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Harnack Inequality for a Class of Degenerate Elliptic Operators

J. D. Fernandes, J. Groisman and S. T. Melo

Abstract. We prove a Harnack inequality for a class of two-weight degenerate elliptic operators. The metric distance is induced by continuous Grushin-type vector fields. It is not know whether there exist cutoffs fitting the metric balls. This obstacle is bypassed by means of a covering argument that allows the use of rectangles in the Moser iteration.

Keywords: *Harnack-type inequality, degenerate elliptic equations*

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1. Introduction

Perhaps inspired by David and Semmes' work [5], Franchi, Gutierrez and Wheeden proved in [10] a very deep generalization of the classical Sobolev-Poincaré inequality, unifying several other previous results. The importance of Sobolev-Poincaré-type inequalities to the study of elliptic equations has been well known for decades [18]. In particular, the so-called *Moser iteration* technique [22 - 24] still is the basis upon which are built more recent proofs of Harnack-type inequalities for non-negative solutions of degenerate elliptic equations [1, 3, 6, 7, 13 - 15].

The main result in [10] thus paved the way for the proof of a more general Harnack inequality. Indeed, in [11: Theorem II] the same authors stated a result which has as particular cases the Harnack inequalities proven in [3, 7].

José D. Fernandes: Univ. de São Paulo, Inst. de Matem. e Estat., Caixa Postal 66281, 05311-970 São Paulo SP, Brazil; diniz@ime.usp.br
Jorge Groisman: Univ. de la Rep., Fac. de Ing., Julio Herrera y Reissig 567, Montevidéo, C.P. 11300, Uruguay; jorgeg@fing.edu.uy
Severino T. Melo: Univ. de São Paulo, Inst. de Matem. e Estat., Caixa Postal 66281, 05311-970 São Paulo SP, Brazil; melo@ime.usp.br

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As they pointed out, that new version would apply to solutions of the equation

$$\frac{\partial}{\partial x}\left[\left(|x|^{\sigma+1}+|y|\right)^{\frac{\kappa}{\sigma+1}}\frac{\partial f}{\partial x}\right]+\frac{\partial}{\partial y}\left[\left(|x|^{\sigma+1}+|y|\right)^{\frac{\kappa}{\sigma+1}}|x|^{\sigma}\frac{\partial f}{\partial y}\right]=0 \qquad (1)$$

in an open set $\Omega \subset \mathbb{R}^2$ containing the origin, with κ and σ arbitrary positive numbers. None of the other available results includes this example.

The proof of [11: Theorem II], however, is not complete. It depends on the (not proven) existence of certain cut-off functions fitting the metric balls defined by the operator. It is easy to construct (see our Proposition 14 below) cutoffs which are identical to one or non-zero not on metric balls, but on certain "rectangles" which are products of Euclidean balls with variable ratio of the radii. If one insists in using balls contained or containing those rectangles, there remains a gap between the two balls which provokes an explosion of the constants that appear in the iteration process.

In this paper, we prove [11: Theorem II] without using cutoffs addapted to balls, applying instead a covering technique, based on a theorem in [4], already used in the study of degenerate parabolic equations by the first author [8]. The building block of the Moser iteration used here turns out to be not exactly a Sobolev-Poincaré inequality, but rather its consequence stated in Theorem 2; which is a Sobolev-Poincaré inequality for rectangles, with the one on the right ε times larger than the one on the left and with a negative power of ε on the right. The main point of Section 4 is to show that a sequence ε_k can be chosen in such a way that the iteration converges. We show that the Moser-type iteration designed by Chanillo and Wheeden in [3] also works in this context. Propositions which are straighforward addaptions of results in [3] are stated here without proof.

We will assume as a hypothesis that the Sobolev-Poincaré inequality we need is true, without explicitly stating Franchi, Gutierrez and Wheeden's Theorem I of [10], which is nonetheless our main motivation (since it provides the main example). One important aspect of that theorem is that it allows the presence of two (possibly non-comparable and non-Muckenhoupt) weights in the ellipticity condition.

The existence of cutoffs suitable to the study of regularity properties of weak solutions of degenerate elliptic equations has been independently proven by Franchi, Serapioni and Serra Cassano [14], and by Garofalo and Nhieu [17]. Their results would apply in our context, however, only if we required that the function λ, defined in our Section 2, be Lipschitz continuous (for the operator in (1), the natural choice of λ would be $\lambda(x) = |x|^{\sigma}, \sigma > 0$). Under this additional assumption, [17: Theorem 1.3] or [14: Proposition 2.9] (together with, for example, the composition argument in the proof of [17: Theorem 1.5]), would imply the existence of the test functions needed for the proof of [11: Theorem II] to work.

A different approach was taken by Biroli and Mosco [1]. Within a very general framework, they proved the existence of cutoffs which satisfy, instead of a pointwise estimate (as in [17: Theorem 1.5], for example), a weaker requirement, in integral form [1: Proposition 3.3]. That also suffices for the proof of Harnack-type inequalities ([1: Theorem 1.1] and [15: Theorem 1]). Working directly with the bilinear form defined by the elliptic operator, they did not have to to deal with the regularity of the vector fields usually used to define the metric.

2. Preliminaries and statement of the main result

The operators considered in this paper are of type

$$Lf = \sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\left(a_{ij}(z)\frac{\partial f}{\partial z_j}\right) \tag{2}$$

where

$$z = (z_1, \ldots, z_N) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m,$$

the matrix $A = ((a_{ij}))$ is symmetric and the functions a_{ij} are real, measurable and satisfy the (degenerate) ellipticity condition

$$v(z)\big(|\xi|^2 + \lambda(x)^2|\eta|^2\big) \leq \sum_{i,j=1}^{N} a_{ij}(z)\zeta_i\zeta_j \leq u(z)\big(|\xi|^2 + \lambda(x)^2|\eta|^2\big) \tag{3}$$

for all $\zeta = (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m$, with the functions λ, u and v non-negative and satisfying several hypotheses which are especified in what follows.

Throughout this paper, aB will denote, for $a0$ and B a ball in some metric space, another ball with the same center and a-times the radius as B.

We require that the function λ defined on $\mathbb{R}^n$ satisfis the following hypotheses:

- **(H1)** λ is non-negative, continuous, and vanishes possibly only on a set of isolated points.
- **(H2)** λ is *doubling* with respect to the Euclidean metric and the Lebesgue measure, with doubling constant C_1, i.e. $\int_{2B_e} \lambda(x)\,dx \leq C_1 \int_{B_e} \lambda(x)\,dx$ for every Euclidean ball $B_e \subset \mathbb{R}^n$.
- **(H3)** There exists a constant C_2 such that $\sup_{x\in B_e} \lambda(x) \leq C_2 \frac{1}{|B_e|} \int_{B_e} \lambda(x)\,dx$ for every Euclidean ball $B_e \subset \mathbb{R}^n$, with $|\cdot|$ denoting the Lebesgue measure.

