

The Generalized Libera Transform on Hardy, Bergman and Bloch Spaces on the Unit Polydisc

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Abstract. In this note we consider the boundedness of Libera transform Λ_{z^0} on Hardy, Bergman and a -Bloch spaces of analytic functions on the unit polydisk.

Keywords: *Analytic function, Hardy space, Bergman space, Libera transform, polydisk, bounded operator*

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1. Introduction and preliminaries

Let $U^1 = U$ be the unit disk in the complex plane \mathbb{C} , $dm(z) = \frac{1}{\pi} drd\theta$ the normalized Lebesgue measure on U , U^n the unit polydisk in the complex vector space \mathbb{C}^n and $H(U^n)$ the space of all analytic functions on U^n . We write

$$\begin{aligned}z \cdot w &= (z_1 w_1, \dots, z_n w_n) \text{ for } z, w \in \mathbb{C}^n \\e^{i\theta} &= (e^{i\theta_1}, \dots, e^{i\theta_n}) \\dt &= dt_1 \cdots dt_n \\d\theta &= d\theta_1 \cdots d\theta_n\end{aligned}$$

and r, t, s, α, λ are vectors in \mathbb{C}^n . If we write $0 \leq r < 1$, where $r = (r_1, \dots, r_n)$, it means $0 \leq r_j < 1$ for $j = 1, \dots, n$.

For $z^0 \in \overline{U}^n$ fixed and $f \in H(U^n)$ we define the linear operator $f \rightarrow \Lambda_{z^0}(f)$ by

$$\Lambda_{z^0}(f)(z) = \frac{1}{\prod_{j=1}^n (z_j - z_j^0)} \int_{z_1^0}^{z_1} \cdots \int_{z_n^0}^{z_n} f(t) dt \quad (z \in U^n). \quad (1)$$

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This operator is one of the most natural averaging operators on $H(U^n)$. For $n = 1$ and $z^0 = 0$ it is called the *Libera transform*, which is studied in geometrical function theory. The Libera transform on the unit disk was investigated, for example, in [2, 5, 7, 8, 10 - 12, 14] (mostly as formal adjoint of the Cesàro operator). The Cesàro operator and its generalizations on the unit disk and the unit polydisk has been studied extensively by many mathematicians in the past decade. One of the major interests in this operator is its behavior on function spaces (see, for example, [1 - 5, 7, 10 - 14]).

The *Hardy space* $H^p(U^n)$ ($0 < p < \infty$) is defined on U^n by

$$H^p(U^n) = \{f \mid f \in H(U^n) \text{ and } \|f\|_{H^p(U^n)} < \infty\}$$

where

$$\|f\|_{H^p(U^n)}^p = \sup_{0 \leq r < 1} \int_{[0, 2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta.$$

The weighted *Bergman space* $\mathcal{A}_\alpha^p(U^n)$ ($\alpha_1, \dots, \alpha_n > -1, p > 0$) is the space of all analytic functions f on U^n for which

$$\begin{aligned} \|f\|_{\mathcal{A}_\alpha^p(U^n)} &= \left[\int_{U^n} |f(z)|^p \prod_{j=1}^n (1 - |z_j|^2)^{\alpha_j} dm(z_j) \right]^{\frac{1}{p}} \\ &= \left[\frac{1}{\pi^n} \int_{[0, 1]^n} \left(\int_{[0, 2\pi]^n} |f(r \cdot e^{i\theta})|^p d\theta \right) \prod_{j=1}^n (1 - r_j^2)^{\alpha_j} r_j dr_j \right]^{\frac{1}{p}} \\ &< \infty. \end{aligned}$$

The a -Bloch space $\mathcal{B}^a = \mathcal{B}^a(U^n)$ is the space of all analytic functions f on U^n such that

$$b_a(f) = \max_{j=1, \dots, n} \sup_{z \in U^n} (1 - |z_j|^2)^a \left| \frac{\partial f}{\partial z_j}(z) \right| < \infty.$$

It is clear that \mathcal{B}^a is a normed space, modulo constant functions, and $\mathcal{B}^{a_1} \subset \mathcal{B}^{a_2}$ for $a_1 < a_2$.

