ∎

2.2 Necessary conditions for global solvability. Before stating the main results of this subsection, we need to introduce some notations and hypotheses. Namely, let

$$a = \min\Big\{\frac{\alpha+1}{p-1}, \frac{\beta+1}{q-1}\Big\} \quad \text{and} \quad H = \min\Big\{F^{\frac{p'}{p}}, G^{\frac{q'}{q}}, F^{\frac{1}{p-l}}, F^{\frac{1}{p-m}}, G^{\frac{1}{q-n}}\Big\}.$$

Further, we assume the hypothesis

(H5) $\min\{\alpha,\beta\} > -1$.

Theorem 2. *If problem* (P) *has a non-negative global solution, then there exists a constant* $\theta > 0$ *for which the limits*

$$\liminf_{|x|\to+\infty} u_0(x)H(x)|x|^{a\theta} \qquad \textit{and} \qquad \liminf_{|x|\to+\infty} v_0(x)H(x)|x|^{a\theta} \tag{16}$$

are bounded where the real number θ *will be specified in the proof.*

Proof. Inequality (13) implies that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} u_0(x)\Phi\Big(\frac{x}{R}\Big) \le C\mathcal{H}(R,T)\int_{\mathbb{R}^N} \frac{1}{H(x)}\Phi\Big(\frac{x}{R}\Big) \tag{17}$$

where

$$\mathcal{H}(R,T) = \frac{T^{\alpha_1}}{R^{\beta_1}} + \frac{T^{\alpha_2}}{R^{\beta_2}} + \frac{T^{\alpha_3}}{R^{\beta_3}} + \frac{1}{T^{\alpha_4}} + \frac{1}{T^{\alpha_5}}$$

with

$$\alpha_1 = \frac{p-(\alpha+l)}{p-l}$$

$$\alpha_2 = \frac{p-(\alpha+m)}{p-m} \qquad\qquad \beta_1 = \frac{2p}{p-l}$$

$$\alpha_3 = \frac{q-(\beta+n)}{q-n} \qquad \text{and} \qquad \beta_2 = \frac{2p}{p-m}$$

$$\alpha_4 = \frac{1+\alpha}{p-1} \qquad\qquad \beta_3 = \frac{2q}{q-n}.$$

$$\alpha_5 = \frac{1+\beta}{q-1}$$

Note that under hypotheses (H4) - (H5) all parameters α_i and β_i are positive. Now we have to minimize the function $\mathcal{H}$ with respect to T. For this one has

$$\frac{\partial\mathcal{H}}{\partial T}(R,T) = \frac{1}{T}\mathcal{L}(R,T)$$

where

$$\mathcal{L}(R,T) = \alpha_1\frac{T^{\alpha_1}}{R^{\beta_1}} + \alpha_2\frac{T^{\alpha_2}}{R^{\beta_2}} + \alpha_3\frac{T^{\alpha_3}}{R^{\beta_3}} - \alpha_4\frac{1}{T^{\alpha_4}} - \alpha_5\frac{1}{T^{\alpha_5}}. \tag{18}$$

Then $\frac{\partial\mathcal{H}}{\partial T}(R,T) = 0$ if and only if $\mathcal{L}(R,T) = 0$. Moreover, it is clear that the function $\mathcal{L}$ is strictly increasing in $T > 0$ and $\lim_{T\to 0^+}\mathcal{L}(R,T) = -\infty$ and $\lim_{T\to+\infty}\mathcal{L}(R,T) = +\infty$, which imply that for any $R > 0$ there is a unique $T_*(R) > 0$ such that $\mathcal{L}(R,T_*(R)) = 0$. The implicit function theorem asserts that the function T_* is smooth in R and

$$\frac{dT_*}{dR}(R) = -\frac{\frac{\partial\mathcal{L}}{\partial R}}{\frac{\partial\mathcal{L}}{\partial T}}(R,T_*(R)) > 0.$$

Hence T_* is strictly increasing in R and we easily see that $\lim_{R\to+\infty} T_*(R) = +\infty$. Finally,

$$\begin{aligned}\frac{\partial^2\mathcal{H}}{\partial T^2}(R,T_*(R)) &= -\frac{1}{T_*^2(R)}\mathcal{L}(R,T_*(R)) + \frac{1}{T_*(R)}\frac{\partial\mathcal{L}}{\partial T}(R,T_*(R))\\ &= \frac{1}{T_*(R)}\frac{\partial\mathcal{L}}{\partial T}(R,T_*(R))\\ &> 0\end{aligned}$$

implies that for any fixed $R > 0$ the function $\mathcal{H}$ has a unique minimum at $(R, T_*(R))$.

