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Quaternionic Fundamental Solutions for Electromagnetic Scattering Problems and Application

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Abstract. We propose a new class of fundamental solutions for the numerical analysis of boundary value problems for the Maxwell equations. We prove completeness of systems of such fundamental solutions in appropriate Sobolev spaces on a smooth boundary and support the relevancy of our approach by numerical results.

Keywords: *Quaternionic analysis, fundamental solutions, Maxwell equations, Beltrami fields, force-free fields*

AMS subject classification: 30G35, 78M25

1. Introduction

The method of fundamental solutions or, equivalently, the method of discrete sources (we will keep to the first name) is a widely used technique for the numerical solution of elliptic boundary value problems which falls in the class of so-called boundary methods reducing problems in n-dimensional domains to some equations on their $(n-1)$-dimensional boundaries. It is applicable when a fundamental solution of the differential equation of the problem is known and the completeness of an infinite system of such fundamental solutions with singularities (sources) placed outside the domain of the problem is proved. The original idea of the method emerged in the sixties [4, 20, 21] and since then it was successfully used in geophysics, acoustics, elasticity theory, electromagnetism and other fields. We refer the reader to the books [1, 8] and to the review [10] for bibliography and more information about the method.

The aim of the present paper is to provide the theoretical foundation for the application of the method of fundamental solutions to boundary value

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problems of electromagnetic scattering theory, in particular, to introduce a system of fundamental solutions for Maxwell's equations and to prove its completeness in appropriate functional spaces. Here an explanation is needed because all this seems to represent nothing new. Let us start with the concept of a fundamental solution for a system of partial differential equations with n unknowns and n equations of the form $Au = 0$, where A is a differential operator. Usually (see, e.g., [28: p. 179]) it is defined as a $n \times n$ matrix, denoted by Φ, such that

$$A\Phi = \delta E_n \tag{1}$$

where δ is the Dirac delta function and E_n is the $n \times n$ identity matrix. Nevertheless such a definition has no clear physical interpretation as a field generated by a point source, the usual meaning of the fundamental solution. In this sense the electromagnetic fields produced by an electric or a magnetic dipole are closer to the physical meaning of a fundamental solution and sometimes they are called the fundamental solutions of the Maxwell system [7: Section 4.2], but then it is not clear how they can be used for the analytical solution of homogeneous and inhomogeneous Maxwell's equations, usually based on property (1).

We propose another possibility, a fundamental solution which enjoys both properties. It satisfies (1) in a sense explained below and has a clear meaning of a field generated by a point source. Moreover, we prove the completeness of an infinite system of such fundamental solutions in appropriate Sobolev spaces which makes it possible to apply our system to the numerical solution of boundary value problems for Maxwell's equations in chiral media.

The construction of the system of fundamental solutions for the Maxwell equations proposed here is based on some elements of quaternionic analysis which seems to be the most appropriate formalism for this task. The solutions obtained are complex quaternions, that is instead to be a pair of three-component vectors, they have four components. Due to their simple form and lower singularity compared with solutions based on a matrix approach (our solutions increase as $O(\frac{1}{\rho^2(x)})$ near the boundary, where $\rho(x)$ is the distance from the point x to the boundary, against $O(\frac{1}{\rho^3(x)})$ in the case of solutions proposed in [1, 8, 20]), and hence the quaternionic fundamental solutions are in apt agreement with the natural integral representations for the electromagnetic field which are the Stratton-Chu formulas) the numerical application of them is easier and more natural.

The main idea for obtaining the quaternionic fundamental solutions for Maxwell's equations consists in the quaternionic diagonalization of Maxwell's equations proposed in [16] (see also [17, 19]). The Maxwell equations for an isotropic homogeneous medium are reduced to a pair of quaternionic equations in which the unknown functions are separated. For each of these equations a fundamental solution is easily constructed and then linear combinations of them will give the required system of fundamental solutions for Maxwell's equations. The main difficulty constitutes the proof of completeness of this system. We base our proof on the completeness of a system of fundamental solutions of the Helmholtz operator in the kernel of this operator in L_2-norm and make use of a quaternionic decomposition of the kernel of the Helmholtz operator.

Our results are applied to Maxwell's equations for chiral media but they are completely new for a non-chiral case as well.

2. Complex quaternions

We shall denote by $\mathbb{H}(\mathbb{C})$ the set of complex quaternions (= biquaternions). Each element a of $\mathbb{H}(\mathbb{C})$ is represented in the form $a = \sum_{k=0}^{3} a_k i_k$ where $\{a_k\} \subset \mathbb{C}$, i_0 is the unit and $\{i_k : k = 1, 2, 3\}$ are the quaternionic imaginary units, that is the standard basis elements possessing the properties

$$
\begin{aligned}
&i_0^2 = i_0 = -i_k^2, \quad i_0 i_k = i_k i_0 = i_k \qquad (k = 1, 2, 3)\\
&i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2.
\end{aligned}
$$

We denote the imaginary unit in $\mathbb{C}$ by i as usual. By definition, i commutes with i_k $(k = 0, 1, 2, 3)$.

The basic quaternionic imaginary units i_1, i_2, i_3 can be identified with the basic coordinate vectors in a three-dimensional space. In this way a vector $\vec{a}$ from $\mathbb{C}^3$ is identified with the complex quaternion $a_1 i_1 + a_2 i_2 + a_3 i_3$. We will use the so-called vector representation of complex quaternions, i.e. each $a \in \mathbb{H}(\mathbb{C})$ is represented as $a = a_0 + \vec{a}$, where a_0 is the scalar part of a sometimes denoted as $\mathrm{Sc}(a) = a_0$ and $\vec{a}$ is the vector part of a: $\mathrm{Vec}(a) = \vec{a} = \sum_{k=1}^{3} a_k i_k$. Complex quaternions of the form $a = \vec{a}$ will be called purely vectorial.

In vector terms, the multiplication of two arbitrary complex quaternions a and b can be rewritten as follows:

$$
a \cdot b = a_0 b_0 - \langle \vec{a}, \vec{b} \rangle + [\vec{a} \times \vec{b}] + a_0 \vec{b} + b_0 \vec{a}
$$

where

$$
\langle \vec{a}, \vec{b} \rangle = \sum_{k=1}^{3} a_k b_k \in \mathbb{C} \qquad \text{and} \qquad [\vec{a} \times \vec{b}] = \begin{vmatrix} i_1 & i_2 & i_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \in \mathbb{C}^3.
$$

The complex quaternion $\overline{a} = a_0 - \vec{a}$ is called the conjugate of a. Let us denote by $\mathfrak{S}$ the set of zero divisors from $\mathbb{H}(\mathbb{C})$. We recall (see, e.g., [19]) that $a \in \mathfrak{S}$ if and only if $a \cdot \overline{a} = 0$. If $a \notin \mathfrak{S} \cup \{0\}$, then a^{-1} exists and $a^{-1} = \frac{\overline{a}}{a \cdot \overline{a}}$.

3. Quaternionic differential operators

We shall consider $\mathbb{H}(\mathbb{C})$-valued functions depending on three variables x_1, x_2, x_3. On the set of such componentwise continuously differentiable functions the operator $D = \sum_{k=1}^{3} i_k \partial_k$ is defined where $\partial_k = \frac{\partial}{\partial x_k}$. The expression Df, where f is an $\mathbb{H}(\mathbb{C})$-valued function, can be rewritten in a vector form as

$$Df = -\mathrm{div}\vec{f} + \mathrm{grad}f_0 + \mathrm{rot}\vec{f}.$$

That is, $\mathrm{Sc}(Df) = -\mathrm{div}\vec{f}$ and $\mathrm{Vec}(Df) = \mathrm{grad}f_0 + \mathrm{rot}\vec{f}$. The condition $f \in \ker D$ is equivalent to the Moisil-Theodoresco system

$$\left.\begin{aligned} \mathrm{div}\vec{f} = 0 \\ \mathrm{grad}f_0 + \mathrm{rot}\vec{f} = 0 \end{aligned}\right\}$$

which has been studied in hundreds of works (see, e.g., [3, 9]).

