

Exponential Growth for a Fractionally Damped Wave Equation

M. Kirane and N.-e. Tatar

Abstract. We consider a nonlinear wave equation with an internal damping represented by a fractional time derivative and with a polynomial source. It is proved that the solution is unbounded and grows up exponentially in the L_p -norm for sufficiently large initial data. To this end we use some techniques based on Fourier transforms and some inequalities such as the Hardy-Littlewood inequality.

Keywords: *Exponential growth, fractional derivative, internal damping*

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1. Introduction

This paper is concerned with the fractional differential problem

$$\left. \begin{aligned} u_{tt} + \partial_t^{1+\alpha} u &= \Delta u + |u|^{p-1} u & (x \in \Omega, t > 0) \\ u(x, t) &= 0 & (x \in \Gamma, t > 0) \\ u(x, 0) = u_0(x), u_t(x, 0) &= u_1(x) & (x \in \Omega) \end{aligned} \right\} \quad (1.1)$$

where $p > 1$, $-1 < \alpha < 1$, u_0 and u_1 are given functions, Ω is a bounded domain of \mathbb{R}^N with smooth boundary Γ , and $\partial_t^{1+\alpha}$ is Caputo's fractional derivative of order $1 + \alpha$ (see [30: Chapter 2.4.1]) defined by

$$\partial_t^{1+\alpha} w(t) = I^{-\alpha} \frac{d}{dt} w(t) \quad (-1 < \alpha < 0) \quad (2)$$

and

$$\partial_t^{1+\alpha} w(t) = I^{1-\alpha} \frac{d^2}{dt^2} w(t) \quad (0 < \alpha < 1) \quad (3)$$

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where I^β ($\beta > 0$) is the fractional integral

$$I^\beta w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s) ds.$$

This kind of fractional derivatives is shown to be more appropriate to the study of partial differential equations with initial conditions (see [30: p. 78]). See [7, 28, 32] for more on fractional integrals and derivatives.

Due to the strong singularity of the kernel appearing in this fractional derivative it seems very difficult to apply the existing methods to study the asymptotic behavior of such problems.

The linear case (with the Riemann-Liouville fractional derivative) is considered by Matignon et al. in [22]. There they interpreted the equation as a coupled problem between an undamped classical wave equation and a diffusion equation after using an idea of the last two authors to establish the positivity of the damping operator. The well-posedness and the asymptotic stability are then obtained using standard Galerkin methods and LaSalle's invariance principle. Moreover, they proved an algebraic decay result of the classical energy for some values of α .

The case $\alpha = \frac{1}{2}$ is known as the Lokshin model. It appears in the study of propagation of the air inside a duct when taking into account viscothermal losses (see [20, 21]).

In the case $\alpha = -1$, i.e. the wave equation without damping and with smooth u_0, u_1 , it was first proved that the solutions break down in finite time for any $p > 1$ when the initial data are large in some sense. Roughly speaking, if the initial data u_0 and u_1 belong to an "unstable set", then the associated solution blows up in a finite time (see [4, 14, 19]). Then, solutions are shown to blow up also for small initial data provided that the exponent p lies in some "critical range" (see [5, 6, 13, 34]). It is often proved that solutions are bounded and so exist globally beyond a critical power.

In general, the presence of a linear damping, i.e. $\alpha = 0$ allows to prove global existence for small initial data (see [23]; see also [26] for a nonlinear dissipation). Using the potential well method (see [29, 33]) it was proved (see, for instance, [12, 31]) that solutions exist globally when the initial data belong to a "stable" set and blow up in a finite time when the initial data are in an "unstable" set. Another powerful tool to prove blow up in finite time is the concavity method of Levine (see [19]). See also Tsutsumi [35] for another method.

In general, besides their utility in control theory of dynamical systems and propagation problems, fractional derivatives and integrals have many other applications in physics (mechanics, electromagnetism,...), chemistry, biology, etc. (see [7, 28, 30, 32] and references therein). They are used, in particular, to describe memory and hereditary properties of various materials and processes.