Now we have to determine the asymptotic behaviour of $T_*(R)$ as $R \to +\infty$. Recall that $a = \min(\alpha_4, \alpha_5)$ and that the pair $(R, T_*(R))$ verifies the identity

$$\alpha_1\frac{T_*^{\alpha_1}(R)}{R^{\beta_1}} + \alpha_2\frac{T_*^{\alpha_2}(R)}{R^{\beta_2}} + \alpha_3\frac{T_*^{\alpha_3}(R)}{R^{\beta_3}} = \alpha_4\frac{1}{T_*^{\alpha_4}(R)} + \alpha_5\frac{1}{T_*^{\alpha_5}(R)}.$$

Then there is an $\ell > 0$ such that

$$\lim_{R\to+\infty}\left\{\alpha_1\frac{T_*^{\alpha_1+a}(R)}{R^{\beta_1}} + \alpha_2\frac{T_*^{\alpha_2+a}(R)}{R^{\beta_2}} + \alpha_3\frac{T_*^{\alpha_3+a}(R)}{R^{\beta_3}}\right\} = \ell.$$

Setting $k_i = \frac{\beta_i}{a+\alpha_i}$ $(i = 1,2,3)$, $\theta = \min\{k_1,k_2,k_3\}$ and $\Theta = \big\{i \in \{1,2,3\} : k_i = \theta\big\}$, we have

$$\sum_{i=1}^{3}\alpha_i\left(\frac{T_*(R)}{R^{k_i}}\right)^{a+\alpha_i} = \sum_{i\in\Theta}\alpha_i\left(\frac{T_*(R)}{R^{\theta}}\right)^{a+\alpha_i} + \varepsilon(R,T). \tag{19}$$

Since the functions $R \to \frac{T_*^{\alpha_i+a}(R)}{R^{\beta_i}}$ $(i \in \{1,2,3\})$ are bounded uniformly with respect to R, we conclude that

$$\lim_{R\to+\infty}\sum_{i\in\Theta}\alpha_i\Big(\frac{T_*(R)}{R^{\theta}}\Big)^{a+\alpha_i} = \ell \qquad \text{and} \qquad \lim_{R\to+\infty}\varepsilon(R,T) = 0.$$

Set

$$P(X) = \sum_{i\in\Theta}\alpha_i X^{a+\alpha_i}.$$

It is clear that the function P is strictly increasing and $P(0) = 0$. Hence

$$\lim_{R\to+\infty}\frac{T_*(R)}{R^{\theta}} = \lim_{R\to+\infty}P^{-1}\circ P\Big(\frac{T_*(R)}{R^{\theta}}\Big) = P^{-1}(\ell) > 0$$

because P^{-1} is continuous, and consequently $T_*(R) \sim P^{-1}(\ell)R^{\theta}$ for R large enough.

Finally, using (17), there is a constant $K > 0$ such that, for R large enough,

$$\int_{\mathbb{R}^N} u_0(x)\Phi\Big(\frac{x}{R}\Big) \le \frac{K}{R^{a\theta}} \int_{\mathbb{R}^N} \frac{1}{H(x)}\Phi\Big(\frac{x}{R}\Big). \tag{20}$$

Using the fact that $\operatorname{supp}(\Phi) \subset \{x \in \mathbb{R}^N : 1 < |x| < 2\}$, we conclude for R large enough that

$$\begin{aligned}\Big\{\inf_{|x|>R} u_0(x)H(x)|x|^{a\theta}\Big\} &\int_{\mathbb{R}^N} \frac{|x|^{-a\theta}}{H(x)}\,\Phi\Big(\frac{x}{R}\Big)\\ &\le \frac{K}{R^{a\theta}} \int_{\mathbb{R}^N} \frac{|x|^{a\theta}|x|^{-a\theta}}{H(x)}\,\Phi\Big(\frac{x}{R}\Big)\\ &\le \frac{(2R)^{a\theta}K}{R^{a\theta}} \int_{\mathbb{R}^N} \frac{|x|^{-a\theta}}{H(x)}\,\Phi\Big(\frac{x}{R}\Big).\end{aligned}$$

Whence the boundedness of $\liminf_{|x|\to+\infty} u_0(x)H(x)|x|^{a\theta}$ is established. The boundedness of the second limit is similar as above. This achieves the proof ∎

Before stating the last result of this section, we will assume $\gamma_j > -1$ $(j = 1,2)$ and distinguish the following two hypotheses

(H6) $\min\{\alpha_1, \alpha_2, \alpha_3\} > \gamma_j + 1$

(H7) $\max\{\alpha_1, \alpha_2, \alpha_3\} < \gamma_j + 1$

for $j = 1$ or $j = 2$.