Denote $D_\alpha = D + \alpha I$, where α is a complex constant and I is the identity operator. As we will see in the subsequent pages, α has the meaning of a wave number. Having this in mind we will assume that $\alpha \neq 0$ and $\mathrm{Im}\alpha \geq 0$.

We have the factorization of the Helmholtz operator

$$\Delta + \alpha^2 = -D_\alpha D_{-\alpha} = -D_{-\alpha} D_\alpha \tag{2}$$

which in particular means that any function satisfying the equation

$$D_\alpha f = 0 \tag{3}$$

or

$$D_{-\alpha} f = 0 \tag{4}$$

also satisfies the Helmholtz equation $(\Delta + \alpha^2) f = 0$.

Remark 1. Purely vectorial solutions of equations (3) or (4) are known as Beltrami fields or linear force-free fields and are intensively studied in different branches of modern physics (see, e.g., [2, 5, 6, 11, 12, 22, 24, 29, 32, 33]).

We will use the fundamental solution of the Helmholtz operator

$$\theta_\alpha(x) = -\frac{e^{i\alpha|x|}}{4\pi|x|}$$

which fulfills the Sommerfeld radiation condition at infinity. Fundamental solutions $\mathcal{K}_\alpha$ and $\mathcal{K}_{-\alpha}$ for the operators D_α and $D_{-\alpha}$, respectively, can be obtained easily using (2). We have that the functions $\mathcal{K}_\alpha = -(D - \alpha)\theta_\alpha$ and

$\mathcal{K}_{-\alpha} = -(D+\alpha)\theta_\alpha$ satisfy the equations $D_{\pm\alpha}\mathcal{K}_{\pm\alpha} = \delta$. More explicitly, we have

$$\mathcal{K}_{\pm\alpha}(x) = \big(\pm\alpha + \tfrac{x}{|x|^2} - i\alpha\tfrac{x}{|x|}\big)\theta_\alpha(x)$$

where $x = \sum_{k=1}^{3} x_k i_k$. Note that $\mathcal{K}_{\pm\alpha}$ are complex quaternionic functions with $\mathrm{Sc}(\mathcal{K}_{\pm\alpha}(x)) = \pm\alpha\theta_\alpha(x)$ and $\mathrm{Vec}(\mathcal{K}_{\pm\alpha}(x)) = -\mathrm{grad}\theta_\alpha(x) = (\frac{x}{|x|^2} - i\alpha\frac{x}{|x|})\theta_\alpha(x)$.

Let us introduce the operators

$$\Pi_{\pm\alpha} = \mp\tfrac{1}{2\alpha}D_{\mp\alpha}$$

considering them on $\mathbb{H}(\mathbb{C})$-valued functions from $\ker(\Delta+\alpha^2)$. Then we have the following statement (see the proof in [19: p. 36]).

Proposition 2. *The following relations hold:*

1. $\Pi_{\pm\alpha}^2 = \Pi_{\pm\alpha}$

2. $\Pi_\alpha\Pi_{-\alpha} = \Pi_{-\alpha}\Pi_\alpha = 0$

3. $\Pi_\alpha + \Pi_{-\alpha} = I$

4. *As* $\mathrm{im}\Pi_{\pm\alpha} = \ker D_{\pm\alpha}$ *we have* $\ker(\Delta+\alpha^2) = \ker D_\alpha \oplus \ker D_{-\alpha}$.

4. Relation between quaternionic differential operators and Maxwell's system for chiral media

As we will see, our approach works not only for homogeneous, isotropic, achiral media but also for chiral media. This last case is more general. When the chirality measure of a medium β is equal to zero, we obtain the non-chiral or achiral situation. This is why we show our results for the case of a chiral medium – transition to a non-chiral case is quite easy.

For the sake of simplicity we consider a sourceless situation. Then Maxwell's equations for time-harmonic electromagnetic fields in a chiral medium have the form (see, e.g., [23, 25])

$$\left.\begin{aligned} \mathrm{div}\widetilde{E}(x) &= \mathrm{div}\widetilde{H}(x) = 0 \\ \mathrm{rot}\widetilde{E}(x) &= i\omega\widetilde{B}(x) \\ \mathrm{rot}\widetilde{H}(x) &= -i\omega\widetilde{D}(x) \end{aligned}\right\} \tag{5}$$

with the Drude-Born-Fedorov constitutive relations [23]

$$\widetilde{D} = \varepsilon\big(\widetilde{E}(x) + \beta\mathrm{rot}\widetilde{E}(x)\big)$$
$$\widetilde{B} = \mu\big(\widetilde{H}(x) + \beta\mathrm{rot}\widetilde{H}(x)\big)$$

where ω is the frequency, ε and μ are complex permittivity and permeability of a medium and β is its chirality measure. The Maxwell equations (5) can be also written as

$$\left.\begin{aligned} \operatorname{rot}\widetilde{E}(x) &= i\omega\mu\big(\widetilde{H}(x) + \beta\operatorname{rot}\widetilde{H}(x)\big) \\ \operatorname{rot}\widetilde{H}(x) &= -i\omega\varepsilon\big(\widetilde{E}(x) + \beta\operatorname{rot}\widetilde{E}(x)\big). \end{aligned}\right\}$$

Introducing the notations

$$\begin{aligned} \widetilde{E}(x) &= -\sqrt{\mu}\cdot\vec{E}(x) \\ \widetilde{H}(x) &= \sqrt{\varepsilon}\cdot\vec{H}(x) \end{aligned}$$

we obtain the equations

$$\left.\begin{aligned} \operatorname{rot}\vec{E}(x) &= -i\alpha\big(\vec{H}(x) + \beta\operatorname{rot}\vec{H}(x)\big) \\ \operatorname{rot}\vec{H}(x) &= i\alpha\big(\vec{E}(x) + \beta\operatorname{rot}\vec{E}(x)\big) \end{aligned}\right\} \tag{6}$$

where as before $\alpha = \omega\sqrt{\varepsilon\mu}$ and, in the case of $\beta = 0$, α is the wave number. When $\beta \neq 0$, as it will be seen below, α does not have the same physical meaning. There appear two wave numbers α_1 and α_2 instead.

Now, following [15] we obtain a more convenient quaternionic form of system (6). For this let us consider the purely vectorial biquaternionic functions

$$\begin{aligned} \vec{\varphi}(x) &= \vec{E}(x) + i\vec{H}(x) \\ \vec{\psi}(x) &= \vec{E}(x) - i\vec{H}(x). \end{aligned}$$

We have

$$D\vec{\varphi}(x) = \operatorname{rot}\vec{E}(x) + i\operatorname{rot}\vec{H}(x).$$

Using (6) we obtain

$$D\vec{\varphi}(x) = -\big(i\alpha\vec{H}(x) + \alpha\vec{E}(x)\big) - \alpha\beta\big(D\vec{E}(x) + iD\vec{H}(x)\big).$$

That is,

$$D\vec{\varphi}(x) = -\alpha\vec{\varphi}(x) - \alpha\beta D\vec{\varphi}(x).$$

Thus the complex quaternionic function $\vec{\varphi}$ satisfies the equation

$$\big(D + \tfrac{\alpha}{1+\alpha\beta}\big)\vec{\varphi}(x) = 0. \tag{7}$$

By analogy we obtain the equation for $\vec{\psi}$

$$\big(D - \tfrac{\alpha}{1-\alpha\beta}\big)\vec{\psi}(x) = 0. \tag{8}$$

Introducing the notations $\alpha_1 = \frac{\alpha}{1+\alpha\beta}$ and $\alpha_2 = \frac{\alpha}{1-\alpha\beta}$ we rewrite equations (7) - (8) in the form

$$(D + \alpha_1)\vec{\varphi}(x) = 0$$
$$(D - \alpha_2)\vec{\psi}(x) = 0.$$

When $\beta = 0$, we arrive at the quaternionic form of the Maxwell equations in the non-chiral case, but in general the wave numbers α_1 and α_2 are different and physically characterize the propagation of waves of opposing circular polarizations.