2. Main and auxiliary results

In this paper we prove the following three results.

Theorem 1. *The generalized Libera operator is bounded on $H^p(U^n)$ if $p > 1$.*

Theorem 2. *The generalized Libera operator is bounded on $\mathcal{A}_\alpha^p(U^n)$ if $\alpha_j + 2 < p$ for all $j = 1, \dots, n$.*

Theorem 3. *The generalized Libera operator is bounded on $\mathcal{B}^a(U^n)$ if $a \in (0, 2)$.*

In the case $n = 1$, Theorems 1 - 3 were proved in [10, 14, 5], respectively. In order to prove in Section 3 the main results we need in the present section several auxiliary results which are incorporated in the following lemmas.

Lemma 1. *Let $f \in H^p(U^n)$ and $\phi_j : U \rightarrow U$ ($j = 1, \dots, n$) be analytic and non-constant, $\phi = (\phi_1, \dots, \phi_n)$ and $p > 0$. Then*

$$\|f \circ \phi\|_{H^p(U^n)}^p \leq \prod_{j=1}^n \left(\frac{1 + |\phi_j(0)|}{1 - |\phi_j(0)|} \right) \|f\|_{H^p(U^n)}^p. \tag{2}$$

Proof. By [15: Theorem XVII 5.16], for almost all $\theta_1 \in (0, 2\pi)$ and all $z_2, \dots, z_n \in U$, the function $f(z_1, z_2, \dots, z_n)$ converges to a regular function $f(e^{i\theta_1}, z_2, \dots, z_n)$ on U^{n-1} , as z_1 tends non-tangentially to $e^{i\theta_1}$. Hence

$$f(e^{i\theta_1}, \phi_2(z_2), \dots, \phi_n(z_n)) \in H(U^{n-1}) \quad \text{for a.a. } \theta_1 \in (0, 2\pi).$$

By [9: Theorem 1], for fixed $\theta_2, \dots, \theta_n$,

$$\begin{aligned} & \int_0^{2\pi} |f(\phi_1(\rho e^{i\theta_1}), \phi_2(\rho e^{i\theta_2}), \dots, \phi_n(\rho e^{i\theta_n}))|^p d\theta_1 \\ & \leq \frac{1 + |\phi_1(0)|}{1 - |\phi_1(0)|} \int_0^{2\pi} |f(e^{i\theta_1}, \phi_2(\rho e^{i\theta_2}), \dots, \phi_n(\rho e^{i\theta_n}))|^p d\theta_1. \end{aligned} \tag{3}$$

Applying [15: Theorem XVII 5.16] in the second coordinate, then integrating (3) from 0 to 2π in θ_2 , applying Fubini's theorem and [9: Theorem 1] on the right-hand side, in the second coordinate, we obtain

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} |f(\phi_1(\rho e^{i\theta_1}), \phi_2(\rho e^{i\theta_2}), \dots, \phi_n(\rho e^{i\theta_n}))|^p d\theta_1 d\theta_2 \\ & \leq \prod_{j=1}^2 \frac{1 + |\phi_j(0)|}{1 - |\phi_j(0)|} \int_0^{2\pi} \int_0^{2\pi} |f(e^{i\theta_1}, e^{i\theta_2}, \phi_3(\rho e^{i\theta_n}), \dots, \phi_n(\rho e^{i\theta_n}))|^p d\theta_1 d\theta_2. \end{aligned}$$

Repeating the above arguments and using [15: Theorem XVII 5.24], we obtain the result ■

Lemma 2. *Let $f \in H^p(U^n)$ ($p > 0$). Then*

$$|f(z)| \leq \frac{C \|f\|_{H^p(U^n)}}{\prod_{i=1}^n (1 - |z_i|)^{\frac{1}{p}}} \quad (z \in U^n) \tag{4}$$

for some constant $C > 0$ independent of f .

