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On Positive-off-Diagonal Operators on Ordered Normed Spaces

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Abstract. On a normed space X ordered by a cone K we consider a continuous linear operator $A: X \to X$ of the following kind: If a positive continuous functional f attains 0 on some positive element x, then $f(Ax) \geq 0$. If X is a vector lattice, then such operators can be represented as $sI + B$, where B is a positive operator, I is the identity and $s \in \mathbb{R}$. We generalize this assertion for weaker assumptions on X, using the Riesz decomposition property.

Keywords: *Positive-off-diagonal operators, ordered normed spaces, Riesz decomposition property*

AMS subject classification: 47B60, 46B40, 47D03

1. Introduction

In the present paper let $(X, K, \|\cdot\|)$ be an ordered normed space, i.e. X is a real vector space, $\|\cdot\|$ is a norm on X and K is a cone in X, i.e. K is a wedge (i.e. $x, y \in K$ and $\lambda, \mu \geq 0$ imply $\lambda x + \mu y \in K$) and $K \cap (-K) = \{0\}$. Furthermore, let K be closed. By means of the cone K a partial order is introduced in X. We will use the notations $x \in K$ and $x \geq 0$ synonymously and write $x > 0$ instead of $0 \neq x \geq 0$. As usual, X' denotes the vector space of all continuous linear functionals on X and $\mathcal{L}(X)$ the vector space of all continuous linear operators on X. An operator $B \in \mathcal{L}(X)$ is called positive if $B(K) \subseteq K$; a functional $f \in X'$ is called positive if $f(K) \subseteq [0, +\infty)$. We write $B \geq 0$ and $f \geq 0$, correspondingly. The wedge of all positive functionals in X' is denoted by K'. On $(X, K, \|\cdot\|)$ operators of the following kind are considered.

Definition 1.1. An operator $A \in \mathcal{L}(X)$ is called *positive-off-diagonal* if $x \in K$ and $f \in K'$ with $f(x) = 0$ imply $f(Ax) \geq 0$.

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The notion "positive-off-diagonal" is motivated as follows: For $X = \mathbb{R}^n$ and the standard cone $K = \mathbb{R}^n_+$, a matrix $A = (a_{ij})_{n,n}$ is a positive-off-diagonal operator if and only if $a_{ij} \geq 0$ for $i \neq j$. Note that on $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$ an invertible operator A, where the operator $-A$ is positive-off-diagonal and $A^{-1} \geq 0$, can be represented as a non-singular M-matrix and vice versa. [1)] The set of all positive-off-diagonal operators on $(X, K, \|\cdot\|)$ is a wedge in $\mathcal{L}(X)$, but it is not a cone, since the identity I and also $-I$ are both positive-off-diagonal operators. If $A = sI + B$ with $B \geq 0$ and $s \in \mathbb{R}$, then the operator A is positive-off-diagonal. If $X = \mathbb{R}^n$ and K is a polyedral generating cone in $\mathbb{R}^n$, then the converse is also true, i.e. every positive-off-diagonal operator A [2)] can be represented as $A = sI + B$ where $s \in \mathbb{R}$ and $B \geq 0$ (see [6]). For several other cones in $\mathbb{R}^n$, in particular circular ones, this implication is not true (see [6] or Example 4.1 below).

On $(X, K, \|\cdot\|)$ consider for an operator $A \in \mathcal{L}(X)$ the properties

(i) A is a positive-off-diagonal operator

(ii) $\|A\|I + A \geq 0$.

Obviously, (ii) implies (i). If X is a Banach lattice, (i) and (ii) are equivalent (see, e.g., [4: C-II, Theorem 1.11]). We shall prove the implication (i) $\Rightarrow$ (ii) for operators on certain ordered normed spaces X that need not be vector lattices. Note that for any $s \leq -\|A\|$ condition (ii) implies $B = -sI + A \geq 0$, hence $A = sI + B$ with $B \geq 0$.