Theorem 3. *Assume hypotheses* (H4) - (H5) *and either hypothesis* (H6) *or hypothesis* (H7) *holds. If problem* (P) *has a non-negative global solution, then there are constants a_j and θ_j such that the limits*

$$\liminf_{|x|\to+\infty} W_j(x)H(x)|x|^{a_j\theta_j} \qquad (j = 1,2) \tag{21}$$

are bounded.

Proof. As before, inequalities () and () imply that there is a constant $C > 0$ such that for $j = 1,2$ one has

$$T^{\gamma_j+1} \int_{\mathbb{R}^N} W_j(x)\Phi\Big(\frac{x}{R}\Big) \le C\mathcal{H}(R,T) \int_{\mathbb{R}^N} \frac{1}{H(x)}\Phi\Big(\frac{x}{R}\Big). \tag{22}$$

First, if hypotheses (H4), (H5) and (H6) hold, consider the function $\tilde{\mathcal{H}}(R,T) = \frac{\mathcal{H}(R,T)}{T^{\gamma_j+1}}$. The situation then is similar to that of Theorem 2 with $\tilde{\mathcal{H}}$

instead of $\mathcal{H}$. Following its proof we obtain $\theta_j = \theta$ and $a_j = a + 1 + \gamma_j$ where a and θ are defined in that proof.

Second, if hypotheses (H4), (H5) and (H7) hold, let θ_j be the unique positive real number defined by

$$\min\left\{\beta_i + \theta_j(\gamma_j + 1 - \alpha_i) : i \in \{1, 2, 3\}\right\} = (\gamma_j + 1 + a)\theta_j.$$

It follows from (22) that there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^N} W_j(x)\Phi\Big(\frac{x}{R}\Big) \leq \frac{C}{R^{(a+\gamma_j+1)\theta_{4,j}}} \int_{\mathbb{R}^N} \frac{1}{H(x)}\Phi\Big(\frac{x}{R}\Big).$$

Proceeding as in the end of the last proof, we show that

$$\liminf_{|x|\to+\infty} W_j(x)H(x)|x|^{(a+\gamma_j+1)\theta_j} \tag{23}$$

is bounded. This achieves the proof ∎

The next section deals with the non-existence of global (non-trivial) solutions to problem (P) from a different angle: We will present results of Fujita's type. These results will take into account the dimension N instead of the behaviour at infinity of the data and of the non-homogeneous terms. We refer the interested reader to the valuable surveys by Levine [25], Bandle and Brunner [4] and Deng and Levine [11] for some background.

3. Necessary conditions for global solvability: Fujita's type results

The hypotheses considered in this section are (H1) - (H3). In order to simplify the presentation, we initially set $f \equiv g \equiv 1$.

The aim of the following lemma is to show that if problem (P) has a global non-trivial solution (u, v), then neither u nor v is trivial. This result will be used in Theorems 4 and 5 to show that any non-trivial weak solution of problem (P) blows up in finite time (under some hypotheses relating the exponents of the non-linearities and the dimension N).

Lemma 1. *Let (u, v) be a weak solution of problem* (P) *in Q. Then, if $u \equiv 0$ or $v \equiv 0$, one has $u \equiv v \equiv 0$.*

Proof. We show this lemma in a general way in order to use some contained results in the sequel. Let $\varphi_R \in \mathcal{D}(\mathbb{R}^N \times [0, +\infty[)$ be a non-negative function such that

$$\varphi_R(x, t) = \Phi^\lambda\Big(\frac{t + |x|^2}{R}\Big)$$

where $\lambda > 1, R > 0$ and $\Phi \in \mathcal{D}([0,+\infty))$ is the "standard cut-off function", i.e.

$$0 \le \Phi(r) \le 1 \qquad \text{and} \qquad \Phi(r) = \begin{cases} 1 & \text{if } 0 \le r \le 1 \\ 0 & \text{if } r \ge 2. \end{cases} \tag{24}$$

Then equation (5) gives

$$\begin{aligned}
&\int_Q \varphi_R |u|^q dxdt + a(R) \\
&\quad \le \int_Q \left(|v|\,|\varphi_{Rt}| + |v|^l |\Delta\varphi_R|\right) dxdt \\
&\quad \le \left(\int_Q |v|^p \varphi_R dxdt\right)^{\frac{1}{p}} \left(\int_Q |\varphi_{Rt}|^{\frac{p}{p-1}} \varphi_R^{-\frac{1}{p-1}} dxdt\right)^{\frac{p-1}{p}} \\
&\qquad + \left(\int_Q |v|^p \varphi_R dxdt\right)^{\frac{l}{p}} \left(\int_Q |\Delta\varphi_R|^{\frac{p}{p-l}} \varphi_R^{-\frac{l}{p-l}} dxdt\right)^{\frac{p-l}{p}}
\end{aligned}$$