Proof. If $f \in H^p(U)$, then

$$|f(\zeta)| \leq \frac{c_p \|f\|_p}{(1 - |\zeta|)^{\frac{1}{p}}} \quad (\zeta \in U) \quad (5)$$

where the constant $c_p > 0$ depends only of p (see [6: p. 36]). Without loss of generality we may assume $n = 2$. If $f(z_1, z_2) \in H^p(U^2)$, then the function f is analytic in each variable separately on the unit disk. Hence, for each fixed $\zeta_2 \in U$ and every $\zeta_1 \in r_1 U$ ($0 < r_1 < 1$), by (5) we have

$$|f(\zeta_1, \zeta_2)|^p \leq \frac{c_p^p r_1}{r_1 - |\zeta_1|} \int_0^{2\pi} |f(r_1 e^{i\theta_1}, \zeta_2)|^p \frac{d\theta_1}{2\pi}. \quad (6)$$

Similarly, for each fixed $r_1 e^{i\theta_1} \in U$ and every $\zeta_2 \in r_2 U$ ($0 < r_2 < 1$), we have

$$|f(r_1 e^{i\theta_1}, \zeta_2)|^p \leq \frac{c_p^p r_2}{r_2 - |\zeta_2|} \int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^p \frac{d\theta_2}{2\pi}. \quad (7)$$

Combining (6) and (7) and since $f \in H^p(U^2)$, we obtain the result ■

Lemma 3. For each polynomial f , $\Lambda_{z^0}(f)$ can be written in the form

$$\Lambda_{z^0}(f)(z) = \int_{(0,1)^n} f(\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n)) dt \quad (8)$$

where $\psi_{t_j}(z_j) = t_j z_j + (1 - t_j) z_j^0$ with $t_j \in (0, 1]$ and $z_j \in U$. Let $p > 1$. Then for every $f \in H^p(U^n)$ the above integral is finite and defines an analytic function on U .

Proof. It is clear that $\Lambda_{z^0}(f)(z)$ is also polynomial in this case and that integral (8) is finite for each $z \in U^n$. In integral (1) we choose the path of integration between z and z^0 as

$$(t_1 z_1 + (1 - t_1) z_1^0, \dots, t_n z_n + (1 - t_n) z_n^0) \quad (t_1, \dots, t_n \in (0, 1]).$$

Hence

$$\Lambda_{z^0}(f)(z) = \int_{(0,1)^n} f(\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n)) dt.$$

Let $f \in H^p(U^n)$. By (4) we have

$$|f(\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n))| \leq \frac{C \|f\|_{H^p(U^n)}}{\prod_{j=1}^n (1 - |\psi_{t_j}(z_j)|)^{\frac{1}{p}}} \leq \frac{C \|f\|_{H^p(U^n)}}{\prod_{j=1}^n t_j^{\frac{1}{p}} (1 - |z_j|)^{\frac{1}{p}}}$$

for each $z \in U^n$ and $t_j \in (0, 1]$. Hence from the hypothesis it follows that the integral in (8) is finite for each $z \in U^n$ and that it defines an analytic function on U^n ■

Lemma 4. Let $L_t(f)(z) = f(\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n))$ and $1 \leq p < \infty$. Then

$$\|L_t\|_{H^p \rightarrow H^p} \leq C \prod_{j=1}^n \frac{1}{t_j^{1/p}} \quad (t_j \in (0, 1]) \tag{9}$$

for some constant $C > 0$.

Proof. Let $\psi_t(z) = (\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n))$. By Lemma 1,

$$\begin{aligned} \|f \circ \psi_t\|_{H^p(U^n)}^p &\leq \prod_{j=1}^n \left(\frac{1 + |(1-t_j)z_j^0|}{1 - |(1-t_j)z_j^0|} \right) \|f\|_{H^p(U^n)}^p \\ &\leq \prod_{j=1}^n \left(\frac{2-t_j}{t_j} \right) \|f\|_{H^p(U^n)}^p \\ &\leq 2^n \prod_{j=1}^n \frac{1}{t_j} \|f\|_{H^p(U^n)}^p \end{aligned}$$

as desired ■

Remark 1. Let $z_j^0 \in \partial U$ and let λ_j be complex numbers such that $\Re(\lambda_j) < \frac{1}{p}$ ($j = 1, \dots, n$). It is well known that the functions

$$f_\lambda(z) = \prod_{j=1}^n \frac{1}{(z_j^0 - z_j)^{\lambda_j}}$$

belong to $H^p(U^n)$ and

$$f_\lambda(\psi_{t_1}(z_1), \dots, \psi_{t_n}(z_n)) = \prod_{j=1}^n \left(\frac{1}{t_j} \right)^{\lambda_j} f_\lambda(z).$$