There is a close connection between positive-off-diagonal operators and the theory of positive operator semigroups. Namely, consider the condition

(iii) A is the generator of a semigroup $(T(t))_{t\geq 0}$ of positive operators on X.

Obviously, (ii) implies (iii) since

$$T(t) = e^{tA} = e^{t(A+\|A\|)}e^{-t\|A\|} \geq 0 \qquad (t \geq 0).$$

Now assume (iii) and consider $x \in K$ and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t}.$$

1) If $(X, K, \|\cdot\|)$ is an ordered normed space and $A \in \mathcal{L}(X)$ with $A = sI - B$, where $B \geq 0$ and $s > r(B)$ (here $r(B)$ denotes the spectral radius of the operator B), then we call A an M-operator. In the space $X = \mathbb{R}^n$ with the cone $K = \mathbb{R}^n_+$ the notion M-matrix is used.

2) Note that in Matrix Theory instead of "positive-off-diagonal" the notion "cross-positive" is used.

Then for $f \in K'$ with $f(x) = 0$ one has

$$f(Ax) = \lim_{t \downarrow 0} \frac{f(T(t)x) - f(x)}{t} = \lim_{t \downarrow 0} \frac{f(T(t)x)}{t} \geq 0,$$

hence (iii) implies (i). In certain ordered normed spaces, e.g. if K has a non-empty interior [3: Theorem 7.27], condition (i) implies (iii). In general (iii) does not imply (ii) (see Example 4.1). If one has the implication (i) $\Rightarrow$ (ii) for some ordered normed space, then this yields the equivalence of all properties (i), (ii) and (iii).

2. Preliminaries

Recall some definitions and notations of the theory of ordered vector spaces, where our terminology mainly follows that of [1, 7]. Let (X, K) be a real vector space ordered by a cone K. For given $a, b \in X$ with $a \leq b$, let $[a, b] = \{x \in X : a \leq x \leq b\}$. (X, K) is called Archimedean if $nx \in [0, y]$ for all $n \geq 1$ and some $y \in K$ implies $x = 0$. An element $u > 0$ is an order unit if for every $x \in X$ there exists a number $\lambda > 0$ such that $x \in [-\lambda u, \lambda u]$. K is generating if each $x \in X$ can be represented as $x = y - z$ where $y, z \in K$. The ordered vector space (X, K) is said to satisfy the Riesz Decomposition Property, if for every $y, x_1, x_2 \in K$ with $y \leq x_1 + x_2$ there exist $y_1, y_2 \in K$ such that $y = y_1 + y_2$ and $y_i \leq x_i \quad (i = 1, 2)$.

An element $x > 0$ is called an extremal of the cone K, if $y \in K$ and $y \leq x$ imply $y = \lambda x$ for some $\lambda \geq 0$, i.e. x is an extremal of K if and only if it generates an extreme ray of K. A subset D of the cone K is called a base of K if D is a non-empty convex set such that each $x > 0$ has a unique representation $x = \lambda y$ with $y \in D$ and $\lambda > 0$. If K possesses a base D with extreme points, then every extreme point of D is an extremal of K.

Now let (X, K) be a vector lattice. Note that every Dedekind complete vector lattice is Archimedean. A subset S of X is called solid, if $y \in X$, $x \in S$ and $|y| \leq |x|$ imply $y \in S$. The band generated by a singleton $\{x\}$, i.e. the intersection of all bands that contain the element x, will be denoted by B_x. Note that $B_x = \{y \in X : |y| \wedge n|x| \uparrow_n |y|\}$. Two vectors x and y are called disjoint, written $x \perp y$, if $|x| \wedge |y| = 0$. The disjoint complement of a set $S \subset X$ is defined as $S^d = \{x \in X : x \perp y \text{ for all } y \in S\}$. A band B in X is called a projection band if $X = B \oplus B^d$.