Definition 1. Given $z_\circ = (x_\circ, y_\circ) \in \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^N$ and $r' > 0$, we define

$$\Lambda(z_\circ, r) = \sup_{\{x\colon |x-x_\circ|<r\}} \lambda(x)$$

and denote

$$Q(z_\circ, r) = \Big\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : |x - x_\circ| < r \text{ and } |y - y_\circ| < r\Lambda(z_\circ, r)\Big\}.$$

If $Q = Q(z_\circ, r)$ and $t > 0$, tQ will denote $Q(z_\circ, tr)$.

Remark 2. If follows from hypotheses (H2) - (H3) that

$$\Lambda(z_\circ, 2r) \leq \frac{C_1 C_2}{2^n} \Lambda(z_\circ, r) \tag{4}$$

for all $z_\circ \in \mathbb{R}^N$ and all $r > 0$; and, hence, $C_1 C_2 \geq 2^n$ must hold.

Lemma 3. *If $z \in Q(z_\circ, r)$ and $w \in Q(z, s)$, then $w \in Q(z_\circ, r + s)$.*

Definition 4. An absolutely continuous curve in $\mathbb{R}^N$ is *subunit* if, for every $\zeta = (\xi, \eta) \in \mathbb{R}^N$ and for almost every t in its domain, we have

$$\langle \gamma'(t), \zeta \rangle^2 \leq |\xi|^2 + \lambda(\gamma(t))^2 |\eta|^2$$

with $\langle \cdot, \cdot \rangle$ denoting the usual inner product of $\mathbb{R}^N$. Given z and w in $\mathbb{R}^N$, let $\rho(z, w)$ denote the infimum of all $T \geq 0$ such that there is a subunit curve joining the two points with domain $[0, T]$.

The function ρ corresponds to the metric on $\mathbb{R}^N$ associated to the *Grushin-type* vector fields

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda(x)\frac{\partial}{\partial y_1}, \ldots, \lambda(x)\frac{\partial}{\partial y_m}$$

in a way which has by now become standard [12]. If λ is smooth and does not vanish, one can see that ρ is equal to the geodesic distance associated to the Riemannian metric

$$ds^2 = \sum_{i=1}^n dx_i^2 + \lambda(x)^{-2} \sum_{j=1}^m dy_j^2.$$

An elementary proof of the following proposition can be given. For a somewhat different but closely related result, we refer to [9].

Proposition 5. *The function ρ above defines a metric on $\mathbb{R}^N$ and there exists a constant b, depending only on n and m, such that the double inclusion*

$$Q(z_\circ, \tfrac{r}{b}) \subseteq B(z_\circ, r) \subseteq Q(z_\circ, br) \tag{5}$$

holds for every $z_\circ \in \mathbb{R}^N$ and $r > 0$, where $B(z_\circ, r)$ denotes the ball with respect to this new metric with center $z_\circ$ and radius r.

Remark 6. Only hypothesis (H1) is required for the proof of Proposition 5. As shown in [19: Proposition 2.1.1], one may take $b = \max\{3, \sqrt{m}, \sqrt{n}\}$. Proposition 5 and hypothesis (H1) imply that the metric ρ induces in $\mathbb{R}^N$ its usual topology.

We require that u and v be *weights* on $\mathbb{R}^N$ (non-negative non-trivial locally integrable functions), which are *doubling* with respect to the ρ-metric and the Lebesgue measure, i.e. such that there are constants $C_3 > 0$ and $C_4 > 0$ with

$$\int_{2B} u(z)\,dz \le C_3 \int_B u(z)\,dz \qquad \text{and} \qquad \int_{2B} v(z)\,dz \le C_4 \int_B v(z)\,dz \tag{6}$$

holding for all ρ-balls B. For every measurable $E \subseteq \mathbb{R}^N$, we will denote by $u(E)$ and $v(E)$ the integrals over E of u and v, respectively. Notice that (6) and Proposition 5 imply that $u(E)$ and $v(E)$ are positive if E has non-empty interior.

For every locally integrable function g, we will denote by $m_E(g)$ the u-average $u(E)^{-1} \int_E gu$.

Last we state the strongest hypothesis we impose on u, v and λ: that the following Sobolev-Poincaré inequality holds. For sufficient conditions for its validity see, for example, the papers [2, 7, 9, 10, 12, 16, 21, 25] and their references.

(SP) There exist $q > 2$ and $C_5 > 0$, constants depending only on u, v, λ, n and m, such that the inequality

$$\left[\frac{1}{u(B)} \int_B |g(z) - m_B(g)|^q u(z)\,dz\right]^{\frac{1}{q}} \le C_5 r \left[\frac{1}{v(B)} \int_B |\nabla_\lambda g(z)|^2 v(z)\,dz\right]^{\frac{1}{2}}$$

holds for every Lipschitz continuous function g and every ball B with respect to the metric ρ induced by λ, with r denoting the radius of B, and $\nabla_\lambda g$ denoting the vector field

$$\nabla_\lambda g(z) = \left(\frac{\partial g}{\partial x_1}(z), \ldots, \frac{\partial g}{\partial x_n}(z), \lambda(x)\frac{\partial g}{\partial y_1}(z), \ldots, \lambda(x)\frac{\partial g}{\partial y_m}(z)\right).$$

Weak solutions of the equation $Lf = 0$ in a bounded open set $\Omega \subset \mathbb{R}^N$ are defined (as in [11]) in $H(\Omega)$, the completion of the space $\mathrm{Lip}(\overline{\Omega})$ of Lipschitz continuous functions on $\overline{\Omega}$, the closure of Ω, with respect to the norm

$$||f||_H^2 = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(z) \frac{\partial f}{\partial z_i}(z) \frac{\partial f}{\partial z_j}(z)\, dz + \int_\Omega f(z)^2 u(z)\, dz. \tag{7}$$

Using (3) and (6), one can show, similarly as in [3], that the equation above indeed defines a norm. Moreover, if we denote by $H_\circ(\Omega)$ the closure in $H(\Omega)$ of the space $\mathrm{Lip}_\circ(\Omega)$ of the Lipschitz continuous functions of compact support in Ω, it can be proven, and for that hypothesis (SP) is required, that the bilinear form $a_\circ$ on $\mathrm{Lip}_\circ(\Omega)$,

$$a_\circ(f, g) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(z) \frac{\partial f}{\partial z_i}(z) \frac{\partial g}{\partial z_j}(z)\, dz,$$

induces on $H(\Omega)o$ an inner product whose corresponding norm is equivalent to $|| \cdot ||_H$.

Definition 7. An element $f \in H(\Omega)$ is a *weak solution* of the equation $Lf = 0$ if $a_\circ(f, \theta) = 0$ for all $\theta \in H_\circ(\Omega)$.

Applying the Lax-Milgram's theorem, existence and uniqueness of a suitably defined weak version of the Dirichlet problem on Ω can be proven, in exactly the same way as in [3].

We still need two more definitions. The inequality

$$\int_\Omega f(x)^2 u(x)\, dx \leq ||f||_H^2$$

follows from condition (3) and the definition of $|| \cdot ||_H$. A natural mapping $H(\Omega) \to L^2(\Omega, u(z)\, dz)$, $f \mapsto \tilde{f}$, is then defined. We stress we are not claiming that this is an injection, even though that could be proven under additional hypotheses. Finally, we will call an $f \in H(\Omega)$ *non-negative*, and denote this by $f \geq 0$, if there is a sequence of non-negative functions $f_k \in \mathrm{Lip}(\overline{\Omega})$ converging to f in $H(\Omega)$.