In [16, 17], the authors have considered a semilinear problem with a boundary damping (instead of internal damping) and a strongly positive definite kernel (see [27])

$$\int_0^t k(t-s)u_t(s) ds.$$

In particular, they have examined kernels of the forms e^{-t} . Stability, non-existence and blow up in finite time results have been proved using appropriately chosen modified "energy", the concavity method of Levine, the potential well method and an argument due to Tsutsumi. They have used in a crucial way the following inequality which holds for all continuous, non-negative, non-increasing and convex kernels:

$$|(k * w)(t)|^2 \leq 2k(0^+) \operatorname{Re} \int_0^t w(\tau) \overline{(k * w)(\tau)} d\tau.$$

In this form, this is not valid in our case. Instead, we shall use a somewhat similar relation which may be found in the book of Gripenberg et al. [8]. A crucial use is also made of the Hardy-Littlewood-Sobolev inequality.

In [3], an unboundedness result is proved in the case of a dissipation of polynomial type. A cubic spatial convolution of the form

$$u_t \int_{\mathbb{R}^N} \frac{u_t^2(y, t)}{|x - y|^\gamma} dy \quad (0 < \gamma < N)$$

has been considered in [25]. The authors proved a polynomial decay result in the absence of sources.

It is then clear that our present situation is much different from the previous ones (except for the last one, in some sense) in that our kernel is strongly singular and not integrable.

In this paper, using a technique combining some ideas which are similar in spirit to those in [3, 9, 10, 18, 25] and some continuity properties of the fractional derivative (see [7, 32]) we shall prove that the solution of the problem grows up exponentially in the L_{p+1} -norm for sufficiently large initial data.

2. Exponential growth

In this section we state and prove our result. In the proof we make use of Fourier transforms, in particular the Parseval's theorem (see, for instance, [11: Theorem 7.1.6]). We also need a Sobolev inequality which is sometimes referred to as the Hardy-Littlewood inequality or the Hardy-Littlewood-Sobolev inequality (see [11: p. 117/Theorem 4.5.3] or [24: p. 378/Corollary]). We refer to [22] for the issue of well-posedness (see also A.A. Kilbas et al. [15] for more on existence results on ordinary differential equations). As we will be concerned only by those values of α between -1 and 0 , the appropriate definition is given by (2). In the notation of Hörmander [11], this definition is written as

$$I^{-\alpha} v = \chi_+^{-(\alpha+1)} * v.$$

In order to avoid working with both powers and subscripts, we will denote our kernel simply by k with the appropriate power (with a minus sign) as a subscript.

Let us first define

$$G(t) = \int_{\Omega} \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \varepsilon u u_t - \frac{1}{p+1} |u|^{p+1} \right\} dx \tag{4}$$

where $0 < \varepsilon < 1$ is a small constant to be determined later.

Lemma 1 (Hardy-Littlewood-Sobolev inequality, see [11] or [24]). *Let $u \in L^p(\mathbb{R})$ ($p > 1$), $0 < \lambda < 1$ and $\lambda > 1 - \frac{1}{p}$. Then $\frac{1}{|x|^\lambda} * u \in L^q(\mathbb{R})$ with $\frac{1}{q} = \lambda + \frac{1}{p} - 1$. Also, the mapping from $u \in L^p(\mathbb{R})$ into $\frac{1}{|x|^\lambda} * u \in L^q(\mathbb{R})$ is continuous.*

Theorem 2. *Let $u = u(x, t)$ be a regular solution of problem (1) with $-1 < \alpha < 0$. If the initial data are large enough (they will be determined in the proof according to the cases $-\alpha < \frac{1}{p+1}$, $-\alpha > \frac{1}{p+1}$ and $-\alpha = \frac{1}{p+1}$), then the solution u grows up exponentially in the L_{p+1} -norm.*

Proof. Let us multiply equation (1)₁ by $u_t - \varepsilon u$ and integrate over Ω . We get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left\{ \frac{1}{2} u_t^2 + \frac{1}{2} |\nabla u|^2 - \varepsilon u u_t - \frac{1}{p+1} |u|^{p+1} \right\} dx \\ & + \frac{1}{\Gamma(-\alpha)} \int_{\Omega} u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s) ds dx \\ & = \varepsilon \int_{\Omega} |\nabla u|^2 dx + \frac{\varepsilon}{\Gamma(-\alpha)} \int_{\Omega} u \int_0^t (t-s)^{-(\alpha+1)} u_t(s) ds dx \\ & - \varepsilon \int_{\Omega} u_t^2 dx - \varepsilon \int_{\Omega} |u|^{p+1} dx. \end{aligned} \tag{5}$$