It follows that $\|L_t\|_{H^p \rightarrow H^p} \geq \prod_{j=1}^n \frac{1}{t_j^{1/p}}$. From this and Lemma 4 we obtain that in this case

$$\prod_{j=1}^n \frac{1}{t_j^{1/p}} \leq \|L_t\|_{H^p \rightarrow H^p} \leq C \prod_{j=1}^n \frac{1}{t_j^{1/p}}.$$

Lemma 5. Let $\psi_j : U \rightarrow U$ ($j = 1, \dots, n$) be analytic and non-constant, $\psi = (\psi_1, \dots, \psi_n)$. Then the norm of the operator $f \rightarrow T_\psi(f) = f \circ \psi$ on \mathcal{A}_α^p satisfies

$$\|T_\psi\|_{\mathcal{A}_\alpha^p \rightarrow \mathcal{A}_\alpha^p} \leq \prod_{j=1}^n C_j \left(\frac{\|\psi_j\|_\infty + |\psi_j(0)|}{\|\psi_j\|_\infty - |\psi_j(0)|} \right)^{\frac{\alpha_j+2}{p}}$$

where

$$C_j = \begin{cases} 1 & \text{if } \alpha_j \geq 0 \\ \prod_{j=1}^n (\|\psi_j\|_\infty + |\psi_j(0)|)^{\frac{\alpha_1}{p}} (\|\psi_j\|_\infty + 3|\psi_j(0)|)^{-\frac{\alpha_j}{p}} & \text{if } -1 < \alpha_j < 0. \end{cases}$$

Proof. Let $f \in \mathcal{A}_\alpha^p$. Then by [11: Lemma 1], for almost all fixed z_2, \dots, z_n ,

$$\begin{aligned} & \int_U |f(\psi_1(z_1), \psi_2(z_2), \dots, \psi_n(z_n))|^p (1 - |z_1|^2)^{\alpha_1} dm(z_1) \\ & \leq C_1 \left(\frac{\|\psi_1\|_\infty + |\psi_1(0)|}{\|\psi_1\|_\infty - |\psi_1(0)|} \right)^{\alpha_1 + 2} \\ & \quad \times \int_U |f(z_1, \psi_2(z_2), \dots, \psi_n(z_n))|^p (1 - |z_1|^2)^{\alpha_1} dm(z_1) \end{aligned} \quad (10)$$

where

$$C_1 = \begin{cases} 1 & \text{if } \alpha_1 \geq 0 \\ (\|\psi_1\|_\infty + |\psi_1(0)|)^{\alpha_1} (\|\psi_1\|_\infty + 3|\psi_1(0)|)^{-\alpha_1} & \text{if } -1 < \alpha_1 < 0. \end{cases}$$

Multiplying (10) by $(1 - |z_2|^2)^{\alpha_2}$, integrating obtained inequality over U in z_2 , then applying Fubini's theorem and [11: Lemma 1] on the right-hand side, in the second coordinate, we obtain that for almost all z_3, \dots, z_n

$$\begin{aligned} & \int_{U^2} |f(\psi_1(z_1), \psi_2(z_2), \dots, \psi_n(z_n))|^p (1 - |z_1|^2)^{\alpha_1} (1 - |z_2|^2)^{\alpha_2} dm(z_1) dm(z_2) \\ & \leq \prod_{j=1}^2 C_j \left(\frac{\|\psi_j\|_\infty + |\psi_j(0)|}{\|\psi_j\|_\infty - |\psi_j(0)|} \right)^{\alpha_j + 2} \\ & \quad \times \int_{U^2} |f(z_1, z_2, \psi_3(z_3), \dots, \psi_n(z_n))|^p (1 - |z_1|^2)^{\alpha_1} (1 - |z_2|^2)^{\alpha_2} dm(z_1) dm(z_2). \end{aligned}$$

Repeating this procedure we obtain the result ■

3. Proofs of the main results

In this section, we prove the main results of the paper.