Remark 8. If $U \subset \Omega$ is open and $f \in H(\Omega)$ is a weak solution of the equation $Lf = 0$ in Ω, the restriction $f|_U \in H(U)$ is then a weak solution of $Lf = 0$ in U. We also have $\tilde{f}|_U = \widetilde{f|_U}$.

We are ready to state our main result.

Theorem 1. *Suppose that λ satisfies hypotheses* (H1) - (H3)*, u and v are doubling weights and also that hypothesis* (SP) *holds. Then there is a constant K, depending only on $C_1, \ldots, C_5$,q, m and n, such that, if Ω is a bounded open subset of $\mathbb{R}^N$ and $f \in H(\Omega)$ is a non-negative weak solution of the equation $Lf = 0$, with L satisfying* (2) *and* (3)*, then*

$$\mathrm{esssup}_B \tilde{f} \leq e^{K\mu} \mathrm{essinf}_B \tilde{f} \tag{8}$$

for every ρ-ball B such that $2b^4 B \subseteq \Omega$, where $\mu = u(B)^{\frac{1}{2}} v(B)^{-\frac{1}{2}}$.

3. Application of a covering technique

All hypotheses of Theorem 1 are assumed to be true for the rest of the paper, even if not explicitly. By a "constant" we will always mean a positive number which may depend only on the constants that arise in the hypotheses of Theorem 1: $C_1, \ldots, C_5$, q, m and n.

We start with a Sobolev-Poincaré inequality for the rectangles Q of Definition 1.

Proposition 9. *There exists a constant C_6 such that*

$$\begin{aligned} &\left[\frac{1}{u(Q)} \int_Q |g(z)|^q u(z)\, dz\right]^{\frac{1}{q}} \\ &\quad \leq C_6 r \left[\frac{1}{v(Q)} \int_{b^2 Q} |\nabla_\lambda g(z)|^2 v(z)\, dz\right]^{\frac{1}{2}} + \left[\frac{1}{u(Q)} \int_{b^2 Q} g(z)^2 u(z)\, dz\right]^{\frac{1}{2}} \end{aligned} \tag{9}$$

holds for every Lipschitz function g and every $Q = Q(z, r)$, where $q > 2$ is the constant provided by hypothesis (SP)*.*

Proof. Using (5), we see that

$$\left[\frac{1}{u(Q)} \int_Q |g(z) - m_{bB}(g)|^q u(z)\, dz\right]^{\frac{1}{q}}$$

is bounded by

$$\left[\frac{u(bB)}{u(\frac{1}{b}B)} \frac{1}{u(bB)} \int_{bB} |g(z) - m_{bB}(g)|^q u(z)\, dz\right]^{\frac{1}{q}}.$$

Using that u is doubling and inequality (SP) for the ball bB, we get

$$\left[\frac{1}{u(Q)} \int_Q |g - m_{bB}(g)|^q u\right]^{\frac{1}{q}} \leq C_6 r \left[\frac{1}{v(Q)} \int_{b^2 Q} |\nabla_\lambda g|^2 v\right]^{\frac{1}{2}} \tag{10}$$

with $C_6 = bC_3^{\frac{l}{q}} C_5$, where l is an integer such that $b^2 < 2^l$. To prove (9), we start by applying to $g = [g - m_{bB}(g)] + m_{bB}(g)$ the triangle inequality in $L^p(Q, u(z)\, dz)$, followed by (10), then by the Cauchy-Schwarz inequality for $L^2(Q, u(z)\, dz)$ and finally (5) ∎

We will call a metric space *homogeneous* if it can be equipped with a Borel measure ν such that $\nu(2B) \le D\nu(B)$ for every ball B, for some *doubling-factor* D.

The following proposition is a particular case of [4: Theorem 1.2].

Proposition 10. *If $\{B(x,r)\}$ is a family of balls of constant radius covering a subset E of a homogeneous metric space X, then there is a finite sub-family $\{B(x_i,r) : i = 1,\dots,m\}$ of disjoint balls such that $\{B(x_i,4r) : i = 1,\dots,m\}$ still covers E.*

Proposition 11. *The metric space $(\mathbb{R}^N,\rho)$ is homogeneous.*

Proof. Let $z_\circ = (x_\circ, y_\circ) \in \mathbb{R}^n \times \mathbb{R}^m$ and $r > 0$ be given. By (4), we have

$$\Lambda(z_\circ,t) \le C_7^l \Lambda(z_\circ, \tfrac{t}{2^l}) \tag{11}$$

for every non-negative integer l and every $t > 0$, with $C_7 = 2^{-n}C_1C_2$. Using Proposition 5, we then get

$$|B(z_\circ,2r)| \le \omega_n\omega_m(2br)^N\Lambda(z_\circ,2br)^m \le C_7^{ml}(2b^2)^N|Q(z_\circ,\tfrac{r}{b})|$$

if l is chosen so that $2b^2 \le 2^l$, with ω_k denoting the volume of the unit ball in $\mathbb{R}^r$. Since $Q(z_\circ,\frac{r}{b}) \subseteq B(z_\circ,r)$, this shows that the Lebesgue measure is doubling with doubling-factor $C_7^{ml}(2b^2)^N$ $\blacksquare$

Proposition 12. *Given $z \in \mathbb{R}^N$ and $0 < r < s$, there exist $z_1,\dots,z_p \in Q(z,s)$ such that the family $\{Q(z_1,r),\dots,Q(z_p,r)\}$ covers $Q(z,s)$, with $Q(z_j,\frac{r}{4b^2})$ and $Q(z_k,\frac{r}{4b^2})$ disjoint when $j \ne k$. Moreover, there are constants β and C_8 such that*

$$p \le C_8\left(\frac{s}{r}\right)^\beta. \tag{12}$$

Proof. The first statement of this proposition follows straightforwardly from Proposition 5, Proposition 10 (with $\frac{r}{4b}$ replacing r) and Proposition 11. In order to prove (12), let us first remark that there is a constant β such that the inequality

$$|Q(w,\theta t)| \ge C_7^{-m}\theta^\beta|Q(w,t)| \tag{13}$$

holds for all $0 < \theta < 1$, $t > 0$ and $w \in \mathbb{R}^N$. Indeed, let β be defined by $\beta = N + m\frac{\log C_7}{\log 2}$. Using $|Q(w,t)| = \omega_n\omega_m t^N\Lambda(w,t)^m$, we get (13) by applying (11) to the integer l such that $\frac{\theta}{2} < 2^{-l} \le \theta$. It follows from Remark 2 that $C_7 \ge 1$ and thus β is positive.