5. Quaternionic integral operators

Let Γ be a closed Liapunov surface in $\mathbb{R}^3$. The corresponding interior and exterior domains we denote by Ω^+ and Ω^-, respectively. Let $\vec{n}$ be the outward unitary normal on Γ with respect to Ω^+ in quaternionic form: $\vec{n} = \sum_{k=1}^{3} n_k i_k$. Denote

$$(K_{\pm\alpha} f)(x) = -\int_\Gamma \mathcal{K}_{\pm\alpha}(x-y)\vec{n}(y)f(y)\,d\Gamma_y \qquad (x \in \mathbb{R}^3 \setminus \Gamma)$$

where f is an $\mathbb{H}(\mathbb{C})$-valued function and all the products under the integral are quaternionic. The following important result is well known (see, e.g., [19: p. 70]).

Theorem 3. *Let $f \in C^1(\Omega^+) \cap C(\overline{\Omega^+})$ and $f \in \ker D_{\pm\alpha}(\Omega^+)$. Then $f(x) = (K_{\pm\alpha} f)(x)$ for all $x \in \Omega^+$.*

Remark 4. In this paper the belonging of a complex quaternionic function f to some functional space means that each of its components f_k belongs to that space.

For the consideration of equations (3) - (4) in the domain Ω^- one needs appropriate radiation conditions at infinity. Such conditions were introduced in [26] (see also [18]). Solutions of equation (3) are required to satisfy the equality

$$\Big(1 + \frac{ix}{|x|}\Big) \cdot f(x) = o\Big(\frac{1}{|x|}\Big) \qquad (|x| \to \infty) \tag{9}$$

uniformly in all directions. For solutions of equation (4) in Ω^- the corresponding radiation condition has the form

$$\Big(1 - \frac{ix}{|x|}\Big) \cdot f(x) = o\Big(\frac{1}{|x|}\Big) \qquad (|x| \to \infty). \tag{10}$$

Then we have the following result [18]:

Theorem 5. *Let $f \in C^1(\Omega^-) \cap C(\overline{\Omega^-})$, $f \in \ker D_\alpha(\Omega^-)$ and satisfying condition* (9) *or $f \in \ker D_{-\alpha}(\Omega^-)$ and satisfying condition* (10)*. Then*

$$f(x) = \begin{cases} -K_\alpha f(x) \\ or \\ -K_{-\alpha} f(x) \end{cases}$$

respectively for any $x \in \Omega^-$.

As we will consider $\mathbb{H}(\mathbb{C})$-valued functions satisfying the Helmholtz equation

$$(\Delta + \alpha^2)u = 0 \tag{11}$$

in unbounded domains it will be convenient to obtain a radiation condition at infinity in a quaternionic form for such functions. If a solution u_0 of the Helmholtz equation is a scalar function, then the corresponding radiation condition is the well known Sommerfeld condition

$$i\alpha u_0(x) - \left\langle \tfrac{x}{|x|}, \mathrm{grad} u_0(x) \right\rangle = o\left(\tfrac{1}{|x|}\right) \qquad \text{when } |x| \to \infty. \tag{12}$$

For a vector solution $\vec{u}$ of the Helmholtz equation the corresponding radiation condition has the form [7: Section 4.2]

$$\left[\mathrm{rot}\vec{u} \times \tfrac{x}{|x|}\right] + \tfrac{x}{|x|}\mathrm{div}\vec{u} - i\alpha\vec{u} = o\left(\tfrac{1}{|x|}\right) \qquad \text{when } |x| \to \infty. \tag{13}$$

Let us notice that a vector solution $\vec{u}$ of the Helmholtz equation fulfills this condition if and only if each Cartesian component of $\vec{u}$ fulfills the Sommerfeld radiation condition [7: Section 4.2]. Thus, our quaternionic radiation condition must include both conditions (12) for u_0 and (13) for $\vec{u}$.

It is easy to obtain such a condition using Proposition 2 and radiation conditions (9) - (10). From Proposition 2, an $\mathbb{H}(\mathbb{C})$-valued solution $u = u_0 + \vec{u}$ of equation (11) has the form $u = \Pi_\alpha u + \Pi_{-\alpha} u$ where $\Pi_\alpha u$ fulfills (9) and $\Pi_{-\alpha} u$ fulfills (10). Thus we obtain

$$u(x) = -i\tfrac{x}{|x|} \cdot \Pi_\alpha u(x) + i\tfrac{x}{|x|} \cdot \Pi_{-\alpha} u(x) + o\left(\tfrac{1}{|x|}\right) \qquad (|x| \to \infty).$$

From the definition of $\Pi_{\pm\alpha}$ we have

$$2\alpha u(x) = i\tfrac{x}{|x|} \cdot (D-\alpha)u(x) + i\tfrac{x}{|x|} \cdot (D+\alpha)u(x) + o\left(\tfrac{1}{|x|}\right) \qquad (|x| \to \infty).$$

Finally we arrive at the following radiation condition at infinity for complex quaternionic solutions of the Helmholtz equation (11):

$$i\alpha u(x) + \tfrac{x}{|x|} \cdot Du(x) = o\left(\tfrac{1}{|x|}\right) \qquad \text{when } |x| \to \infty. \tag{14}$$

As is easy to see, when $u = u_0$, then the scalar part of this equality gives us exactly the Sommerfeld condition (12) and the vector part

$$\left[\tfrac{x}{|x|} \times \operatorname{grad} u_0(x)\right] = o\left(\tfrac{1}{|x|}\right) \tag{15}$$

is a redundant equality because it is a simple consequence of the fact that a scalar solution of the Helmholtz equation satisfying the Sommerfeld condition at infinity can be represented as a single layer potential which satisfies (15).

When $u = \vec{u}$, the vector part of (14) gives us (13) and the scalar part

$$\left\langle \tfrac{x}{|x|}, \operatorname{rot}\vec{u}(x)\right\rangle = o\left(\tfrac{1}{|x|}\right)$$

is again a simple consequence from the integral representation of $\vec{u}$ (see [7: Section 4.2] or [19: p. 120]). Thus (14) in special cases reduces to (12) and (13) and in general represents the radiation condition at infinity for the quaternionic Helmholtz equation. Note that (9) - (10) follow from (14) immediately if one assumes that $u \in \ker D_\alpha$ or $u \in \ker D_{-\alpha}$, respectively.

We will need the operators

$$\begin{aligned} (S_\alpha f)(x) &= -2\int_\Gamma \mathcal{K}_\alpha(x-y)\vec{n}(y)f(y)\,d\Gamma_y \quad (x \in \Gamma) \\ P_\alpha &= \tfrac{1}{2}(I + S_\alpha) \quad \text{and} \quad Q_\alpha = \tfrac{1}{2}(I - S_\alpha) \end{aligned}$$

defined for example on Hölder functions in the sense of the Cauchy principal value. It is well known that S_α is a singular integral operator of Calderon-Zygmund type (see [13: Section 2.5]). This implies the boundedness of the operators $P_\alpha, Q_\alpha, S_\alpha$ in Sobolev spaces $H^s(\Gamma)$ for all real s.

The following important properties of the operators $P_\alpha, Q_\alpha, S_\alpha$ will be widely used in this work.