Proof of Theorem 1. Let $p > 1$. Then, by Minkowski inequality and Lemma 4,

$$\begin{aligned} \|\Lambda_{z^0}(f)\|_{H^p(U^n)} &= \sup_{0 \leq r < 1} \left(\int_{[0,2\pi]^n} |\Lambda_{z^0}(f)(r \cdot e^{i\theta})|^p d\theta \right)^{\frac{1}{p}} \\ &= \sup_{0 \leq r < 1} \left(\int_{[0,2\pi]^n} \left| \int_{(0,1]^n} f(\psi_t(r \cdot e^{i\theta})) dt \right|^p d\theta \right)^{\frac{1}{p}} \\ &\leq \sup_{0 \leq r < 1} \int_{(0,1]^n} \left(\int_{[0,2\pi]^n} |f(\psi_t(r \cdot e^{i\theta}))|^p d\theta \right)^{\frac{1}{p}} dt \\ &\leq C \|f\|_{H^p(U^n)} \int_{(0,1]^n} \prod_{j=1}^n t_j^{-\frac{1}{p}} dt \\ &= C_1 \|f\|_{H^p(U^n)} \end{aligned}$$

as desired ■

Remark 2. Theorem 1 is not true in the case $p \in (0, 1]$. Indeed, if $p \in (0, 1)$ and $z_j^0 \in \partial U$ for $j \in S \subseteq \{1, \dots, n\}$, then $f_1(z) = \prod_{j \in S} \frac{1}{z_j^0 - z_j} \in H^p(U^n)$. On the other hand, it is easy to see that Λ_{z^0} have no sense on f_1 , that is, Λ_{z^0} is unbounded on $H^p(U^n)$ for $p \in (0, 1)$.

Let now $p = 1$ and $z_j^0 \in \partial U$ for $j \in S \subseteq \{1, \dots, n\}$. It is known [10] that the operator

$$F_{z_0}(f)(z) = \frac{1}{z - z_0} \int_{z_0}^z f(t) dt \quad (z \in U)$$

is not bounded on $H^1(U)$. Using this fact we see that the operator Λ_{z^0} is not bounded on $H^1(U^n)$.

Proof of Theorem 2. Let $\alpha_j + 2 < p$ for $j = 1, \dots, n$. It is clear that $p > 1$ in this case. By the Minkowski inequality and Lemma 5, where $\psi = \psi_t$,

we obtain

$$\begin{aligned} \|\Lambda_{z^0}(f)\|_{\mathcal{A}_\alpha^p} &= \left(\int_{U^n} \left| \int_{(0,1)^n} f(\psi_t(z)) dt \right|^p \prod_{j=1}^n (1 - |z_j|)^{\alpha_j} dm(z_j) \right)^{\frac{1}{p}} \\ &\leq \int_{(0,1)^n} \|f \circ \psi_t\|_{\mathcal{A}_\alpha^p} dt \\ &\leq C \|f\|_{\mathcal{A}_\alpha^p} \int_{(0,1)^n} \prod_{j=1}^n \left(\frac{2-t_j}{t_j} \right)^{\frac{\alpha_j+2}{p}} dt_j \\ &\leq C \|f\|_{\mathcal{A}_\alpha^p} \prod_{j=1}^n \int_0^1 t_j^{-\frac{\alpha_j+2}{p}} dt_j. \end{aligned}$$

Since the last integrals converge for $\alpha_j + 2 < p$, we obtain the result in this case ■