Then, from definition (4) of G , it is clear that

$$\begin{aligned} & G(t) - G(0) + \frac{1}{\Gamma(-\alpha)} \int_0^t \int_{\Omega} u_t \int_0^s (s-z)^{-(\alpha+1)} u_t(z) dz dx ds \\ & = \varepsilon \int_0^t \int_{\Omega} |\nabla u|^2 dx ds \\ & + \frac{\varepsilon}{\Gamma(-\alpha)} \int_0^t \int_{\Omega} u \int_0^s (s-z)^{-(\alpha+1)} u_t(z) dz dx ds \\ & - \varepsilon \int_0^t \int_{\Omega} u_t^2 dx ds - \varepsilon \int_0^t \int_{\Omega} |u|^{p+1} dx ds. \end{aligned} \tag{6}$$

We want to estimate the second term in the right-hand side of (6). Let us denote $\frac{1}{\Gamma(1-\beta)} t^{-\beta}$ by $k_{\beta}(t)$ and for fixed $t = T > 0$ define

$$\begin{aligned} L_T w(\tau) &= \begin{cases} w(\tau) & \text{for } \tau \in [0, T] \\ 0 & \text{for } \tau \in \mathbb{R} \setminus [0, T] \end{cases} \\ Lk_{\beta}(\tau) &= \begin{cases} k_{\beta}(\tau) & \text{for } \tau > 0 \\ 0 & \text{for } \tau \leq 0. \end{cases} \end{aligned}$$

(To simplify the notation we shall drop the subscript T in L_T). Then we can easily see that

$$\begin{aligned} & \frac{1}{\Gamma(-\alpha)} \int_0^T u(s) \int_0^s (s-z)^{-(\alpha+1)} u_t(z) dz ds \\ & = \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\alpha+1}(s-z) Lu_t(z) dz ds. \end{aligned}$$

By Parseval' theorem we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\alpha+1}(s-z)Lu_t(z) dz ds \\ &= \int_{-\infty}^{+\infty} \mathcal{F}(Lu)(\sigma) \overline{\mathcal{F}(Lk_{\alpha+1} * Lu_t)}(\sigma) d\sigma \end{aligned} \tag{7}$$

where $\mathcal{F}(f)$ denotes the usual Fourier transform of f . By the convolution here we mean the integrand in the left-hand side.

It is well known (see, e.g., [30: p. 7] or [36]) that k_β satisfies the convolution property $k_{\beta+\gamma-1} = k_\beta * k_\gamma$. Therefore, by (7), the Cauchy-Schwarz inequality and the generalized Hölder inequality, we see that for $\delta > 0$

$$\begin{aligned} & \int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\alpha+1}(s-z)Lu_t(z) dz ds \\ & \leq \left(\int_{-\infty}^{+\infty} |\mathcal{F}(Lu)\mathcal{F}(Lk_{\frac{\alpha+2}{2}})|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}})\mathcal{F}(Lu_t)|^2 d\sigma \right)^{\frac{1}{2}} \\ & \leq \delta \int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}})\mathcal{F}(Lu_t)|^2 d\sigma + \frac{1}{4\delta} \int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}})\mathcal{F}(Lu)|^2 d\sigma. \end{aligned} \tag{8}$$

By [8: Theorem 16.5.1], we have

$$I = \int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}})\mathcal{F}(Lu)|^2 d\sigma \leq \frac{1}{\cos \frac{\alpha\pi}{2}} \int_{-\infty}^{+\infty} Lu(s)(Lk_{\alpha+1} * Lu)(s) ds.$$

Then, an application of Hölder's inequality yields

$$I \leq \frac{1}{\cos \frac{\alpha\pi}{2}} \left(\int_{-\infty}^{+\infty} |Lu|^{p+1} ds \right)^{\frac{1}{p+1}} \left(\int_{-\infty}^{+\infty} (Lk_{\alpha+1} * Lu)^{\frac{p+1}{p}}(s) ds \right)^{\frac{p}{p+1}}. \tag{9}$$

This will be our reference inequality for the discussion below.