Theorem 6. *Let $f \in L_2(\Gamma)$. Then for almost every point $\tau \in \Gamma$ the non-tangential limits*

$$\lim_{\Omega^\pm \ni x \to \tau \in \Gamma} K_\alpha[f](x) = K_\alpha[f]^\pm(\tau) \tag{16}$$

exist and the formulas

$$\begin{aligned} K_\alpha[f]^+(\tau) &= P_\alpha[f](\tau) = \tfrac{1}{2}(I + S_\alpha)f(\tau) \\ K_\alpha[f]^-(\tau) &= -Q_\alpha[f](\tau) = -\tfrac{1}{2}(I - S_\alpha)f(\tau) \end{aligned} \tag{17}$$

hold.

Let now Γ be a sufficiently smooth surface in order that the Sobolev space $H^s(\Gamma)$ for a given s be defined.

Remark 7. Since $H^s(\Gamma) \subset L_2(\Gamma)$ for $s \geq 0$ and S_α is bounded in $H^s(\Gamma)$, equalities (16) - (17) hold for $f \in H^s(\Gamma)$ $(s \geq 0)$.

Corollary 8.

1. *The equalities* $S_\alpha^2 = I, P_\alpha^2 = P_\alpha, Q_\alpha^2 = Q_\alpha, P_\alpha Q_\alpha = Q_\alpha P_\alpha = 0$ *hold on* $H^s(\Gamma)$ $(s \geq 0)$.

2. *In order for* $f \in H^s(\Gamma)$ $(s \geq 0)$ *to be a boundary value of a function* $F \in \ker D_\alpha(\Omega^+)$, *the condition* $f \in \operatorname{im} P_\alpha(H^s(\Gamma))$ *is necessary and sufficient.*

3. *In order for* $f \in H^s(\Gamma)$ $(s \geq 0)$ *to be a boundary value of a function* $F \in \ker D_\alpha(\Omega^-)$ *satisfying condition* (9) *at infinity, the condition* $f \in \operatorname{im} Q_\alpha(H^s(\Gamma))$ *is necessary and sufficient.*

The proof of these facts in $L_2(\Gamma)$ can be found in [19: Chapter 5]. By Theorem 6 and Remark 7 these assertions also hold in the space $H^s(\Gamma)$ $(s \geq 0)$. Needless to say that the same facts are valid for $D_{-\alpha}$.

6. Complete systems of fundamental solutions of the Helmholtz operator

Let Γ be a closed surface in $\mathbb{R}^3$ which is a boundary of a bounded domain Ω^+ and of an unbounded domain $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. By Γ^- we denote a closed surface enclosed in Ω^+ and enclosing the domain V and by Γ^+ a closed surface enclosing $\overline{\Omega^+}$ as shown in Figure 1.

By $\{y_n^-\}_{n=1}^\infty$ we denote a set of points distributed on Γ^- and dense on Γ^-, and by $\{y_n^+\}_{n=1}^\infty$ a set of points distributed on Γ^+ and dense on Γ^+. To each of these sets a system of fundamental solutions $\{\theta_\alpha(x - y_n^-)\}_{n=1}^\infty$ or $\{\theta_\alpha(x - y_n^+)\}_{n=1}^\infty$ is related. Denote $\theta_{\alpha,n}^-(x) = \theta_\alpha(x - y_n^-)$ and $\theta_{\alpha,n}^+(x) = \theta_\alpha(x - y_n^+)$. Singularities of functions of the first system are distributed on the interior surface Γ^- and consequently every such function is a solution of the Helmholtz equation in Ω^- satisfying the Sommerfeld radiation condition at infinity. Functions from the second system have their singularities on the exterior surface Γ^+ and solve the Helmholtz equation in Ω^+.

We start with the following theorem due to V. Kupradze [20]. Its proof can be found, for example, in [8: p. 51].

Theorem 9. *Let Γ be a closed surface of class C^2. Then the system of functions $\{\theta_\alpha(x - y_n^+)\}_{n=1}^\infty$ is complete in $L_2(\Gamma)$. Assume additionally that α^2 is not an eigenvalue of the Dirichlet problem in V. Then the system of functions $\{\theta_\alpha(x - y_n^-)\}_{n=1}^\infty$ is complete in $L_2(\Gamma)$ also.*

Our aim is to obtain a similar result for the Sobolev spaces $H^s(\Gamma)$. This will require a sequence of steps. We will show first that these systems of functions are complete in $L_2(\Omega) \cap \ker(\Delta + \alpha^2)$. Then this result will be extended to $H^s(\Omega) \cap \ker(\Delta + \alpha^2)$. Finally, as $H^s(\Gamma)$ can be considered as a space of traces of corresponding solutions of the Helmholtz equation, we will be able to prove the completeness of our systems of fundamental solutions for the Helmholtz operator in this space.

6.1 Interior domain. First, let us consider the case of a bounded domain Ω^+ and then of an unbounded domain Ω^-.

Theorem 10. *Let Ω^+ be a bounded domain in $\mathbb{R}^3$ with a Liapunov boundary Γ. The system of functions $\{\theta_{\alpha,n}^+\}_{n=1}^\infty$ is complete in $L_2(\Omega^+) \cap \ker(\Delta + \alpha^2)$.*

Proof. We consider $\Delta + \alpha^2$ as an unbounded operator in $L_2(\Omega^+)$ with domain $H^2(\Omega^+)$. This operator is closed and the set $L_2(\Omega^+) \cap \ker(\Delta + \alpha^2)$ is a subspace. Thus it is sufficient to prove that the system $\{\theta_{\alpha,n}^+\}_{n=1}^\infty$ is closed in $L_2(\Omega^+) \cap \ker(\Delta + \alpha^2)$. Assume that there exists a non-trivial function $f \in L_2(\Omega^+) \cap \ker(\Delta + \alpha^2)$ with the property $\langle \theta_{\alpha,n}^+, f \rangle_{L_2(\Omega^+)} = 0$ for all $n \in \mathbb{N}$ or, in explicit form,

$$\int_{\Omega^+} \theta_\alpha(x - y_n^+) f^*(x)\, dx = 0 \qquad (n \in \mathbb{N})$$

where * stands for the usual complex conjugation. Denote

$$V_{\Omega^+} f(y) = \int_{\Omega^+} \theta_\alpha(x - y) f(x)\, dx.$$

We have $V_{\Omega^+} f^*(y_n^+) = 0$ for all $n \in \mathbb{N}$. These equalities and the continuity of $V_{\Omega^+} f^*$ imply the equality $V_{\Omega^+} f^* = 0$ on Γ^+.

The function $V_{\Omega^+} f^*$ satisfies the Helmholtz equation in Ω^- and fulfills the Sommerfeld radiation condition at infinity. Consequently, $V_{\Omega^+} f^* \equiv 0$ in Ω^-. Moreover, all the derivatives of $V_{\Omega^+} f^*$ in Ω^- are equal to zero. Thus we obtain that the function $V_{\Omega^+} f^*$ and all its derivatives are equal to zero on Γ. Taking this into account and using the fact that $V_{\Omega^+} f^* \in \ker(\Delta + \alpha^{*2})(\Delta + \alpha^2)$ in Ω^+ due to the uniqueness of continuation for the null solutions of this elliptic operator [27: Theorem 6.14] we obtain $V_{\Omega^+} f^* \equiv 0$ in Ω^+ and hence $f \equiv 0$ in Ω^+ ∎

Theorem 11. *Under the conditions of Theorem* 10 *the system of functions* $\{\theta^+_{\alpha,n}\}_{n=1}^{\infty}$ *is complete in* $H^s(\Omega^+) \cap \ker(\Delta + \alpha^2)$ $(s \geq 0)$.