and (4) gives also

$$\begin{aligned}
&\int_Q \varphi_R |v|^p dxdt + b(R) \\
&\quad \le \int_Q \Big(|u|\,|\varphi_{Rt}| + |u|^n |\Delta\varphi_R| + |v|^m |\Delta\varphi_R|\Big) dxdt \\
&\quad \le \left(\int_Q |u|^q \varphi_R dxdt\right)^{\frac{1}{q}} \left(\int_Q |\varphi_{Rt}|^{\frac{q}{q-1}} \varphi_R^{-\frac{1}{q-1}} dxdt\right)^{\frac{q-1}{q}} \\
&\qquad + \left(\int_Q |u|^q \varphi_R dxdt\right)^{\frac{n}{q}} \left(\int_Q |\Delta\varphi_R|^{\frac{q}{q-n}} \varphi_R^{-\frac{n}{q-n}} dxdt\right)^{\frac{q-n}{q}} \\
&\qquad + \left(\int_Q |v|^p \varphi_R dxdt\right)^{\frac{m}{p}} \left(\int_Q |\Delta\varphi_R|^{\frac{p}{p-m}} \varphi_R^{-\frac{m}{p-m}} dxdt\right)^{\frac{p-m}{p}}
\end{aligned}$$

where

$$\begin{aligned}
a(R) &= \int_{\mathbb{R}^N} v_0(x)\varphi_R(x,0)\, dx + \int_Q w_2(x,t)\varphi_R(x,t)\, dxdt \\
b(R) &= \int_{\mathbb{R}^N} u_0(x)\varphi_R(x,0)\, dx + \int_Q w_1(x,t)\varphi_R(x,t)\, dxdt.
\end{aligned}$$

If we set

$$X(R) = \Big(\int_Q |u|^q \varphi_R dxdt \Big)^{\frac{1}{q}}$$

$$Y(R) = \Big(\int_Q |v|^p \varphi_R dxdt \Big)^{\frac{1}{p}}$$

$$A = \Big(\int_Q |\varphi_{Rt}|^{\frac{p}{p-1}} \varphi_R^{-\frac{1}{p-1}} dxdt \Big)^{\frac{p-1}{p}}$$

$$B = \Big(\int_Q |\Delta\varphi_R|^{\frac{p}{p-l}} \varphi_R^{-\frac{l}{p-l}} dxdt \Big)^{\frac{p-l}{p}}$$

$$C = \Big(\int_Q |\varphi_{Rt}|^{\frac{q}{q-1}} \varphi_R^{-\frac{1}{q-1}} dxdt \Big)^{\frac{q-1}{q}}$$

$$D = \Big(\int_Q |\Delta\varphi_R|^{\frac{q}{q-n}} \varphi_R^{-\frac{n}{q-n}} dxdt \Big)^{\frac{q-n}{q}}$$

$$E = \Big(\int_Q |\Delta\varphi_R|^{\frac{p}{p-m}} \varphi_R^{-\frac{m}{p-m}} dxdt \Big)^{\frac{p-m}{rp}},$$

we then have the system of inequalities

$$\left.\begin{array}{r} X^q(R) + a(R) \le AY(R) + BY^l(R) \\ Y^p(R) + b(R) \le CX(R) + DX^n(R) + EY^m(R) \end{array}\right\}. \tag{25}$$

It is easy to see that if λ is selected sufficiently large, then all integrals $A, ..., E$ are convergent.

On the one hand, if $v \equiv 0$, then $X^q(R)$ is a bounded and increasing function of R. Using the monotone convergence theorem, we deduce that $|u|^q$ is in $L^1(Q)$ and

$$\begin{aligned} \lim_{R\to+\infty} \big(X^q(R) + a(R)\big) &= \int_Q |u|^q dxdt + \int_{\mathbb{R}^N} v_0(x)\, dx + \int_Q w_2(x,t)\, dxdt \\ &= 0. \end{aligned}$$

Then we have necessarily

$$\int_Q |u|^q dxdt = \int_{\mathbb{R}^N} v_0(x)\, dx = \int_Q w_2(x,t)\, dxdt = 0,$$

which implies that $u \equiv 0$ (and also $v_0 \equiv 0$, $w_2 \equiv 0$) and consequently $(u, v) \equiv (0, 0)$. On the other hand, if $u \equiv 0$, then there is a constant $C_0 > 0$ such that