By Lemma 3, and since $s + \frac{r}{4b^2} < (b^2+1)s$, each $Q_j = Q(z_j,\frac{r}{4b^2})$ is contained in $Q(z,(b^2+1)s)$. Since the Q_j's are mutually disjoint, we have

$$|Q(z,(b^2+1)s)| \ge \sum_{j=1}^{p}\left|Q\Big(z_j,\frac{r}{4b^2}\Big)\right|. \tag{14}$$

Now let us apply (13) to $w = z_j, t = (2b^2+1)s$ and $\theta = \frac{r}{8b^4 s + 4b^2 s}$. We get

$$\left|Q\Big(z_j, \frac{r}{4b^2}\Big)\right| \geq \left(\frac{r}{s}\right)^{\beta} \frac{|Q(z_j,(2b^2+1)s)|}{C_7^m(8b^4+4b^2)^{\beta}}. \tag{15}$$

By (5), z is in $Q(z_j, b^2 s)$. Now Lemma 3 implies $Q(z_j,(2b^2+1)s) \supseteq Q(z,(b^2+1)s)$. This, (14) and (15) together imply

$$|Q(z,(b^2+1)s)| \geq p\left(\frac{r}{s}\right)^{\beta} \frac{|Q(z,(b^2+1)s)|}{C_7^m(8b^4+4b^2)^{\beta}}.$$

This proves (12) with $C_8 = C_7^m(8b^4+4b^2)^{\beta}$ ∎

Lemma 13. *There are constants C_9 and γ such that*

$$\frac{u(sQ)}{u(rQ)} \leq C_9\left(\frac{s}{r}\right)^{\gamma} \qquad \text{and} \qquad \frac{v(sQ)}{v(rQ)} \leq C_9\left(\frac{s}{r}\right)^{\gamma} \tag{16}$$

for every "rectangle" Q and for every $0 < r < s$.

Proof. It follows from (5) and (6) that, if l is an integer such that $b^2 < 2^l$, then $u(2Q) \leq C_3^{l+1}u(Q)$ and $v(2Q) \leq C_4^{l+1}v(Q)$ for all Q. Arguing similarly as for the proof of (13), we can get (16) with $C_9 = \max\{C_3^{l+1}, C_4^{l+1}\}$ and $\gamma = \frac{\log C_9}{\log 2}$ ∎

The following theorem plays here the role of Theorem D in [8]. The explicit form of the constants in inequality (17) below, valid for arbitrarily small ε, is needed for an efficient control of the constants that show up in the iteration process.

Theorem 2. *Under the hypotheses of Theorem 1, there are constants α and C_{10} such that the estimate*

$$\frac{\varepsilon^{\alpha}}{C_{10}}\left[\frac{1}{u(Q)}\int_Q |g(z)|^q u(z)\,dz\right] \leq$$
$$\left[\left(\frac{s^2}{v(Q)}\int_{(1+\varepsilon)Q} |\nabla_\lambda g(z)|^2 v(z)\,dz\right)^{\frac{1}{2}} + \left(\frac{1}{u(Q)}\int_{(1+\varepsilon)Q} g(z)^2 u(z)\,dz\right)^{\frac{1}{2}}\right]^q \tag{17}$$

holds for every $Q = Q(z,s)$, for every $0 < \varepsilon < 1$, and for every Lipschitz continuous function g, where $q > 2$ is the constant provided by hypothesis (SP).

Proof. Let us apply Proposition12 with $r = \varepsilon \frac{s}{b^2}$ and let the Q's then obtained be denoted by $Q_j = Q(z_j, r)$ $(j = 1, \ldots, m)$. By (9) we get

$$\int_Q |g(z)|^q u(z)\, dz \leq \sum_{j=1}^{p} u(Q_j) \left[C_6 r \left(\frac{1}{v(Q_j)} \int_{b^2 Q_j} |\nabla_\lambda g(z)|^2 v(z)\, dz \right)^{\frac{1}{2}} \right. \\ \left. + \left(\frac{1}{u(Q_j)} \int_{b^2 Q_j} g(z)^2 u(z)\, dz \right)^{\frac{1}{2}} \right]^q . \tag{18}$$

By Lemma 3, we have $b^2 Q_j \subseteq Q(z, s + b^2 r)$, and hence the integrals on $b^2 Q_j$ inside the brackets in (18) may be replaced by integrals on $(1+\varepsilon)Q$. We then estimate $\frac{u(Q)}{u(Q_j)}$ and $\frac{v(Q)}{v(Q_j)}$ using (16) and $Q(z,s) \subseteq Q(z_j, (b^2+1)s)$ (which follows from Lemma 3 and Proposition 5). This way we see that the expression between brackets in (18) is bounded by the expression between brackets in (17) times $C_9^{\frac{1}{2}} \max\{C_6, 1\}[\frac{(b^2+1)s}{r}]^{\frac{\gamma}{2}}$.

Next we use that $Q_j \subseteq 2Q$ (which follows from Lemma 3), to get $u(Q_j) \leq C_9 u(Q)$ (by the proof of Lemma 13). After using (12), we finally get (17) with

$$C_{10} = C_8 C_9^{\frac{2+q}{2}} \max\{C_6, 1\}^q (b^4 + b^2)^{\frac{q\gamma}{2}} b^{2\beta}$$

and $\alpha = \beta + \frac{q\gamma}{2}$ ∎

4. Moser iteration and Harnack inequality

We start this section with the construction of the test functions addapted to rectangles mentioned in the Introduction.

Proposition 14. *Given any $z_\circ \in \mathbb{R}^N$ and any $0 < r_1 < r_2$, there is a smooth function η equal to one everywhere on $Q(z_\circ, r_1)$, with support contained in $Q(z_\circ, r_2)$, and such that $0 \leq \eta(z) \leq 1$ and $|\nabla_\lambda \eta(z)| \leq \frac{C_{11}}{r_2 - r_1}$ for all $z \in \mathbb{R}^N$, with C_{11} denoting the constant $2\sqrt{N}$.*

Proof. Choose ψ a smooth function on $\mathbb{R}$ identical to one on $(-\infty, 0]$, with support contained in $(-\infty, 1)$, and such that $0 \leq \psi(t) \leq 1$ and $|\psi'(t)| \leq 2$ for all $t \in \mathbb{R}$. Given $z_\circ = (x_\circ, y_\circ) \in \mathbb{R}^n \times \mathbb{R}^m$ and $0 < r_1 < r_2$, define

$$\eta(x,y) = \varphi\left(\frac{|x - x_\circ|}{r_2}\right) \varphi\left(\frac{|y - y_\circ|}{r_2 \Lambda(z_\circ, r_2)}\right) \qquad ((x,y) \in \mathbb{R}^n \times \mathbb{R}^m)$$

where $\varphi(t) = \psi(\frac{r_2 t - r_1}{r_2 - r_1})$. It is straightforward to check that this η does it ∎

Definition 15. An element $f \in H(\Omega)$ is a *weak subsolution* of the equation $Lf = 0$ if $a_\circ(f, \theta) \leq 0$ for all non-negative θ in $H_\circ(\Omega)$.

Definition 16. Given $M > 0$ and $d \geq 1$, let the *function* $H_{M,d}$ (continuously differentiable with bounded derivative) be defined by $H_{M,d}(t) = t^d$ if $t \in [0, M]$, and $H_{M,d}(t) = M^d + dM^{d-1}(t - M)$ if $t > M$.