Proof. Here we use the quite general fact proved by N. Tarkhanov (for a general elliptic system, see [30: Section 8.1]) that a function from $H^s(\Omega^+) \cap \ker(\Delta + \alpha^2)$ belongs to the closure of the subspace $\operatorname{sol}(\Omega^+)$ in $H^s(\Omega^+)$ consisting of all C^∞ solutions of the Helmholtz equation in a neighborhood of $\overline{\Omega^+}$. That is, for any function $f \in H^s(\Omega^+) \cap \ker(\Delta + \alpha^2)$ and for any $\varepsilon > 0$ we can find such a function $f_0 \in \ker(\Delta + \alpha^2)$ in $\widetilde{\Omega}^+$, where $\overline{\Omega^+} \subset \widetilde{\Omega}^+$, that $\|f - f_0\|_{H^s(\Omega^+)} < \frac{\varepsilon}{2}$. The domain $\widetilde{\Omega}^+$ can be chosen enclosed by Γ^+.

For all solutions u of the Helmholtz equation in $\widetilde{\Omega}^+$ we have the estimate (see, e.g., [31: Theorem 11.1]) $\|u\|_{H^s(\Omega^+)} \leq C\|u\|_{L_2(\widetilde{\Omega}^+)}$ where the constant C does not depend on u. Due to Theorem 10, for any $\varepsilon_1 > 0$ the function f_0 can be approximated by a linear combination $f_N = \sum_{n=1}^N a_n \theta^+_{\alpha,n}$ in $\widetilde{\Omega}^+$ in such a way that $\|f_0 - f_N\|_{L_2(\widetilde{\Omega}^+)} < \varepsilon_1$. Choose $\varepsilon_1 = \frac{\varepsilon}{2C}$ and consider

$$\begin{aligned}
\|f - f_N\|_{H^s(\Omega^+)} &= \|f - f_0 + f_0 - f_N\|_{H^s(\Omega^+)} \\
&\leq \|f - f_0\|_{H^s(\Omega^+)} + \|f_0 - f_N\|_{H^s(\Omega^+)} \\
&< \tfrac{\varepsilon}{2} + C\|f_0 - f_N\|_{L_2(\widetilde{\Omega}^+)} \\
&< \varepsilon.
\end{aligned}$$

Thus the statement is proved ∎

Theorem 12. *Let* Γ *be a sufficiently smooth* (*the space* $H^s(\Gamma)$ *is defined*) *closed surface. The system of functions* $\{\theta^+_{\alpha,n}\}_{n=1}^{\infty}$ *is complete in* $H^s(\Gamma)$ $(s \in \mathbb{R})$.

Proof. For $s \leq 0$ the result follows from Theorem 10. Let $s > 0$. Given $\varepsilon > 0$, for any $u \in H^s(\Gamma)$ there exists (probably not unique) a solution of the Dirichlet problem $(\Delta + \alpha^2)U = 0$ in Ω^+, $U|_\Gamma = u$ in $H^{s+\frac{1}{2}}(\Omega^+)$. Due to Theorem 11, for any $\varepsilon_1 > 0$ we can approximate it by a linear combination $U_N = \sum_{n=1}^N a_n \theta^+_{\alpha,n}$ in such a way that $\|U - U_N\|_{H^{s+\frac{1}{2}}(\Omega^+)} < \varepsilon_1$. Denoting $u_N = \gamma U_N$ and using the continuity of the trace operator γ we obtain

$$\|u - u_N\|_{H^s(\Gamma)} = \|\gamma(U - U_N)\|_{H^s(\Gamma)} \leq C\|U - U_N\|_{H^{s+\frac{1}{2}}(\Omega^+)} < C\varepsilon_1.$$

Choosing $\varepsilon_1 = \frac{\varepsilon}{C}$ we finish the proof ∎

Remark 13. Theorem 12 was proved for scalar functions from $H^s(\Gamma)$. Nevertheless, it is obviously valid also for $\mathbb{H}(\mathbb{C})$-valued functions from $H^s(\Gamma)$ which in this case are approximated by linear combinations $\sum_{n=1}^{N} c_n^+ \theta_{\alpha,n}^+$ where c_n^+ are complex quaternions.

6.2 Exterior domain. Let B_R be an arbitrary ball with a sufficiently large radius R such that $\Omega^+ \subset B_R$. Denote $\Omega_R^- = \Omega^- \cap B_R$. Thus Ω_R^- is a domain in $\mathbb{R}^3$ with a boundary consisting of Γ and of the sphere ∂B_R.

Theorem 14. *Let Γ be a closed Liapunov surface and α^2 be not an eigenvalue of the Dirichlet problem in V. The system of functions $\{\theta_{\alpha,n}^-\}_{n=1}^{\infty}$ is complete in $L_2(\Omega_R^-) \cap \ker(\Delta + \alpha^2)$.*

Proof. Assume that there exists a non-trivial function $f \in L_2(\Omega_R^-) \cap \ker(\Delta + \alpha^2)$ with the property $\langle \theta_{\alpha,n}^-, f \rangle_{L_2(\Omega_R^-)} = 0$ for all $n \in \mathbb{N}$ or, in explicit form,

$$\int_{\Omega_R^-} \theta_\alpha(x - y_n^-) f^*(x)\, dx = 0 \qquad (n \in \mathbb{N}).$$

Denote

$$V_{\Omega_R^-} f(y) = \int_{\Omega_R^-} \theta_\alpha(x - y) f(x)\, dx.$$

We have $V_{\Omega_R^-} f^*(y_n^-) = 0$ for all $n \in \mathbb{N}$ and hence $V_{\Omega_R^-} f^* = 0$ on Γ^-.

The function $V_{\Omega_R^-} f^*$ satisfies the Helmholtz equation in Ω^+. It is equal to zero in V due to the uniqueness of a solution of the Dirichlet problem in V, and it is zero with all its derivatives in $\overline{\Omega^+}$ due to the uniqueness of continuation for the solutions of the Helmholtz equation. Moreover, in $\overline{\Omega^+} \cup \Omega_R^-$ the function $V_{\Omega_R^-} f^*$ belongs to $\ker(\Delta + \alpha^{*2})(\Delta + \alpha^2)$. Thus due to the uniqueness of continuation for the solutions of this elliptic operator we obtain $V_{\Omega_R^-} f^* \equiv 0$ in Ω_R^- and hence $f \equiv 0$ in Ω_R^- ∎

Theorem 15. *Under the conditions of Theorem* 14 *the system of functions $\{\theta_{\alpha,n}^-\}_{n=1}^{\infty}$ is complete in $H^s(\Omega_R^-) \cap \ker(\Delta + \alpha^2)$ $(s \geq 0)$.*

Proof. First, for a given $\varepsilon > 0$ we choose such a function $f_0 \in \ker(\Delta + \alpha^2)$ in $\widetilde{\Omega}_R^-$ that $\|f - f_0\|_{H^s(\Omega_R^-)} < \frac{\varepsilon}{2}$, where $\widetilde{\Omega}_R^-$ is a domain containing $\overline{\Omega_R^-}$ and such that $\Gamma^- \cap \widetilde{\Omega}_R^- = \emptyset$. For all solutions u of the Helmholtz equation in $\widetilde{\Omega}_R^-$ we have the estimate $\|u\|_{H^s(\Omega_R^-)} \leq C \|u\|_{L_2(\widetilde{\Omega}_R^-)}$ where the constant C does not depend on u. The proof finishes by analogy with that of Theorem 11 ∎

Theorem 16. *Let Γ be a sufficiently smooth closed surface and α^2 be not an eigenvalue of the Dirichlet problem in V. The system of functions $\{\theta^-_{\alpha,n}\}_{n=1}^{\infty}$ is complete in $H^s(\Gamma)$ $(s \in \mathbb{R})$.*

The proof is completely analogous to that of Theorem 12.