$Y^p(R) \leq C_0 + EY^m(R)$. Since $m < p$, there is a constant $C_1 > 0$ such that $Y(R) \leq C_1$. Similarly, the function $|v|^p$ is in $L^1(Q)$. Note that instead of (25) we have more precisely the system of inequalities

$$\left.\begin{array}{l} X^q(R) + a(R) \leq A\tilde{Y}(R) + B\tilde{Y}^l(R) \\ Y^p(R) + b(R) \leq C\tilde{X}(R) + D\tilde{X}^n(R) + E\tilde{Y}^m(R) \end{array}\right\} \tag{26}$$

where

$$\tilde{X}(R) = \left(\int_{\Omega_R} |u|^q \varphi_R dxdt\right)^{\frac{1}{q}} \qquad \text{and} \qquad \tilde{Y}(R) = \left(\int_{\Omega_R} |v|^p \varphi_R dxdt\right)^{\frac{1}{p}}$$

and $\Omega_R = \big\{(x,t) \in Q : \ R \leq t + |x|^2 \leq 2R\big\}$. Indeed, as before, equation (5) gives

$$\begin{aligned} &\int_Q \varphi_R |u|^q dxdt + a(R) \\ &\qquad \leq \int_Q \big(|v|\,|\varphi_{Rt}| + |v|^l |\Delta\varphi_R|\big) dxdt \\ &\qquad = \int_{\Omega_R} \big(|v|\,|\varphi_{Rt}| + |v|^l |\Delta\varphi_R|\big) dxdt \\ &\qquad \leq \left(\int_{\Omega_R} |v|^p \varphi_R dxdt\right)^{\frac{1}{p}} \left(\int_{\Omega_R} |\varphi_{Rt}|^{\frac{p}{p-1}} \varphi_R^{-\frac{1}{p-1}} dxdt\right)^{\frac{p-1}{p}} \\ &\qquad\quad + \left(\int_{\Omega_R} |v|^p \varphi_R dxdt\right)^{\frac{l}{p}} \left(\int_{\Omega_R} |\Delta\varphi_R|^{\frac{p}{p-l}} \varphi_R^{-\frac{l}{p-l}} dxdt\right)^{\frac{p-l}{p}}. \end{aligned}$$

This implies

$$X^q(R) + a(R) \leq A\tilde{Y}(R) + B\tilde{Y}^l(R).$$

Similarly, we obtain the second inequality

$$Y^p(R) + b(R) \leq C\tilde{X}(R) + D\tilde{X}^n(R) + E\tilde{Y}^m(R).$$

Now we return to system (26). Using the dominated convergence theorem, we obtain $\lim_{R\to+\infty} \tilde{Y}(R) = 0$. Hence,

$$\begin{aligned} \lim_{R\to+\infty} \big(Y^p(R) + b(R)\big) &= \int_Q |v|^p dxdt + \int_{\mathbb{R}^N} u_0(x)\,dx + \int_Q w_1(x,t)\,dxdt \\ &= 0 \end{aligned}$$

which implies, as before, that $v \equiv 0$ (and also $u_0 \equiv 0$, $w_1 \equiv 0$). This completes the proof ∎

The following lemma gives a generalization of Lemma 1.

Lemma 2. *Using the same notations as before, if $X(R)$ or $Y(R)$ is bounded, then $u \equiv v \equiv 0$.*

Proof. If $X(R)$ or $Y(R)$ is bounded, it follows via (25) that $X(R)$ and $Y(R)$ are bounded. Then $(|u|^q, |v|^p)$ is in $L^1(Q) \times L^1(Q)$. Finally, using (26) and the dominated convergence theorem, we obtain the result ∎

Theorem 4. *Assume that $p > \max\{m, l\}$ and $q > n$ with $1 < \min\{m, n, l\}$. If one of the conditions*

a) $\frac{p+1}{q+1} < \min\left(\frac{l-1}{n-1}, \frac{p}{n}, \frac{m}{n}\right)$ *and* $\frac{N}{2} \leq \frac{1+p}{p-n+pq-pn}$

b) $\frac{m}{n} < \frac{p+1}{q+1} < \min\left(\frac{l-1}{n-1}, \frac{p}{n}\right)$ *and* $\frac{N}{2} \leq \frac{p}{p-m} - \frac{pq-1}{p-n+pq-pn}$

c) $\frac{l-1}{n-1} < \frac{p+1}{q+1} < \frac{p}{n}, \frac{p+1}{q+1} \neq \frac{m}{n}, \frac{p+l}{q+n} > \frac{m}{n} \frac{N}{2} \leq \frac{p}{p-m} - \frac{pq-1}{p-n+pq-pn}$

d) $\frac{l-1}{n-1} < \frac{p+1}{q+1} < \frac{p}{n}, \frac{p+1}{q+1} \neq \frac{m}{n}, \frac{p+l}{q+n} < \frac{m}{n}, \frac{N}{2} \leq \frac{p(n+q)}{pq-nl} - \frac{pq-1}{p-n+pq-pn}$