Essentially we will make use of the following assertion (see [1: Theorem 3.8]):

Proposition 2.1. *Every band in a Dedekind complete vector lattice is a projection band.*

Proposition 2.2. *Let (X,K) be an Archimedean vector lattice and $0 < x \in X$. The element x is an extremal of K if and only if $B_x = \{\lambda x : \lambda \in \mathbb{R}\}$.*

Proof. Let x be an extremal of K and $y \in B_x$. Then $|y| = \sup\{|y| \wedge n|x| : n \in \mathbb{N}\}$. Since x is an extremal, $0 \le |y| \wedge n|x| \le n|x| = nx$ implies the existence of a number $\alpha_n \in \mathbb{R}$ such that $|y| \wedge n|x| = \alpha_n x$. We show that the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is bounded. If the contrary is assumed, then for each $m \in \mathbb{N}$ there exists an $n \in \mathbb{N}$ such that $mx \le \alpha_n x = |y| \wedge n|x| \le |y|$. Since X is Archimedean, we conclude $x = 0$ which is a contradiction. If C denotes an upper bound of $(\alpha_n)_{n\in\mathbb{N}}$, then

$$|y| = \sup\{|y| \wedge n|x| : n \in \mathbb{N}\} = \sup\{\alpha_n x : n \in \mathbb{N}\} \le Cx.$$

Since x is an extremal, we get $|y| = \alpha x$ for some $\alpha \ge 0$. Finally, y^+ and y^- are multiples of x as well because of $0 \le y^+, y^- \le |y| = \alpha x$. Hence $y = y^+ - y^- = \lambda x$ for some $\lambda \in \mathbb{R}$.

Vice versa, let $x \in K$ and $B_x = \{\lambda x \colon \lambda \in \mathbb{R}\}$. Obviously, since B_x is solid, $0 \le y \le x$ implies $y \in B_x$, hence $y = \lambda x$ ∎

Now let $(X, K, \|\cdot\|)$ be an ordered normed space. A cone K is called non-flat, if there exists a constant $\kappa > 0$ such that each $x \in X$ possesses a representation $x = y - z$ with $y, z \in K$ and $\|y\|, \|z\| \le \kappa\|x\|$. If K has a non-empty interior, then K is non-flat (and generating, obviously). K is called normal, if the norm in X is semi-monotone on K, i.e. there exists a constant (of semi-monotony) N such that $0 \le x \le y$ implies $\|x\| \le N\|y\|$. Note that K' is a cone in X' if and only if X is the norm closure of $K - K$. We will call a non-empty subset $M \subseteq K'$ total if $x \in X$ and $f(x) \ge 0$ for every $f \in M$ imply $x \in K$.

A norm $\|\cdot\|$ on a vector lattice is a lattice norm if $|x| \le |y|$ implies $\|x\| \le \|y\|$. If $(X, K, \|\cdot\|)$ is a normed vector lattice, i.e. a vector lattice equipped with a lattice norm, then for any two disjoint elements $x, y \in X$ one has

$$\|x\| \le \|x + y\|. \tag{1}$$

This follows immediately from [1: Theorem 1.4] since

$$0 = |x| \wedge |y| = \frac{1}{2}(|x+y| - |x-y|)$$

implies $|x - y| = |x + y|$ and

$$|x| \le |x| \vee |y| = \frac{1}{2}(|x+y| + |x-y|) = \frac{1}{2}(2|x+y|) = |x+y|.$$

Since $\|\cdot\|$ is a lattice norm we get (1) ∎

Recall the following result of Riesz and Kantorovich [7: Theorem V.3.1]:

Proposition 2.3. *If an ordered normed space $(X, K, \|\cdot\|)$ with a non-flat and normal cone K satisfies the Riesz decomposition property, then (X', K') is a Dedekind complete vector lattice.*

In the case of a Banach space X and a closed cone K, due to a theorem of Krein [7: Theorem III.2.1] the condition on K to be non-flat can be replaced by the condition on K to be generating. Note that because of $\|x^+\|, \|x^-\| \leq \| \, |x| \, \| = \|x\|$ any normed vector lattice satisfies all assumptions of Proposition 2.3.