7. Extensions into exterior domains

Let us consider the exterior Dirichlet problem for the Helmholtz equation

$$\left.\begin{array}{c}(\Delta+\alpha^2)U = 0 \ \text{ in } \Omega^- \\ U|_\Gamma = u\end{array}\right\} \tag{18}$$

and let U satisfy the Sommerfeld radiation condition (12) at infinity. For $u \in H^s(\Gamma)$ $(s > 0)$ it is known that the solution of this problem exists, is unique and belongs to a weighted Sobolev space in Ω^- (see [28: Section 2.6]). For our purposes the important fact will be that the solution belongs to $H^{s+\frac{1}{2}}(\Omega^-_R)$ where Ω^-_R is an intersection of Ω^- with a ball B_R of radius R chosen large enough to enclose the interior domain Ω^+. We denote by $H^{s+\frac{1}{2}}_{loc}(\Omega^-)$ the union of all such $H^{s+\frac{1}{2}}(\Omega^-_R)$.

The same will be valid if in (18) we assume the functions u and U to be $\mathbb{H}(\mathbb{C})$-valued and each Cartesian component of U to satisfy (12) or, which is equivalent, the whole function U to satisfy (14). The operator transforming u into U we denote by Λ. As we have just seen, Λ acts from $H^s(\Gamma)$ to $H^{s+\frac{1}{2}}_{loc}(\Omega^-)$.

The operators $\Pi_{\pm\alpha}$ introduced above act obviously from $H^s_{loc}(\Omega^-)$ to $H^{s-1}_{loc}(\Omega^-)$. Consider a function $\Pi_\alpha \Lambda u$. For $u \in H^s(\Gamma)$ $(s > 1)$ it will belong to $H^{s-\frac{1}{2}}_{loc}(\Omega^-)$ and its trace (see, e.g., [28: p. 50]) is $\gamma\Pi_\alpha\Lambda u \in H^{s-1}(\Gamma)$. As the operators $Q_{\pm\alpha}$ are bounded in $H^s(\Gamma)$, we can introduce two new operators $\widetilde{Q}_\alpha$ and $\widetilde{Q}_{-\alpha}$ by

$$\widetilde{Q}_{\pm\alpha} = Q_{\pm\alpha}\gamma\Pi_{\pm\alpha}\Lambda : \ H^s(\Gamma) \to H^{s-1}(\Gamma) \qquad (s > 1).$$

Proposition 17. *Let an $\mathbb{H}(\mathbb{C})$-valued function u belongs to $H^s(\Gamma)$ $(s > 1)$. Then*

$$u = \widetilde{Q}_\alpha u + \widetilde{Q}_{-\alpha} u. \tag{19}$$

Proof. Consider $U = \Lambda u$. We have

$$U = \Pi_\alpha U + \Pi_{-\alpha} U = -(K_\alpha\gamma\Pi_\alpha U + K_{-\alpha}\gamma\Pi_{-\alpha} U) \tag{20}$$

for any point $x \in \Omega^-$. Taking the limit of this equality when x tends to the boundary and using Theorem 6 we obtain

$$u = Q_\alpha\gamma\Pi_\alpha\Lambda u + Q_{-\alpha}\gamma\Pi_{-\alpha}\Lambda u = \widetilde{Q}_\alpha u + \widetilde{Q}_{-\alpha} u$$

and the statement is proved ∎

Remark 18. As seen, the operators $\widetilde{Q}_{\pm\alpha}$ act from $H^s(\Gamma)$ to $H^{s-1}(\Gamma)$ $(s > 1)$. So equality (19) can appear a little bit surprising. Nevertheless, this is a reflection of the corresponding fact inside the domain Ω^- (20), where the differential operators $\Pi_{\pm\alpha}$ also reduce the smoothness of a function but the derivatives in (20) are cancelled.

Proposition 19. *Let $f \in \mathrm{im}Q_\alpha(H^s(\Gamma))$ $(s > 0)$. Then $Q_\alpha f = \widetilde{Q}_\alpha f$.*

Proof: Let $f \in \mathrm{im}Q_\alpha$, that is $f = Q_\alpha f$. We have that the function Λf satisfies equation (3) in Ω^- and belongs to $H^{s+\frac{1}{2}}_{loc}(\Omega^-)$. Moreover, due to the uniqueness of the solution of the Dirichlet problem for the Helmholtz operator in Ω^- we obtain $\Pi_\alpha \Lambda f = \Lambda f$. Thus $\Lambda f = -K_\alpha \gamma \Pi_\alpha \Lambda f$ which on the boundary due to Theorem 6 gives $f = \widetilde{Q}_\alpha f$ ∎

Let us introduce the systems of functions

$$\{\mathcal{K}^{\pm}_{\alpha,n}(x) = (-D + \alpha)\theta_\alpha(x - y_n^{\pm})\}_{n=1}^{\infty}$$
$$\{\mathcal{K}^{\pm}_{-\alpha,n}(x) = -(D + \alpha)\theta_\alpha(x - y_n^{\pm})\}_{n=1}^{\infty}$$

where the sets of points $\{y_n^+\}_{n=1}^\infty$ and $\{y_n^-\}_{n=1}^\infty$ are defined as in Section 6. We are ready to prove one of the central facts of this work.

Theorem 20. *Let α^2 be not an eigenvalue of the Dirichlet problem in V. Then the systems of functions $\{\mathcal{K}^-_{\pm\alpha,n}\}_{n=1}^\infty$ are complete in $\mathrm{im}Q_{\pm\alpha}(H^s(\Gamma))$ $(s > 1)$, respectively, by the norm of $H^{s-1}(\Gamma)$.*

Proof. Let us consider the system $\{\mathcal{K}^-_{\alpha,n}\}_{n=1}^\infty$. Due to Proposition 19, any function $f \in \mathrm{im}Q_\alpha(H^s(\Gamma))$ can be represented as $f = \widetilde{Q}_\alpha f$. Due to Theorem 16, for any $\varepsilon_1 > 0$ there exists such a linear combination $f_N = \sum_{j=1}^N a_j \theta^-_{\alpha,j}$ that $\|f - f_N\|_{H^s(\Gamma)} < \varepsilon_1$ where a_j are constant complex quaternions. Due to the boundedness of $\widetilde{Q}_\alpha$ we have

$$\|f - \widetilde{Q}_\alpha f_N\|_{H^{s-1}(\Gamma)} = \|\widetilde{Q}_\alpha f - \widetilde{Q}_\alpha f_N\|_{H^{s-1}(\Gamma)} \le C\|f - f_N\|_{H^s(\Gamma)} < C\varepsilon_1$$

where $C > 0$ is a constant. Choosing $\varepsilon = C\varepsilon_1$ we obtain $\|f - \widetilde{Q}_\alpha f_N\|_{H^{s-1}(\Gamma)} < \varepsilon$. Thus the function $\widetilde{Q}_\alpha f_N$ approximates f in the norm of $H^{s-1}(\Gamma)$. Consider

$$\widetilde{Q}_\alpha f_N = Q_\alpha \gamma \Pi_\alpha \Lambda \sum_{j=1}^N a_j \theta^-_{\alpha,j}.$$

It is obvious that the extension $\Lambda\theta^-_{\alpha,j}$ coincides with the values of $\theta_\alpha(x - y_j^-)$ for all $x \in \Omega^-$. We obtain

$$\widetilde{Q}_\alpha f_N(x) = \sum_{j=1}^N Q_\alpha \gamma \Pi_\alpha(\theta_\alpha(x - y_j^-))a_j = \frac{1}{2\alpha}\sum_{j=1}^N Q_\alpha \gamma \mathcal{K}^-_{\alpha,j}(x) a_j = \frac{1}{2\alpha}\sum_{j=1}^N \mathcal{K}^-_{\alpha,j}(x) a_j.$$