(a) If $-\alpha < \frac{1}{p+1}$, then $r = \frac{p+1}{p-\alpha(p+1)} > 1$ and we may apply the Hardy-Littlewood-Sobolev inequality to get

$$\left(\int_{-\infty}^{+\infty} (Lk_{\alpha+1} * Lu)^{\frac{p+1}{p}}(s) ds \right)^{\frac{p}{p+1}} \leq \frac{C(p, \alpha)}{\Gamma(-\alpha)} \left(\int_{-\infty}^{+\infty} |Lu|^r ds \right)^{\frac{1}{r}}. \tag{10}$$

Note that $r < 2$ and $C(p, \alpha)$ depends only on p and α (it does not depend on T).

By the Hölder inequality the integral term in the right-hand side of (10) is estimated as

$$\left(\int_{-\infty}^{+\infty} |Lu|^r ds \right)^{\frac{1}{r}} \leq T^{\frac{p+1-r}{p+1}} \left(\int_{-\infty}^{+\infty} |Lu|^{p+1} ds \right)^{\frac{1}{p+1}}.$$

From this inequality, (10) and (9), we obtain

$$\begin{aligned}
 I &\leq \frac{C(p, \alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} T^{\frac{p+1-r}{p+1}} \left(\int_{-\infty}^{+\infty} |Lu|^{p+1} ds \right)^{\frac{2}{p+1}} \\
 &\leq \frac{2}{p+1} \left(\frac{C(p, \alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} \right)^{\frac{p+1}{2}} \int_{-\infty}^{+\infty} |Lu|^{p+1} ds + \frac{p-1}{p+1} T^{\frac{p+1-r}{p-1}}.
 \end{aligned}
 \tag{11}$$

Taking into account relation (11) in (8), and choosing

$$\delta = \frac{1}{p-1} \left(\frac{C(p, \alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} \right)^{\frac{p+1}{2}}$$

we find

$$\begin{aligned}
 &\int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\alpha+1}(s-z) Lu_t(z) dz ds \\
 &\leq \delta \int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}}) \mathcal{F}(Lu_t)|^2 d\sigma \\
 &\quad + \frac{p-1}{2(p+1)} \int_{-\infty}^{+\infty} |Lu|^{p+1} ds + \frac{(p-1)^2}{4(p+1)} \left(\frac{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}}{C(p, \alpha)} \right)^{\frac{p+1}{2}} T^{\frac{p+1-r}{p-1}}.
 \end{aligned}
 \tag{12}$$

Back to (6), from (12) and [8: Theorem 16.5.1] we find

$$\begin{aligned}
 G(T) &+ \left[\cos \frac{\alpha\pi}{2} - \varepsilon\delta \right] \int_{\Omega} \int_0^T |k_{\frac{\alpha+2}{2}} * u_t|^2 ds dx \\
 &\leq G(0) + \varepsilon \int_0^T \int_{\Omega} |\nabla u|^2 dx ds - \varepsilon \int_0^T \int_{\Omega} u_t^2 dx ds \\
 &\quad - \frac{\varepsilon(p+3)}{2(p+1)} \int_0^T \int_{\Omega} |u|^{p+1} dx ds + C_1 T^{\sigma_1}
 \end{aligned}
 \tag{13}$$

where

$$C_1 = \frac{\varepsilon(p-1)^2 |\Omega|}{4(p+1)} \left(\frac{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}}{C(p, \alpha)} \right)^{\frac{p+1}{2}} \quad \text{and} \quad \sigma_1 = \frac{p+1-r}{p-1}.$$

Next, adding and subtracting the term

$$\frac{\varepsilon^2(p+3)}{2} \int_0^T \int_{\Omega} uu_t dx ds$$

to the right-hand side of (13) and using the fact that

$$\int_{\Omega} uu_t dx \leq \frac{\tilde{C}}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{1}{2} \int_{\Omega} u_t^2 dx
 \tag{14}$$

where \tilde{C} is the Poincaré constant, we obtain

$$\begin{aligned}
 G(T) &+ \left[\cos \frac{\alpha\pi}{2} - \varepsilon\delta \right] \int_{\Omega} \int_0^T |k_{\frac{\alpha+2}{2}} * u_t|^2 ds dx \\
 &\leq G(0) + \frac{\varepsilon(p+3)}{2} \int_0^T G(s) ds \\
 &\quad + \varepsilon \left(1 + \frac{\varepsilon\tilde{C}(p+3)}{4} - \frac{p+3}{4} \right) \int_0^T \int_{\Omega} |\nabla u|^2 dx ds \\
 &\quad - \varepsilon \left(1 - \frac{\varepsilon(p+3)}{4} + \frac{p+3}{4} \right) \int_0^T \int_{\Omega} u_t^2 dx ds + C_1 T^{\sigma_1}.
 \end{aligned} \tag{15}$$