Proposition 17. *Let $f \in H(Q)$, $Q = Q(z_\circ, h)$, be a non-negative subsolution of the equation $Lf = 0$ and let f_k be a sequence of non-negative Lipschitz continuous functions on $\overline{Q}$ converging to f in $H(Q)$. Given $\frac{1}{2} \leq s < t \leq 1, M > 0$ and $\beta \geq 1$, there are a subsequence f_{k_j} of f_k and a sequence $\delta_j \geq 0$, $\delta_j \to 0$, such that for all j we have*

$$\int_{sQ} |\nabla_\lambda (H_{M,d} \circ f_{k_j})|^2 v \leq \delta_j + \frac{4C_{11}^2}{(t-s)^2 h^2} \int_{tQ} |f_{k_j} \cdot (H'_{M,d} \circ f_{k_j})|^2 u. \tag{19}$$

Proposition 17 can be given a proof almost identical to the first part of the proof of [3: Lemma 3.1/pages 1117 - 1119]. One only needs to replace their Euclidean ball B by our rectangle Q, and their ellipticity condition (1.1) by ours (3). When (3) is applied, our ∇_λ will show up, replacing their ∇. Also, one should take η as the test function constructed in Proposition 14, with $r_1 = hs$ and $r_2 = ht$. Since the support of the chosen η is contained in the open set $Q(z_\circ, ht)$, we may allow t to be equal to one (this fact is needed in our iteration).

An inequality to be derived from (17) and (19) will be iterated in the proof of the next proposition, which corresponds to a weaker version of Lemma 3.1 in [3].

Proposition 18. *If $f \in H(Q)$, $Q = Q(z_\circ, h)$, is a non-negative subsolution of the equation $Lf = 0$, then the estimate*

$$\left(\operatorname*{ess\,sup}_{aQ} \tilde{f} \right)^p \leq \frac{C_{12}}{(1-a)^\delta} [p\mu(Q)]^{\frac{2q}{q-2}} \frac{1}{u(Q)} \int_Q \tilde{f}^p u \tag{20}$$

holds for every $a \in [\frac{1}{2}, 1)$ and every $p \geq 2$, with δ and C_{12} denoting constants explicitly defined below (*at the end of the proof*), *and $\mu(Q) = u(Q)^{\frac{1}{2}} v(Q)^{-\frac{1}{2}}$.* (*We recall that q arises in hypothesis* (SP).)

Proof. Given $\frac{1}{2} \leq s < t \leq 1$ and $d \geq 1$, let us first use Proposition 17 to extract a subsequence f_{k_j} of a sequence f_k of non-negative Lipschitz continuous functions on $\overline{Q}$ converging to f in $H(Q)$ for which (19) is true. Then, let us apply (17) to the rectangle sQ, for some ε satisfying $(1 + \varepsilon)s < t$ and for $g = H_{M,d} \circ f_{k_j}$. Then let us apply (19) with $(1+\varepsilon)s$ replacing s. Next, we use (16) with t and s replacing s and r, respectively, and taking advantage of the fact that $1 < \frac{t}{s} \le 2$. Finally, after using that $H_{M,d}(\varphi) \le \varphi H'_{M,d}(\varphi)$ for all $\varphi \in \mathbb{R}$, we get

$$\begin{aligned} \left[\frac{1}{u(sQ)}\int_{sQ} |H_{M,d}\circ f_{k_j}|^q u\right]^{\frac{1}{q}} \le \frac{C_{10}^{\frac{1}{q}}\varepsilon^{-\frac{\alpha}{q}} sh}{v(sQ)^{\frac{1}{2}}}\delta_j^{\frac{1}{2}} &+ 2^{\frac{\gamma}{2}} C_9^{\frac{1}{2}} C_{10}^{\frac{1}{q}}\, e^{-\frac{\alpha}{q}} \\ &\times \left[2C_{11}\mu(sQ)\frac{s}{t-(1+\varepsilon)s}+1\right] \\ &\times \left[\frac{1}{u(tQ)}\int_{tQ} |f_{k_j}\cdot (H'_{M,d}\circ f_{k_j})|^2 u\right]^{\frac{1}{2}}. \end{aligned} \tag{21}$$

Now we want to let j first, and then M, go to infinity. We may suppose, passing to another subsequence if necessary, that f_{k_j} converges to $\tilde{f}$ pointwise, almost everywhere with respect to the measure $u(z)\,dz$. Using Fatou's lemma on the left-hand side and Lebesgue's convergence theorem on the right (again, this is the same argument as Chanillo and Wheeden's, on page 1120 of [3]), one can see that it is legitimate to replace f_{k_j} by $\tilde{f}$ in (21), and then $H_{M,d}\circ\tilde{f}$ by $\tilde{f}^d$ and $H'_{M,d}\circ\tilde{f}$ by $\tilde{f}^{d-1}$.

Since $\frac{1}{2} \le s < (1+\varepsilon)s < t \le 1$, then $\frac{s}{t-(1+\varepsilon)s}$ is greater than one. By (3), it follows that $\mu(sQ) \ge 1$. Hence, the "+1" inside the first pair of brackets at the right-hand side of inequality (21) may be absorbed by the constant at its left, which will then be multiplied by two. Next we raise to the $\frac{1}{d}$-th power both sides of the inequality and change notation, writing $r = 2d$ and $q = 2\sigma$. After all that is taken into account, we will have deduced from (21) the estimate

$$\left[\frac{1}{u(sQ)}\int_{sQ}\tilde{f}^{r\sigma}u\right]^{\frac{1}{r\sigma}} \le \left[\frac{C_{13}\varepsilon^{-A}\mu(sQ)rs}{t-(1+\varepsilon)s}\right]^{\frac{2}{r}} \left[\frac{1}{u(tQ)}\int_{tQ}\tilde{f}^{r}u\right]^{\frac{1}{r}} \tag{22}$$

for all $r \ge 2$, with $C_{13} = 2^{\frac{2+\gamma}{2}} C_9^{\frac{1}{2}} C_{10}^{\frac{1}{q}} C_{11}$ and $A = \frac{\alpha}{q}$.

Let $a \in [\frac{1}{2}, 1)$ and $p \ge 2$ be given and define $a_j = a + \frac{1-a}{j+1}$. For each non-negative integer j, let us apply (22) with $t = a_j, s = a_{j+1}, \varepsilon = \varepsilon_j = \frac{a_{j+1}-a_{j+2}}{a_{j+1}}$ and $r = \sigma^j p$. Let us apply (22) again to the right-hand side of the inequality thus obtained, but with $t = a_{j-1}, s = a_j, \varepsilon = \varepsilon_{j-1}$ and $r = \sigma^{j-1}p$. By repeating this procedure, after $j+1$ steps we will get

$$\begin{aligned} &\left[\frac{1}{u(a_{j+1}Q)}\int_{a_{j+1}Q}\tilde{f}^{p\sigma^{j+1}}u\right]^{\frac{1}{p\sigma^{j+1}}} \\ &\le \left\{\prod_{k=0}^{j}\left[\frac{C_{13}\varepsilon_k^{-A}\mu(a_{k+1}Q)p\sigma^k a_{k+1}}{a_k-(1+\varepsilon_k)a_{k+1}}\right]^{\frac{2}{p\sigma^k}}\right\}\left[\frac{1}{u(Q)}\int_Q \tilde{f}^p u\right]^{\frac{1}{p}}. \end{aligned} \tag{23}$$