Thus the statement is proved ∎

8. Extensions into interior domains

The results of this section and their proofs are similar to those of Section 7 and we present them more briefly. Here a new and natural assumption will be that α^2 is not an eigenvalue of the Dirichlet problem in Ω^+. Then for each $\mathbb{H}(\mathbb{C})$-valued function $u \in H^s(\Gamma)$ $(s > 0)$ there exists its unique Helmholtz extension, an $\mathbb{H}(\mathbb{C})$-valued function U satisfying the Helmholtz equation (11) in Ω^+ and coinciding with u on the boundary. As before, the operator transforming u into U we denote by Λ. By analogy with the operators $\widetilde{Q}_{\pm\alpha}$ we introduce the operators

$$\widetilde{P}_{\pm\alpha} = P_{\pm\alpha}\gamma\Pi_{\pm\alpha}\Lambda : \; H^s(\Gamma) \to H^{s-1}(\Gamma) \qquad (s > 1).$$

Proposition 21. *Let α^2 be not an eigenvalue of the Dirichlet problem in Ω^+ and let an $\mathbb{H}(\mathbb{C})$-valued function u belong to $H^s(\Gamma)$ $(s > 1)$. Then $u = \widetilde{P}_\alpha u + \widetilde{P}_{-\alpha} u$.*

The proof is analogous to that of Proposition 17.

Proposition 22. *Let α^2 be not an eigenvalue of the Dirichlet problem in Ω^+ and let an $\mathbb{H}(\mathbb{C})$-valued function f belong to* $\mathrm{im} P_\alpha(H^s(\Gamma))$ $(s > 0)$. *Then $P_\alpha f = \widetilde{P}_\alpha f$.*

The proof is analogous to that of Proposition 19.

Finally, by analogy with Theorem 20 the following statement can be proved.

Theorem 23. *Let α^2 be not an eigenvalue of the Dirichlet problem in Ω^+. Then the systems of functions $\{\mathcal{K}^{+}_{\pm\alpha,n}\}_{n=1}^{\infty}$ are complete in* $\mathrm{im} P_{\pm\alpha}(H^s(\Gamma))$ $(s > 1)$ *respectively by the norm of $H^{s-1}(\Gamma)$.*

Remark 24. For $\alpha = 0$ a similar result can be found in [14: p. 284] (see also the references therein). Unfortunately, the scheme of the proof proposed in that work is not applicable for complex quaternion-valued functions due to the difficulty of introduction of an L_2 space which would correspond to the complex quaternionic multiplication.

9. Complete systems for Maxwell's equations

As was shown in preceding sections, the functions $\vec{\varphi}$ and $\vec{\psi}$ can be approximated by right linear combinations of functions $\{\mathcal{K}^{\pm}_{\alpha_1,n}\}_{n=1}^{\infty}$ and $\{\mathcal{K}^{\pm}_{-\alpha_2,n}\}_{n=1}^{\infty}$, respectively. The vectors $\vec{E}$ and $\vec{H}$ from (6) are easily recovered from $\vec{\varphi}$ and $\vec{\psi}$ by

$$\vec{E} = \tfrac{1}{2}(\vec{\varphi} + \vec{\psi}) \qquad \text{and} \qquad \vec{H} = \tfrac{1}{2i}(\vec{\varphi} - \vec{\psi}). \tag{21}$$

Consequently, all the results of preceding sections are applicable to the electromagnetic field.

As before we start with exterior domains. The radiation condition for the vectors $\vec{E}$ and $\vec{H}$ is the Silver-Müller condition

$$\vec{E} - \left[\tfrac{x}{|x|} \times \vec{H}\right] = o\left(\tfrac{1}{|x|}\right) \tag{22}$$

or, in an equivalent form,

$$\vec{H} + \left[\tfrac{x}{|x|} \times \vec{E}\right] = o\left(\tfrac{1}{|x|}\right) \tag{23}$$

uniformly for all directions. Note that (22) and (23) are fulfilled automatically if as before $\vec{\varphi}$ and $\vec{\psi}$ satisfy (9) amd (10), respectively. We have

$$\begin{aligned}\vec{E} &= \tfrac{1}{2}(\vec{\varphi} + \vec{\psi}) \\ &= \tfrac{1}{2}(-i\tfrac{x}{|x|} \cdot \vec{\varphi} + i\tfrac{x}{|x|} \cdot \vec{\psi}) + o\left(\tfrac{1}{|x|}\right) \\ &= \tfrac{x}{|x|} \cdot \tfrac{1}{2i}(\vec{\varphi} - \vec{\psi}) + o\left(\tfrac{1}{|x|}\right) \\ &= \tfrac{x}{|x|} \cdot \vec{H} + o\left(\tfrac{1}{|x|}\right).\end{aligned}$$

The vector part of this equality gives us (22) and the scalar is a simple consequence of (23). Starting with $\vec{H}$ instead of $\vec{E}$ we arrive at (23).

From Theorem 6, equalities (21) and the last observation concerning the relation between the radiation conditions for $\vec{\varphi}$ and $\vec{\psi}$ from one side and for $\vec{E}$ and $\vec{H}$ from the other, we obtain the following criterion.

Theorem 25. *Let complex vectors $\vec{e}$ and $\vec{h}$ belong to $H^s(\Gamma)$ $(s > 0)$. Then in order for $\vec{e}$ and $\vec{h}$ to be boundary values of $\vec{E}$ and $\vec{H}$ satisfying Maxwell equations* (6) *in Ω^- and* (22) *at infinity, the condition*

$$\begin{aligned}(\vec{e} + i\vec{h}) &\in \operatorname{im} Q_{\alpha_1} \\ (\vec{e} - i\vec{h}) &\in \operatorname{im} Q_{-\alpha_2}\end{aligned} \tag{24}$$

is necessary and sufficient.

Now from Theorem 20 we obtain immediately the following important result opening the possibility to apply the systems of quaternionic fundamental solutions $\{\mathcal{K}^{-}_{\alpha_1,n}\}_{n=1}^{\infty}$ and $\{\mathcal{K}^{-}_{-\alpha_2,n}\}_{n=1}^{\infty}$ to approximation of the electromagnetic field in exterior domains.

Theorem 26. *Let both α_1^2 and α_2^2 be no eigenvalues of the Dirichlet problem in V. Then if condition (24) is fulfilled, the vectors $\vec{e}$ and $\vec{h}$ belonging to $H^s(\Gamma)$ $(s > 1)$ can be approximated with an arbitrary precision (in the norm of $H^{s-1}(\Gamma)$) by right linear combinations of the form*

$$\vec{e}_N = \tfrac{1}{2}\Bigg(\sum_{j=1}^{N} \mathcal{K}^{-}_{\alpha_1,j} a_j + \sum_{j=1}^{N} \mathcal{K}^{-}_{-\alpha_2,j} b_j\Bigg)$$

$$\vec{h}_N = \tfrac{1}{2i}\Bigg(\sum_{j=1}^{N} \mathcal{K}^{-}_{\alpha_1,j} a_j - \sum_{j=1}^{N} \mathcal{K}^{-}_{-\alpha_2,j} b_j\Bigg)$$

where a_j and b_j are constant complex quaternions.

Proof. Due to Theorem 20, there exist such a_j and b_j that

$$\Bigg\|\vec{e} + i\vec{h} - \sum_{j=1}^{N} \mathcal{K}^{-}_{\alpha_1,j} a_j\Bigg\|_{H^{s-1}(\Gamma)} < \varepsilon \quad \text{and} \quad \Bigg\|\vec{e} - i\vec{h} - \sum_{j=1}^{N} \mathcal{K}^{-}_{-\alpha_2,j} b_j\Bigg\|_{H^{s-1}(\Gamma)} < \varepsilon.$$

From these two inequalities we obtain the necessary result ∎

In a similar way we obtain the corresponding result for interior domains.