is satisfied, then problem (P) *has no non-trivial global weak solution.*

Proof. Let (u, v) be a non-trivial weak solution of problem (P) and $\varphi_R \in \mathcal{D}(\mathbb{R}^N \times [0, +\infty))$ be a non-negative function such that

$$\varphi_R(x, t) = \Phi^\lambda\Big(\frac{t + |x|^\delta}{R^\delta}\Big)$$

where $\lambda > 1, \delta > 0, R > 0$ and $\Phi \in \mathcal{D}([0, +\infty))$ is the cut-off function defined before. Following the same method described in the previous proof, we deduce the two systems (25) and (26). We precise that the different terms appearing in those two systems depend on δ. Using the fact that $\lim_{R\to+\infty} a(R) \geq 0$ and $\lim_{R\to+\infty} b(R) \geq 0$ and applying Young's inequality in system (25) one has, for some $0 < \varepsilon < 1$

$$\begin{aligned}
(1-\varepsilon)Y^{pq} \leq\, & c_{1,\varepsilon}(AC^q)^{\frac{pq}{pq-1}} + c_{2,\varepsilon}(BC^q)^{\frac{pq}{pq-l}} \\
& + c_{3,\varepsilon}(E^q)^{\frac{p}{p-m}} + c_{4,\varepsilon}(A^n D^q)^{\frac{pq}{pq-n}} + c_{5,\varepsilon}(B^n D^q)^{\frac{pq}{pq-nl}} \\
(1-\varepsilon)X^{pq} \leq\, & c'_{1,\varepsilon}(A^p C)^{\frac{pq}{pq-1}} + c'_{2,\varepsilon}(A^p D)^{\frac{pq}{pq-n}} + c'_{3,\varepsilon}(B^p C^l)^{\frac{pq}{pq-l}} \\
& + c'_{4,\varepsilon}(B^p D^l)^{\frac{pq}{pq-nl}} + c'_{5,\varepsilon} A^p E^{\frac{p}{p-m}} + c'_{6,\varepsilon} B^p E^{\frac{lp}{p-m}}.
\end{aligned}$$

At this stage we introduce the scaled variables $\tau = R^{-\delta}t$ and $y = R^{-1}x$. It is easy to check that for R large enough

$$A \leq c_1 R^{\alpha_1}, \quad B \leq c_2 R^{\alpha_2}, \quad C \leq c_3 R^{\alpha_3}, \quad D \leq c_4 R^{\alpha_4}, \quad E \leq c_5 R^{\alpha_5}$$

where

$$\alpha_1 = \Big(N + \delta - \frac{\delta p}{p-1}\Big)\frac{p-1}{p}$$
$$\alpha_2 = \Big(N + \delta - \frac{2p}{p-l}\Big)\frac{p-l}{p}$$
$$\alpha_3 = \Big(N + \delta - \frac{\delta q}{q-1}\Big)\frac{q-1}{q}$$
$$\alpha_4 = \Big(N + \delta - \frac{2q}{q-n}\Big)\frac{q-n}{q}$$
$$\alpha_5 = \Big(N + \delta - \frac{2p}{p-m}\Big)\frac{p-m}{p}.$$

Finally, we have the system of inequalities

$$\left.\begin{aligned} (1-\varepsilon)Y(R)^{pq} &\le c_\varepsilon\Big\{R^{r_1(\delta)} + R^{r_2(\delta)} + R^{r_3(\delta)} + R^{r_4(\delta)} + R^{r_5(\delta)}\Big\} \\ (1-\varepsilon)X^{pq} &\le c'_\varepsilon\Big\{R^{s_1(\delta)} + R^{s_2(\delta)} + R^{s_3(\delta)} + R^{s_4(\delta)} + R^{s_5(\delta)} + R^{s_6(\delta)}\Big\} \end{aligned}\right\} \tag{27}$$

where

$$r_1(\delta) = Nq + \frac{q(-\delta - \delta p)}{pq-1}$$
$$r_2(\delta) = Nq + \frac{q(-l\delta - 2p)}{pq-l}$$
$$r_3(\delta) = Nq + \frac{q(-\delta n + \delta pq - \delta pn - 2pq)}{pq-n}$$
$$r_4(\delta) = Nq + \frac{q(\delta pq - \delta nl - 2pn - 2pq)}{pq-nl}$$
$$r_5(\delta) = Nq + \frac{q(\delta p - \delta m - 2p)}{p-m}$$
$$s_1(\delta) = Np + \frac{p(-\delta q - \delta)}{pq-1}$$
$$s_2(\delta) = Np + \frac{p(-\delta n - 2q)}{pq-n}$$
$$s_3(\delta) = Np + \frac{p(\delta pq - lq\delta - l\delta - 2pq)}{pq-l}$$
$$s_4(\delta) = Np + \frac{p(\delta pq - \delta nl - 2pq - 2lq)}{pq-nl}$$
$$s_5(\delta) = Np - 2\frac{p}{p-m}$$
$$s_6(\delta) = Np + \frac{p(\delta p - \delta m - 2l - 2p + 2m)}{p-m}.$$