Since $a < a_{j+1} < 2a$ for all j, it follows from Lemma 13 that the left-hand side of (23) is greater than or equal to

$$\left[\frac{2^\gamma C_9}{u(aQ)}\int_{aQ}\tilde{f}^{p\sigma^{j+1}}u\right]^{\frac{1}{p\sigma^{j+1}}}$$

which converges to $\operatorname{ess\,sup}_{aQ}\tilde{f}$ as j tends to infinity. On the right-hand side of (23) we may replace $\mu(a_{j+1}Q)$ by $\sqrt{2^\gamma C_9}\mu(Q)$, due to Lemma 13. Hence, all we need is to find a precise estimate for the product

$$\prod_{k=0}^{\infty}\left[C_{13}\sqrt{2^\gamma C_9}\mu(Q)p\sigma^k\right]^{\frac{2}{p\sigma^k}}\prod_{k=0}^{\infty}\left[\frac{\varepsilon_k^{-A}a_{k+1}}{a_k-(1+\varepsilon_k)a_{k+1}}\right]^{\frac{2}{p\sigma^k}}. \tag{24}$$

The first of these products equals

$$\left[C_{13}\sqrt{2^\gamma C_9}\mu(Q)p\sigma^{\frac{1}{\sigma-1}}\right]^{\frac{2\sigma}{p(\sigma-1)}}.$$

The second expression between brackets is equal to the left side of

$$\frac{(k+1)(k+3)^{A+1}(ak+a+1)^{A+1}}{2(1-a)^{A+1}}\leq\frac{(k+3)^{2A+3}}{2(1-a)^{A+1}}.$$

Hence, the second product in (24) is bounded by

$$2^{-\frac{2\sigma}{p(\sigma-1)}}(1-a)^{-\frac{2(A+1)\sigma}{p(\sigma-1)}}\left[\exp\sum_{k=0}^{\infty}\frac{\log(k+3)}{\sigma^k}\right]^{\frac{4A+6}{p}}.$$

Defining $\delta=\frac{2(A+1)\sigma}{\sigma-1}$ and

$$C_{12}=\left[\sqrt{2^{\gamma-2}C_9}\,C_{13}\sigma^{\frac{1}{\sigma-1}}\right]^{\frac{2\sigma}{\sigma-1}}\left[\exp\sum_{k=0}^{\infty}\frac{\log(k+3)}{\sigma^k}\right]^{4A+6}$$

finishes the proof ∎

Proposition 19. *Let $f\in H(Q)$, $Q=Q(z_\circ,h)$, be a strictly positive ($f\geq\varepsilon_\circ>0$) solution of the equation $Lf=0$ and let f_k, $f_k\geq\varepsilon_\circ$, be a sequence in* $\mathrm{Lip}(\overline{Q})$ *converging to f in $H(Q)$. Given $\frac{1}{2}\leq s<t\leq 1$ and $\beta\leq 1$, with $-1\neq\beta\neq 0$, there are a subsequence f_{k_j} of f_k and a sequence of non-negative reals $\delta_j\to 0$ such that for all j we have*

$$\int_{sQ}|\nabla_\lambda(f_{k_j}^{\frac{\beta+1}{2}})|^2v\leq\delta_j+\frac{(\beta+1)^2}{\beta^2}\frac{C_{11}^2}{(t-s)^2h^2}\int_{tQ}f_{k_j}^{\beta+1}u. \tag{25}$$

A proof for Proposition 19 can be given following exactly the same steps as in the first half of the proof of Lemma (3.11) in [3: pages 1121 – 1122], making the addaptations already described after the statement of Proposition 17.

The proof of the following proposition follows the steps of [3: Lemma 3.11] for $p < 0$ or $p \geq 2$. For $0 < p < 2$, we use a technique of Hardy and Littlewood, as in [20: Lemma 3.17].

Proposition 20. *If $f \in H(Q)$, $Q = Q(z_\circ, h)$, is a non-negative solution of the equation $Lf = 0$, then the estimate*

$$\left(\operatorname{esssup}_{aQ} \tilde{f}\right)^p \leq \frac{C_{14}}{(1-a)^\delta} \left[1 + |p|\mu(Q)\right]^{\frac{2q}{q-2}} \frac{1}{u(Q)} \int_Q \tilde{f}^p u \tag{26}$$

holds for every $a \in [\frac{1}{2}, 1)$ and every $0 \neq p \in \mathbb{R}$, with δ and $\mu(Q)$ as defined in Proposition 18 and C_{14} denoting the constant explicitly defined below, at the end of the proof.

Proof. We may suppose that $f \geq \varepsilon_\circ > 0$ and later let $\varepsilon_\circ$ tend to zero, as long as we make sure that none of the constants depends on $\varepsilon_\circ$.

Given $\beta \leq 1$, $-1 \neq \beta \neq 0$, $\frac{1}{2} \leq s < t \leq 1$ and $\varepsilon > 0$ such that $(1+\varepsilon)s < t$, we may combine (17) for $g = f_{k_j}^{\frac{\beta+1}{2}}$ and (25), and then let j go to infinity. Similarly as just before (22), with $r = \beta + 1$ and $\sigma = \frac{q}{2}$, we get

$$\begin{aligned} &\left[\frac{1}{u(sQ)} \int_{sQ} \tilde{f}^{r\sigma} u\right]^{\frac{1}{|r|\sigma}} \\ &\leq \left(\frac{C_{13}\varepsilon^{-A}}{2}\right)^{\frac{2}{|r|}} \left[\frac{|r|}{|r-1|} \frac{s\mu(sQ)}{t-(1+\varepsilon)s} + 1\right]^{\frac{2}{|r|}} \left[\frac{1}{u(tQ)} \int_{tQ} \tilde{f}^r u\right]^{\frac{1}{|r|}} \end{aligned} \tag{27}$$

for all $r \leq 2$, $0 \neq r \neq 1$.