Remark 3. Theorem 2 is not true if there is a $j \in \{1, \dots, n\}$ such that $\alpha_j + 2 \geq p$. Indeed, first let $\alpha_j + 2 > p$ and $z_j^0 \in \partial U$ for a $j \in \{1, \dots, n\}$, and $f_1(z_j) = \frac{1}{z_j^0 - z_j}$. Then by polar coordinates and [6: p. 84/Lemma 3] we obtain

$$\begin{aligned} &\int_{U^n} \frac{1}{|z_j^0 - z_j|^p} \prod_{k=1}^n (1 - |z_k|)^{\alpha_k} dm(z_k) \\ &= C \int_0^1 (1 - r_j^2)^{\alpha_j} \int_{-\pi}^\pi |1 - r_j e^{i\theta_j}|^{-p} d\theta_j r_j dr_j \\ &< C \int_0^1 (1 - r_j)^{\alpha_j+1-p} dr_j \\ &< \infty \end{aligned}$$

for $\alpha_j + 2 > p$. Hence $f_1 \in \mathcal{A}_\alpha^p$ if $\alpha_j + 2 > p$ for a $j \in \{1, \dots, n\}$. On the other hand, Λ_{z^0} have no sense on the function f_1 and consequently $\Lambda_{z^0}(f_1)$ is not in \mathcal{A}_α^p in this case.

Now let $\alpha_j + 2 = p$ and $z_j^0 \in \partial U$ for a $j \in \{1, \dots, n\}$. Note that $p > 1$ in this case. Let

$$f_2(z_j) = \frac{1}{z_j^0 - z_j} \left(\frac{1}{z_j} \log \frac{1}{z_j^0 - z_j} \right)^{-1}.$$

The only singularity of f_2 is at $z_j = z_j^0$. By polar coordinates centered at $z_j = z_j^0$, we see that the integral

$$\int_{U^n} |f_2(z)|^p \prod_{j=1}^n (1 - |z_j|)^{\alpha_j} dm(z_j) = C \int_U |f_2(z_j)|^p (1 - |z_j|)^{p-2} dm(z_j)$$

is equiconvergent to

$$\int_0^{\frac{1}{2}} \frac{d\rho}{\rho(\ln \frac{1}{\rho})^p} < \infty$$

since $p > 1$. Hence $f_2 \in \mathcal{A}_\alpha^p$, where $\alpha = (\alpha_1, \dots, \alpha_{j-1}, p - 2, \alpha_{j+1}, \dots, \alpha_n)$. On the other hand, it is easy to see that Λ_{z^0} have no sense on the function f_2 and consequently $\Lambda_{z^0}(f_2)$ is not in \mathcal{A}_α^p for $\alpha_j + 2 = p$.

Proof of Theorem 3. Let $f \in \mathcal{B}^a$ for $a \in (0, 2)$. Then

$$\begin{aligned} b_a(\Lambda_{z^0}(f)) &= \max_{j=1, \dots, n} \sup_{z \in U^n} (1 - |z_j|^2)^a \left| \frac{\partial \Lambda_{z^0}(f \circ \psi_t)}{\partial z_j}(z) \right| \\ &\leq \max_{j=1, \dots, n} \int_{(0,1]^n} \sup_{z \in U^n} (1 - |z_j|^2)^a \left| \frac{\partial (f \circ \psi_t)(z)}{\partial z_j} \right| dt \\ &\leq 2^a \max_{j=1, \dots, n} \int_{(0,1]^n} \sup_{z \in U^n} \left(\frac{1 - |\psi_j(z_j)|^2}{t_j} \right)^a \left| \frac{\partial f}{\partial z_j}(\psi_t(z)) \right| |\psi_j'(z_j)| dt \\ &\leq 2^a b_a(f) \max_{j=1, \dots, n} \int_{(0,1]^n} t_j^{1-a} dt \\ &= \frac{2^{2a} b_a(f)}{2 - a} \end{aligned}$$

and hence $\Lambda_{z^0}(f) \in \mathcal{B}^a$ ■

Remark 4. Theorem 3 is not true in the case $a \in [2, \infty)$. Indeed, if $a \in [2, \infty)$ and $z_j^0 \in \partial U$ for a $j \in \{1, \dots, n\}$, then the function $f_1(z_j) = \frac{1}{z_j^0 - z_j}$ belongs to $\mathcal{B}^a(U^n)$. However, this function is not in the domain of the definition of Λ_{z^0} .

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