Observe that as $\varepsilon < 1$, the coefficient of $\int_0^T \int_{\Omega} u_t^2 dx ds$ is negative. Furthermore, choosing

$$\varepsilon \leq \min \left\{ 1, \frac{p-1}{\tilde{C}(p+3)}, \frac{1}{\delta} \cos \frac{\alpha\pi}{2} \right\}$$

it appears from (15) that

$$G(T) \leq G(0) + \frac{\varepsilon(p+3)}{2} \int_0^T G(s) ds + C_1 T^{\sigma_1}.$$

We define $\Psi = -G$. Clearly,

$$\Psi(T) \geq \Psi(0) + \frac{\varepsilon(p+3)}{2} \int_0^T \Psi(s) ds - C_1 T^{\sigma_1}. \tag{16}$$

From this inequality we deduce (by an argument similar to that for the classical Gronwall inequality) that

$$\begin{aligned}
 \Psi(T) &\geq \Psi(0) e^{\frac{\varepsilon(p+3)}{2} T} - \sigma_1 C_1 e^{\frac{\varepsilon(p+3)}{2} T} \int_0^T s^{\sigma_1-1} e^{-\frac{\varepsilon(p+3)}{2} s} ds \\
 &\geq \left\{ \Psi(0) - \sigma_1 C_1 \left(\frac{\varepsilon(p+3)}{2} \right)^{-\sigma_1} \Gamma(\sigma_1) \right\} e^{\frac{\varepsilon(p+3)}{2} T}.
 \end{aligned} \tag{17}$$

The initial data u_0 and u_1 are chosen so that

$$\Psi(0) - \sigma_1 C_1 \left(\frac{\varepsilon(p+3)}{2} \right)^{-\sigma_1} \Gamma(\sigma_1) > 0.$$

On the other hand, from the definition of Ψ and inequality (14), we see that

$$\begin{aligned}
 \Psi(T) &\leq \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx - \frac{1}{2} \int_{\Omega} u_t^2 dx \\
 &\quad - \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \frac{\varepsilon}{2} \int_{\Omega} u_t^2 dx - \frac{\varepsilon\tilde{C}}{2} \int_{\Omega} |\nabla u|^2 dx
 \end{aligned}$$

or

$$\Psi(T) \leq \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx - \frac{1}{2}(1-\varepsilon) \int_{\Omega} u_t^2 dx - \frac{1}{2}(1-\varepsilon\tilde{C}) \int_{\Omega} |\nabla u|^2 dx.$$

Note that from our choice of ε we have $\varepsilon < \frac{1}{C}$. It follows that

$$\Psi(T) \leq \frac{1}{p+1} \int_{\Omega} |u|^{p+1} dx. \tag{18}$$

From inequalities (17) and (18) we conclude the exponential growth of the solution in the L_{p+1} -norm.

(b) The case $-\alpha > \frac{1}{p+1}$ can be treated in the following manner:

$$(Lk_{\alpha+1} * Lu)(s) \leq |s|^{(1-\frac{(\alpha+1)(p+1)}{p})\frac{p}{p+1}} \left(\int_0^s |u|^{p+1} dz \right)^{\frac{1}{p+1}}.$$

Note here that $1 - \frac{(\alpha+1)(p+1)}{p} > 0$. Taking this estimate into account in (9) we find

$$\begin{aligned} I &\leq \frac{1}{\cos \frac{\alpha\pi}{2}} \left(\int_0^T |s|^{1-\frac{(\alpha+1)(p+1)}{p}} ds \right)^{\frac{p}{p+1}} \left(\int_0^T |u|^{p+1} ds \right)^{\frac{2}{p+1}} \\ &\leq CT^{\sigma_2} + \frac{2(\cos \frac{\alpha\pi}{2})^{-\frac{p+1}{2}}}{p+1} \int_0^T |u|^{p+1} ds \end{aligned}$$

where

$$C = \frac{p-1}{p+1} \left(2 - \frac{(\alpha+1)(p+1)}{p} \right)^{-\frac{p}{p-1}} \quad \text{and} \quad \sigma_2 = 1 - \frac{\alpha(p+1)}{p-1}.$$