Theorem 27. *Let both α_1^2 and α_2^2 be no eigenvalues of the Dirichlet problem in Ω^+ and let the condition*

$$\begin{aligned} &(\vec{e} + i\vec{h}) \in \mathrm{im}P_{\alpha_1}(H^s(\Gamma)) \\ &(\vec{e} - i\vec{h}) \in \mathrm{im}P_{-\alpha_2}(H^s(\Gamma)) \end{aligned} \qquad (s > 1)$$

be fulfilled which is a necessary and sufficient condition of the extendability of the vectors $\vec{e}$ and $\vec{h}$ into Ω^+ in such a way that their extensions satisfy (6). Then $\vec{e}$ and $\vec{h}$ can be approximated with an arbitrary precision (in the norm of $H^{s-1}(\Gamma)$) by right linear combinations of the form

$$\vec{e}_N = \tfrac{1}{2}\Bigg(\sum_{j=1}^{N} \mathcal{K}^{+}_{\alpha_1,j} a_j + \sum_{j=1}^{N} \mathcal{K}^{+}_{-\alpha_2,j} b_j\Bigg)$$

$$\vec{h}_N = \tfrac{1}{2i}\Bigg(\sum_{j=1}^{N} \mathcal{K}^{+}_{\alpha_1,j} a_j - \sum_{j=1}^{N} \mathcal{K}^{+}_{-\alpha_2,j} b_j\Bigg)$$

where a_j and b_j are constant complex quaternions.

10. Numerical realization

Let $\Gamma = \partial\Omega^-$ be a closed sufficiently smooth surface in $\mathbb{R}^3$. Consider the following exterior boundary value problem for the Maxwell equations:

$$\left.\begin{aligned} \mathrm{rot}\vec{E}(x) &= -i\alpha\big(\vec{H}(x) + \beta\mathrm{rot}\vec{H}(x)\big) && (x \in \Omega^-) \\ \mathrm{rot}\vec{H}(x) &= i\alpha\big(\vec{E}(x) + \beta\mathrm{rot}\vec{E}(x)\big) && (x \in \Omega^-) \\ \big[\vec{E}(x) \times \vec{n}(x)\big] &= \vec{f}(x) && (x \in \Gamma) \end{aligned}\right\} \tag{25}$$

where $\vec{f}(x)$ is a given tangential field and at infinity the vectors $\vec{E}$ and $\vec{H}$ satisfy the Silver-Müller radiation condition (22) or (23). As before $\vec{n}$ stands for the unit outward normal to Γ.

Using results of the preceding section, this problem can be rewritten in the equivalent form

$$\left.\begin{aligned} (D + \alpha_1)\varphi(x) &= 0 && (x \in \Omega^-) \\ (D - \alpha_2)\psi(x) &= 0 && (x \in \Omega^-) \\ \tfrac{1}{2}\big[\big(\varphi(x) + \psi(x)\big) \times \vec{n}(x)\big] &= \vec{f}(x) && (x \in \Gamma) \\ \mathrm{Sc}\varphi(x) = \mathrm{Sc}\psi(x) &= 0 && (x \in \Gamma) \end{aligned}\right\} \tag{26}$$

and at infinity the functions φ and ψ satisfy the conditions

$$\begin{aligned} &\big(1 + \tfrac{ix}{|x|}\big) \cdot \varphi(x) = o\big(\tfrac{1}{|x|}\big) \\ &\big(1 - \tfrac{ix}{|x|}\big) \cdot \psi(x) = o\big(\tfrac{1}{|x|}\big) \qquad (|x| \to \infty). \end{aligned} \tag{27}$$

Note that due to the uniqueness of the solution of the exterior Dirichlet problem for the Helmholtz equation condition $(26)_4$ implies that φ and ψ will be purely vectorial on the whole domain Ω^-.

We look for an approximate solution of problem (26) - (27) in the form

$$\varphi_N(x) = \sum_{j=1}^{N} \mathcal{K}^-_{\alpha_1,j}(x)a_j \qquad \text{and} \qquad \psi_N(x) = \sum_{j=1}^{N} \mathcal{K}^-_{-\alpha_2,j}(x)b_j \tag{28}$$

where a_j and b_j are constant complex quaternions. In order to find the coefficients $a_j$and b_j we use the collocation method. In (28) we have $8N$ unknown complex quantities, the components of a_j and b_j. Thus it is necessary to obtain $8N$ linearly independent equations with respect to components of a_j and b_j. Every collocation point generates four linearly independent equations, two of which correspond to the boundary condition $(26)_3$ and the other two correspond to $(26)_4$. Consequently, in order to determine the coefficients a_j

and b_j we need $2N$ collocation points. After having solved the corresponding system of linear algebraic equations we obtain the approximate solution of problem (25) as

$$\left.\begin{aligned} \vec{E}(x) &= \tfrac{1}{2}\big(\varphi_N(x) + \psi_N(x)\big) \\ \vec{H}(x) &= \tfrac{1}{2i}\big(\varphi_N(x) - \psi_N(x)\big) \end{aligned}\right. \qquad (x \in \Omega^-).$$

The method described above was tested using the following exact solution. Let $\beta = 0$ and consequently $\alpha = \alpha_1 = \alpha_2$. The vectors

$$\vec{E}^m(x) = \operatorname{rot}\vec{c}\theta_\alpha(x) = \begin{pmatrix} c_3\partial_2\theta_\alpha(x) - c_2\partial_3\theta_\alpha(x) \\ c_1\partial_3\theta_\alpha(x) - c_3\partial_1\theta_\alpha(x) \\ c_2\partial_1\theta_\alpha(x) - c_1\partial_2\theta_\alpha(x) \end{pmatrix}$$

and

$$\vec{H}^m(x) = -\tfrac{1}{i\alpha}\operatorname{rot}\vec{E}^m(x) \qquad (x \in \mathbb{R}^3 \setminus \{0\})$$

with $\vec{c} \in \mathbb{R}^3$ constant represent the electromagnetic field of a magnetic dipole situated at the origin [7: Section 4.2]. They satisfy $(25)_{1-2}$ (for $\beta = 0$) as well as the Silver-Müller conditions at infinity.

Let Γ be a unit sphere with its centre at the origin. Then $\vec{E}^m$ and $\vec{H}^m$ give us the solution of the boundary value problem

$$\left.\begin{aligned} \operatorname{rot}\vec{E}(x) &= -i\alpha\vec{H}(x) && (x \in \Omega^-) \\ \operatorname{rot}\vec{H}(x) &= i\alpha\vec{E}(x) && (x \in \Omega^-) \\ \big[\vec{E}(x) \times \vec{n}(x)\big] &= \vec{f}(x) && (x \in \Gamma) \end{aligned}\right\}$$

where

$$\vec{f}(x) = \left[\begin{pmatrix} c_3\partial_2\theta_\alpha(x) - c_2\partial_3\theta_\alpha(x) \\ c_1\partial_3\theta_\alpha(x) - c_3\partial_1\theta_\alpha(x) \\ c_2\partial_1\theta_\alpha(x) - c_1\partial_2\theta_\alpha(x) \end{pmatrix} \times \vec{n}(x)\right].$$

As the auxiliary surface Γ^- containing points y_n^- we have chosen the sphere with centre at the origin and radius 0.15. In the following table we present the results for different values of N. The corresponding errors represent the absolute maximum difference between the exact and the approximate solutions at the points on the sphere with centre at the origin and radius 5.

A quite fast convergence of the method can be appreciated (all numerical results were obtained on a PC Pentium 3).

Let us notice that the approximation by linear combinations of quaternionic fundamental solutions can be applied to other classes of boundary value problems for the Maxwell system like for example the impedance problem.

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