Now let $a \in [\frac{1}{2}, 1)$ and $p < 0$ be given and let a_j and ε_j be defined as in the proof of Proposition 18. For each integer j, let us then apply (27) with $r = \sigma^k p, t = a_k, s = a_{k+1}$ and $\varepsilon = \varepsilon_k$, for $k = 0, 1, \ldots, j$. Iterating the $j+1$ inequalities just obtained and letting j tend to infinity, similarly as before, we get

$$\operatorname{esssup}_{aQ} \tilde{f}^{-1} \leq K_0 \left[\frac{1}{u(Q)} \int_Q \tilde{f}^p u\right]^{\frac{1}{|p|}} \tag{28}$$

with

$$K_0 = \prod_{k=0}^{\infty} \left[\frac{C_{13}\varepsilon_k^{-A}}{2} \left(\frac{\sigma^k |p| \mu(a_{k+1}Q) a_{k+1}}{|\sigma^k p - 1| \left[a_k - (1+\varepsilon_k)a_{k+1}\right]} + 1\right)\right]^{\frac{2}{|p|\sigma^k}}.$$

Since at this point we are assuming $p < 0$, we have $|\sigma^k p - 1| \geq 1$ for all $k \geq 0$. Taking into account also that $1 \leq \sigma^k$, that $1 \leq \frac{a_{k+1}}{[a_k-(1+\varepsilon_k)a_{k+1}]}$, that $\mu(a_{k+1}Q) \leq \sqrt{2^\gamma C_9}\,\mu(Q)$ and that $1 \leq \sqrt{2^\gamma C_9}$, the infinite product above is seen to be bounded by

$$K_0 \leq \prod_{k=0}^{\infty}\left[\frac{C_{13}\sqrt{2^\gamma C_9}}{2}\big(1+|p|\mu(Q)\big)\sigma^k \frac{\varepsilon_k^{-A}a_{k+1}}{a_k-(1+\varepsilon_k)a_{k+1}}\right]^{\frac{2}{|p|\sigma^k}}.$$

We may here use the estimates obtained at the end of the proof of Proposition 18 to conclude that (26) holds, if there we replace C_{14} by C_{12}. It follows from Proposition 18 that the same is true for $p \geq 2$.

In the case $0 < p < 2$, we have $\sigma^j p$ tending to infinity, but smaller than two for some values of j. Let us first suppose that $\sigma^k p \neq 1$, for every integer $k \geq 0$. Let then l be the integer such that $\sigma^l p < 2 \leq \sigma^{l+1}p$. We may iterate as before, but using (27) at the first $l+1$ steps of the iteration and (22) after that. We get

$$\operatorname{esssup}_{aQ}\tilde{f} \leq K_1\left[\frac{1}{u(Q)}\int_Q \tilde{f}^p u\right]^{\frac{1}{p}}$$

with

$$K_1 = \prod_{k=0}^{l}\left[\frac{C_{13}\varepsilon_k^{-A}}{2}\left(\frac{\sigma^k p\mu(a_{k+1}Q)a_{k+1}}{|\sigma^k p-1|\,[a_k-(1+\varepsilon_k)a_{k+1}]}+1\right)\right]^{\frac{2}{p\sigma^k}}$$
$$\times\prod_{k=l+1}^{\infty}\left[\frac{C_{13}\varepsilon_k^{-A}\sigma^k p\mu(a_{k+1}Q)a_{k+1}}{a_k-(1+\varepsilon_k)a_{k+1}}\right]^{\frac{2}{p\sigma^k}}.$$

In order to get a good estimate for K_1, let us further suppose that $p = \frac{\sigma^j(\sigma+1)}{2}$, for some $j \in \mathbb{Z}$. Then it will hold that $|\sigma^k p - 1| \geq \frac{\sigma-1}{2\sigma}$, for every integer $k \geq 0$. We may proceed as we did for the other infinite products, using in adition that $1 < \frac{2\sigma}{\sigma-1}$, and prove that (26) holds for these values of p, with C_{14} replaced by $C_{15} = C_{12}(\frac{2\sigma}{\sigma-1})^{\frac{2\sigma}{\sigma-1}}$.

By Remark 8, we may apply the result we have just obtained with αQ replacing Q, for any $\alpha \in (0,1)$. Given $\frac{1}{2} \leq \alpha' < \alpha \leq 1$ and $\overline{p}$ belonging $X = \{\frac{\sigma^j(\sigma+1}{2} : j \in \mathbb{Z}\}$, we get

$$\left(\operatorname{esssup}_{\alpha' Q}\tilde{f}\right)^{\overline{p}} \leq \frac{C_{15}(2^\gamma C_9)^{\frac{q}{q-2}}}{(\alpha-\alpha')^\delta}\big[1+\overline{p}\mu(Q)\big]^{\frac{2q}{q-2}}\frac{1}{u(\alpha Q)}\int_{\alpha Q}\tilde{f}^{\overline{p}}u \qquad (29)$$

where we have used Lemma 13 and $1 \leq C_9$ in order to replace $\mu(\alpha Q)$ by $\mu(Q)$ inside the brackets.

Let us define

$$I_p = \frac{[1 + p\mu(Q)]^{\frac{2q}{2q-2}}}{u(Q)} \int_Q \tilde{f}^p u \qquad \big(p \in (0,2)\big)$$

and $E(\alpha) = \operatorname{ess\,sup}_{\alpha Q} \tilde{f}$. Given any $p \in (0,2) \setminus X$, let $\overline{p} \in X$ be such that $\frac{\overline{p}}{\sigma} < p < \overline{p}$. By Lemma 13 and (29), we get

$$E(\alpha')^{\overline{p}} \leq \frac{C_{16}}{(\alpha - \alpha')^{\delta}} \, E(\alpha)^{\overline{p}-p} I_p \tag{30}$$

with $C_{16} = C_{15}(2^{\gamma} C_9)^{\frac{2q-2}{q-2}} \sigma^{\frac{2q}{q-2}}$.

Given $a \in [\frac{1}{2}, 1)$, let α_k be a strictly increasing sequence such that $\alpha_0 = a$, and $\lim \alpha_k < 1$. Let us take the logarithm of (30) and iterate, with $\alpha' = \alpha_k$ and $\alpha = \alpha_{k+1}$ $(k \geq 0)$. With $\theta = \frac{\overline{p}-p}{\overline{p}}$, we get

$$\begin{aligned} \log E(a) \leq \frac{1}{\overline{p}} \sum_{k=0}^{\infty} \theta^k \log \frac{C_{16}}{(\alpha_{k+1} - \alpha_k)^{\delta}} \\ + \limsup_{k \to \infty} \theta^{k+1} \log E(\alpha_{k+1}) + \frac{1}{\overline{p}(1-\theta)} \log I_p; \end{aligned} \tag{31}$$

noting that, since $C_{16} > 1$ and $\alpha_{k+1} - \alpha_k < \frac{1}{2}$, the terms of the series in the above inequality are positive.

It follows from Proposition 18 for $p = 2$ that $E(\lim \alpha_k)$ is finite. Since $\theta < 1$, we then get $\limsup_{k\to\infty} \theta^{k+1} \log E(\alpha_{k+1}) = 0$. To estimate the sum in (31), we need to make a precise choice of α_k. If we let

$$\alpha_k = a + (1-a) \frac{\sum_{j=1}^{k} j^{-2}}{2 \sum_{j=1}^{\infty} j^{-2}} \qquad (k \geq 1)$$

we get $\alpha_{k+1} - \alpha_k \geq \frac{1-a}{[4(k+1)^2]}$. Since $\overline{p}(1-\theta) = p$, we get

$$p \log E(a) \leq \log \frac{C_{16} 4^{\delta}}{(1-a)^{\delta}} + \sum_{k=0}^{\infty} \theta^k \log(k+1)^{2\delta} + \log I_p.$$

Exponentiating both sides of this inequality and defining

$$C_{14} = \max \left\{ C_{12}, C_{15}, 4^{\delta} C_{16} \exp \left[\sum_{k=0}^{\infty} \theta^k \log(k+1)^{2\delta} \right] \right\}$$

finishes the proof ∎

Proposition 21. *Let Ω be a bounded open set of $\mathbb{R}^N$, and let $f \in H(\Omega)$ be a positive weak solution of the equation $Lf = 0$, bounded below by a positive number. Let $z_0 \in \Omega$ and $h > 0$ be such that $bB \subseteq \Omega$, where $B = B(z_0, h)$. For each $\alpha \in [\frac{1}{2}, 1)$, define $k(\alpha, f)$ by $\log k(\alpha, f) = m_{\alpha bB}(\log \tilde{f})$ (see page 4). Then there is a constant C_{17} such that, if z_0 and h are such that $b^2 Q \subseteq \Omega$, where $Q = Q(z_0, h)$, then the inequality*

$$u\left(\left\{x \in \alpha Q : \ \left|\log \frac{\tilde{f}(x)}{k(\alpha, f)}\right| > t\right\}\right) \leq \frac{C_{17}\mu(Q)u(\alpha Q)}{(1-\alpha)t} \tag{32}$$

holds for every $t > 0$ and every $\alpha \in [\frac{1}{2}, 1)$.