Choosing

$$\delta = \frac{(\cos \frac{\alpha\pi}{2})^{-\frac{p+1}{2}}}{p-1}$$

we see that

$$\begin{aligned} &\int_{-\infty}^{+\infty} Lu(s) \int_{-\infty}^{+\infty} Lk_{\alpha+1}(s-z) Lu_t(z) dz ds \\ &\leq \delta \int_{-\infty}^{+\infty} |\mathcal{F}(Lk_{\frac{\alpha+2}{2}}) \mathcal{F}(Lu_t)|^2 d\sigma + \frac{p-1}{2(p+1)} \int_0^T |u|^{p+1} ds + \frac{C}{4\delta} T^{\sigma_2}. \end{aligned} \tag{19}$$

Relations (6) and (19) now imply that

$$\begin{aligned} G(T) &+ \left[\cos \frac{\alpha\pi}{2} - \varepsilon\delta \right] \int_{\Omega} \int_0^T |k_{\frac{\alpha+2}{2}} * u_t|^2 ds dx \\ &\leq G(0) + \varepsilon \int_0^T \int_{\Omega} |\nabla u|^2 dx ds - \varepsilon \int_0^T \int_{\Omega} u_t^2 dx ds \\ &\quad - \frac{\varepsilon(p+3)}{2(p+1)} \int_0^T \int_{\Omega} |u|^{p+1} dx ds + C_2 T^{\sigma_2} \end{aligned}$$

with $C_2 = \frac{\varepsilon C}{4\delta} |\Omega|$. Choosing

$$\varepsilon < \min \left\{ 1, \frac{\cos \frac{\alpha\pi}{2}}{\delta}, \frac{p-1}{(p+3)\tilde{C}} \right\}$$

we get

$$\begin{aligned} G(T) \leq & G(0) + \varepsilon \int_0^T \int_{\Omega} |\nabla u|^2 dx ds - \varepsilon \int_0^T \int_{\Omega} u_t^2 dx ds \\ & - \frac{\varepsilon(p+3)}{2(p+1)} \int_0^T \int_{\Omega} |u|^{p+1} dx ds + C_2 T^{\sigma_2}. \end{aligned} \tag{20}$$

Adding and subtracting the same term $\frac{\varepsilon^2(p+3)}{2} \int_0^T \int_{\Omega} uu_t dx ds$ in the right-hand side of (20), we get with the help of (14)

$$G(T) \leq G(0) + \varepsilon \frac{p+3}{2} \int_0^T G(s) ds + C_2 T^{\sigma_2}.$$

We define $\Psi = -G$. Clearly,

$$\Psi(T) \geq \Psi(0) + \varepsilon \frac{p+3}{2} \int_0^T \Psi(s) ds - C_2 T^{\sigma_2}.$$

The rest of the proof is similar to that in part (a).

(c) If $-\alpha = \frac{1}{p+1}$ (see [10]), we use the estimate

$$I \leq \frac{1}{\cos \frac{\alpha\pi}{2}} \left(\int_{-\infty}^{+\infty} |Lu|^2 ds \right)^{\frac{1}{2}} \left(\int_{-\infty}^{+\infty} (Lk_{\alpha+1} * Lu)^2(s) ds \right)^{\frac{1}{2}}.$$

We may use the Hardy-Littlewood-Sobolev inequality with $r = \frac{2}{1-2\alpha} > 1$ to get

$$I \leq \frac{C(\alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} \left(\int_0^T |u|^2 ds \right)^{\frac{1}{2}} \left(\int_0^T |u|^r ds \right)^{\frac{1}{r}}.$$

Next, by Hölder's inequality and Young's inequality, we see that

$$\begin{aligned} I &\leq \frac{C(\alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} T^{\frac{2p-r}{p+1}} \left(\int_0^T |u|^{p+1} ds \right)^{\frac{2}{p+1}} \\ &\leq \frac{p-1}{p+1} T^{\sigma_3} + \frac{2}{p+1} \left(\frac{C(\alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} \right)^{\frac{p+1}{2}} \int_0^T |u|^{p+1} ds. \end{aligned}$$

Here we take

$$\delta = \frac{1}{1(p-1)} \left(\frac{C(\alpha)}{\Gamma(-\alpha) \cos \frac{\alpha\pi}{2}} \right)^{\frac{p+1}{2}}.$$

The rest of the proof is similar to that in case (a) ■

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