The parameter δ is fixed now such that $r_1(\delta) = r_3(\delta)$, i.e.

$$\delta = \delta_1 = 2\frac{pq-1}{p(q+1)-n(p+1)} \qquad \text{with } p(q+1)-n(p+1) > 0.$$

Therefore,

$$\begin{aligned}
r_1(\delta_1) - r_2(\delta_1) &= 2\,\frac{(n+pn+q-lq-l-p)pq}{(pq-l)(-p+n-pq+pn)}\\
r_1(\delta_1) - r_4(\delta_1) &= -2\,\frac{(n+pn+q-lq-l-p)npq}{(-p+n-pq+pn)(-pq+nl)}\\
r_1(\delta_1) - r_5(\delta_1) &= 2\,\frac{(n+pn-m-qm)pq}{(-p+n-pq+pn)(p-m)}.
\end{aligned}$$

Note that if $\frac{p+1}{q+1} < \min\left\{\frac{l-1}{n-1}, \frac{p}{n}, \frac{m}{n}\right\}$ (case (a) in Theorem 1), then $r_1(\delta_1) = \max_{1\le i\le 5} r_i(\delta_1)$. In this case, if $\frac{N}{2} \le (1+p)(p-n+pq-pn)$, then $r_1(\delta_1) \le 0$ and there is a constant C such that

$$Y(R)^{pq} = \left(\int_Q |v|^p \varphi_R dxdt\right)^q = \left(\int_{Q_R} |v|^p \varphi_R dxdt\right)^q \le C$$

where $Q_R = \{(x,t) \in Q : 0 \le t + |x|^{\delta_1} \le 2R^{\delta_1}\}$. According to Lemma 1, we deduce that $u \equiv v \equiv 0$. This contradicts our assumption. Assertions b) - d) can be showed in the same manner ∎

Theorem 5. *Assume that* $p > \max\{m, l\}$ *and* $q > n$, *with* $1 < \min\{m, n, l\}$. *If one of the conditions*

- **a)** $(m+1) - l > \frac{p+1}{q+1} > \max\left\{\frac{l-1}{n-1}, \frac{l}{q}\right\}$, $\frac{N}{2} \le \frac{q+1}{pq+q-ql-l}$
- **b)** $\frac{p+1}{q+1} > \max\left\{\frac{l-1}{n-1}, \frac{l}{q}, (m+1)-l\right\}$, $\frac{N}{2} \le \frac{pq-1}{-pq-q+ql+l} + \frac{l+p-m}{p-m}$
- **c)** $\frac{l}{q} < \frac{p+1}{q+1} < \min\left\{\frac{l-1}{n-1}, (m+1)-l\right\}$, $\frac{N}{2} \le \frac{pq-1}{-pq-q+ql+l} + \frac{q(p+l)}{pq-nl}$
- **d)** $\frac{l-1}{n-1} > \frac{p+1}{q+1} > \max\left\{\frac{l}{q}, (m+1)-l\right\}$, $\frac{N}{2} \le \min\left\{N_1, N_2\right\}$

is satisfied where

$$\begin{aligned}
N_1 &= \frac{pq-1}{-pq-q+ql+l} + \frac{q(p+l)}{pq-nl}\\
N_2 &= \frac{pq-1}{-pq-q+ql+l} + \frac{l+p-m}{p-m},
\end{aligned}$$

then problem (P) *has no non-trivial global weak solution.*

Proof. We follow the proof of Theorem 4 and choose the parameter δ such that $s_1(\delta) = s_3(\delta)$, i.e.

$$\delta = \delta_2 = \frac{2(pq-1)}{q(p+1) - l(q+1)} \qquad \text{with } q(p+1) - l(q+1) > 0.$$

If

$$(m+1) - l > (p+1)(q+1) > \max\left\{(l-1)(n-1), \frac{l}{q}\right\}$$

(case (a) in Theorem 2), then $s_1(\delta_2) = \max_{1 \le i \le 6} s_i(\delta_2)$. In this case, if $\frac{N}{2} \le (q+1)(pq + q - ql - l)$, then $s_1(\delta_2) \le 0$ and there is a constant C such that

$$X^{pq}(R) = \left(\int_Q |u|^q \varphi_R dxdt\right)^p = \left(\int_{Q_R} |u|^q \varphi_R dxdt\right)^p \le C.$$

The result of Lemma 1 completes the proof. Assertions b) - d) can be proved similarly ∎

Comments.