Proof. This proposition can be given a proof very similar to that of [3: Lemma 3.13]. We are going to highlight a few points, refering to Chanillo and Wheeden's article for more details.

Let f_k denote a sequence of positive Lipschitz continuous functions, uniformly bounded away from zero, converging to f in $H(\Omega)$. With the aid of the test function η (built in Proposition 14 – here we take $r_1 = \alpha h$ and $r_2 = h$), we can extract from f_k a subsequence, which we will still denote by f_k, such that

$$\int_{\alpha Q} |\nabla_\lambda (\log f_k)|^2 v \leq \frac{4C_{11}^2 u(Q)}{(1-\alpha)^2 h^2} + \delta_k \tag{33}$$

for some $\delta_k \to 0$.

With $g = \log f_k$, let us apply (10) with q replaced by 2 (this is allowed by Hölder's inequality) and Q replaced by αQ. Next, let us apply (33) with Q replaced by $b^2 Q$. Using also Lemma 13, we get

$$\int_{\alpha Q} \left|\log(f_k) - m_{\alpha bB}(\log f_k)\right|^2 u \leq \frac{C_{17}^2}{(1-\alpha)^2} \mu(Q)^2 u(\alpha Q) + \delta_k' \tag{34}$$

with $C_{17} = 2C_6 C_9^{\frac{1}{2}} C_{11} b^{\gamma-2}$ and $\delta_k' \to 0$. Using that f_k is uniformly bounded away from zero, one can see that the $\lim_k$ of the left-hand side of (34) is equal to $\int_{\alpha Q} |\log \tilde{f} - \log k(\alpha, f)|^2 u$. The proposition now follows from Chebyshev's and Cauchy-Schwartz's inequalities ∎

The following lemma for $w \equiv 1$ is essentially Lemma 3 of [24], whose proof also works for the case of an arbitrary weight w.

Lemma 22 (Bombieri-Moser). *Let w be a (non-negative) weight on $\mathbb{R}^N$, and let f be a bounded non-negative measurable function defined on a bounded measurable set E. Suppose there is a family $E_t, t \in (0, 1]$, of measurable sets* *with $w(E_t) > 0$ for all t, $E_1 = E$ and $E_s \subset E_t$ if $s < t$. Assume there are $\mu, c, d > 0$ such that*

$$\operatorname{esssup}_{E_s} f^p \leq \frac{c}{(t-s)^d} \frac{1}{w(E_1)} \int_{E_t} f^p w \tag{35}$$

for all $p, s,$ and t such that $0 < p < \mu^{-1}$ and $\frac{1}{2} \leq s < t \leq 1$ and

$$w\big(\{x \in E_1 : \log f(x) > \tau\}\big) \leq \frac{c\mu}{\tau} w(E_1) \tag{36}$$

for all $\tau > 0$. Then there exists $C > 0$, depending only on c, such that

$$\operatorname{esssup}_{E_\alpha} f \leq \exp\left[\frac{C\mu}{(1-\alpha)^{2d}}\right] \tag{37}$$

for all $\alpha \in [\frac{1}{2}, 1)$.

Proof of Theorem 1. We may suppose that $\tilde{f}$ is bounded away from zero, otherwise we could add an $\varepsilon > 0$ and later let $\varepsilon \to 0$.

Let $B = B(z_0, h)$ be such that $2b^3 B \subseteq \Omega$ and let $Q = Q(z_0, h)$. With $w = u$ and $E_t = \frac{3t}{2} Q$, we are going to apply Lemma 22 to the functions $\frac{\tilde{f}}{k}$ and $\frac{k}{\tilde{f}}$, where $k = \exp[m_{\frac{3}{2}bB}(\log \tilde{f})]$. Notice that $\tilde{f}$ is bounded on E_1, since the closure of E_1 is contained in Ω, and we may then apply Proposition 18 with $p = 2$ for a rectangle slightly larger than E_1. Choosing, for example,

$$c = \max\left\{4C_{17}\sqrt{2^\gamma C_9},\, 2^\gamma C_9 C_{14} (2^{\frac{\gamma}{2}+1} C_9)^{\frac{2q}{q-2}}\right\}$$

we can check that (35) and (36) with $d = \delta$ and $\mu = \mu(Q)$ hold for both $\frac{\tilde{f}}{k}$ and $\frac{k}{\tilde{f}}$, by Propositions 20 and 21, and by also using that u is doubling (Lemma 13). We remark that $2b^3 B \subseteq \Omega$ implies that $2b^2 Q \subseteq \Omega$, and we may apply (32) with $2Q$ replacing Q and $\alpha = \frac{3}{4}$. Choosing $\alpha = \frac{2}{3}$ in (37) for $\frac{\tilde{f}}{k}$ and $\frac{k}{\tilde{f}}$, we see that $\operatorname{ess\,sup}_Q \frac{\tilde{f}}{k}$ and $\operatorname{ess\,sup}_Q \frac{k}{\tilde{f}} = [\operatorname{ess\,inf}_Q \frac{\tilde{f}}{k}]^{-1}$ are both bounded by $\exp(3^{2d} C\mu)$. Taking the product of these two inequalities, we get

$$\operatorname{esssup}_Q \tilde{f} \leq \exp(2C 3^{2d} \mu) \operatorname{essinf}_Q \tilde{f}. \tag{38}$$

Now let $B = B(z_0, h)$ be such that $2b^4 \subseteq \Omega$. We may apply (38) for the rectangle bQ. By (5) we thus have

$$\begin{aligned}
\operatorname{esssup}_B \tilde{f} &\leq \operatorname{esssup}_{bQ} \tilde{f} \\
&\leq \exp[2C3^{2d}\mu(bQ)] \operatorname{essinf}_{bQ} \tilde{f} \\
&\leq \exp[2C3^{2d}\mu(bQ)] \operatorname{essinf}_B \tilde{f}.
\end{aligned}$$

This proves (8) with $K = 2C3^{2d}$ but with $\mu = \mu(bQ)$ instead of $\mu = u(B)^{\frac{1}{2}} v(B)^{-\frac{1}{2}}$. ∎
Since u and v are doubling, those two quantities are comparable ∎

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