1) Note that in Section 3 the positivity of the solutions of problem (P) is not guaranteed even if the data are positive. It is then natural that the initial data $u_0(x)$ and $v_0(x)$ may change signs as well as the non-homogeneous terms $w_1(x,t)$ and $w_2(x,t)$. Now, our hypotheses are weaker than those in the literature, i.e. the data may change signs but must have non-negative integral. This difficulty was first solved in the scalar case in [22].

2) We are now able to treat the case where $f \ge 0, g \ge 0, f \sim t^{\gamma_1}|x|^{\theta_1}$ and $g \sim t^{\gamma_2}|x|^{\theta_2}$ for t and $|x|$ large enough. A slight change in the proof (Theorem

4 and Theorem 5) shows that we have systems (25) - (26) with

$$X(R) = \Big(\int_Q |u|^q \varphi_R g \, dxdt\Big)^{\frac{1}{q}}$$

$$Y(R) = \Big(\int_Q |v|^p \varphi_R f \, dxdt\Big)^{\frac{1}{p}}$$

$$A = \Big(\int_Q |\varphi_{Rt}|^{\frac{p}{p-1}} (\varphi_R f)^{-\frac{1}{p-1}} dxdt\Big)^{\frac{p-1}{p}}$$

$$B = \Big(\int_Q |\Delta\varphi_R|^{\frac{p}{p-l}} (\varphi_R f)^{-\frac{l}{p-l}} dxdt\Big)^{\frac{p-l}{p}}$$

$$C = \Big(\int_Q |\varphi_{Rt}|^{\frac{q}{q-1}} (\varphi_R g)^{-\frac{1}{q-1}} dxdt\Big)^{\frac{q-1}{q}}$$

$$D = \Big(\int_Q |\Delta\varphi_R|^{\frac{q}{q-n}} (\varphi_R g)^{-\frac{n}{q-n}} dxdt\Big)^{\frac{q-n}{q}}$$

$$E = \Big(\int_Q |\Delta\varphi_R|^{\frac{p}{p-m}} (\varphi_R f)^{-\frac{m}{p-m}} dxdt\Big)^{\frac{p-m}{p}}.$$

This naturally changes the $\alpha_i \ \ (i = 1, ..., 5)$ into

$$\begin{aligned}
\alpha_1 &= \Big(N + \delta - \frac{\delta p}{p-1}) + \beta_1\Big)\frac{p-1}{p} & & \beta_1 = -\frac{\theta_1 + \gamma_1\delta}{p-1}\\
\alpha_2 &= \Big(N + \delta - \frac{2p}{p-l} + \beta_2\Big)\frac{p-l}{p} & & \beta_2 = -l\frac{\theta_1 + \gamma_1\delta}{p-l}\\
\alpha_3 &= \Big(N + \delta - \frac{\delta q}{q-1} + \beta_3\Big)\frac{q-1}{q} & \text{where} \quad & \beta_3 = -\frac{\theta_2 + \gamma_2\delta}{q-1}\\
\alpha_4 &= \Big(N + \delta - \frac{2q}{q-n} + \beta_4\Big)\frac{q-n}{q} & & \beta_4 = -n\frac{\theta_2 + \gamma_2\delta}{q-n}\\
\alpha_5 &= \Big(N + \delta - \frac{2p}{p-m} + \beta_5\Big)\frac{p-m}{p} & & \beta_5 = -m\frac{\theta_1 + \gamma_1\delta}{p-m}.
\end{aligned}$$

3) This work can be easily generalized to higher order systems with triangular diffusion matrices under the same type of hypotheses.

4) The method described above can also be used for the more general system

$$\left.\begin{aligned}
u_t(x,t) &= |x|^\alpha\big\{(-\Delta)^{\frac{\alpha_1}{2}}(\varphi(u)) + (-\Delta)^{\frac{\alpha_2}{2}}(\psi(v))\big\} + f(x,t)k(u) + w_1(x,t)\\
v_t(x,t) &= |x|^\beta(-\Delta)^{\frac{\alpha_3}{2}}(\chi(v)) + g(x,t)l(v) + w_2(x,t)
\end{aligned}\right\}$$

where $-(-\Delta)^{\frac{\alpha_i}{2}}$ is the fractional power of the Laplacian. A suitable choice of the functions φ, ψ, χ are required.

5) If parabolic problem (P) is replaced by the hyperbolic one, i.e. (u_t, v_t) is replaced by (u_{tt}, v_{tt}), our study remains valid. The non-negativity assumptions on (u_0, v_0) are set on (u_{0_t}, v_{0_t}) and the test function changes slightly; for example, $\varphi_R(x,t) = \Phi^\lambda\big(\frac{t^2+|x|^2}{R}\big)$ with $\lambda >> 1$.

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