If (X', K') is a Dedekind complete vector lattice, then for $f, g \in X'$ and $x \in K$ one has

$$(f \wedge g)(x) = \inf \big\{ f(x') + g(x - x') : \, x' \in [0, x] \big\}$$

and

$$|f|(x) = \sup \big\{ |f(y)| : \, |y| \leq x \big\}.$$

In an ordered normed space $(X, K, \|\cdot\|)$ with a closed cone K we consider for $0 \neq f \in K'$ the following properties:

(I) $f^{-1}(0) = (f^{-1}(0) \cap K) - (f^{-1}(0) \cap K)$. This means that the part $f^{-1}(0) \cap K$ of the boundary of K generates the corresponding hyperplane $f^{-1}(0)$ of X.

(II) f is an extremal of K'.

Property (I) always implies property (II). Indeed, for $g \in K'$ with $0 \leq g \leq f$ from $f(x) = 0$ for some $x \in X$ one has $x = x_1 - x_2$ with $x_1, x_2 \in f^{-1}(0) \cap K$, hence $0 = f(x_1) \geq g(x_1) \geq g(x)$. Similarly, $f(-x) = 0$ yields $0 \geq g(-x)$. This implies $g(x) = 0$. Therefore we can conclude: Either $g = 0$ or g and f have the same kernel. Hence $g = \lambda f$ for some $\lambda \in \mathbb{R}$.

In a Banach lattice property (II) also yields property (I). Indeed, for any $x \in X$ one has $x = x^+ - x^-$, where $x^+ = x \vee 0$. Suppose that f is an extreme element in K' and $f(x) = 0$. Then f is a lattice homomorphism and we get $f(x^+) = f(0) \vee f(x) = 0$. Accordingly, $f(x^-) = 0$. However, in general property (II) does not imply property (I) (see Example 4.1).

We will say that a cone K in an ordered normed space is b-generating if for every extremal f of K' property (I) is satisfied. [3] Note that the Riesz decomposition property does generally not imply that K is b-generating (consider, e.g., the space $X = C^1[0, 1]$ of all continuously differentiable functions on $[0, 1]$, ordered by the cone of non-negative functions).

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[3] This property is used, e.g., in [5].

3. Main results

We start with the main result on positive-off-diagonal operators.

Theorem 3.1. *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property and let K be a closed normal non-flat b-generating cone. Assume that there exists a total set of extremals of K'. Then for any operator $A \in \mathcal{L}(X)$ the conditions*

(i) *A is a positive-off-diagonal operator*

(ii) $\|A\|I + A \geq 0$

are equivalent.

An ordered normed space that satisfies all assumptions of Theorem 3.1 need not be a vector lattice (see Example 4.2 below). For the proof of Theorem 3.1 we need some preliminary results.

Lemma 3.2. *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property. Furthermore, let the cone K be closed, normal and b-generating. Then for every extremal f of K' there exists some constant $C > 0$ such that the following property is satisfied: For every $y \in f^{-1}(1) \cap K$ there exists an element $z \in f^{-1}(1)$ such that $z \leq y$ and $\|z\| \leq C$.*

Proof. Let f be an extremal of K'. Fix some element $y_0 \in f^{-1}(1) \cap K$ and put $C = N\|y_0\|$, where N is the constant of semi-monotony of the norm. Let $y \in f^{-1}(1) \cap K$. The element $x = y - y_0$ lies in $f^{-1}(0)$ and can be decomposed into $x = x_1 - x_2$, where $x_1, x_2 \geq 0$ and $x_1, x_2 \in f^{-1}(0)$, since K is b-generating. Hence we get $0 \leq y \leq y + x_2 = x_1 + y_0$. The Riesz decomposition property yields $y = w + z$, where $0 \leq w \leq x_1$ and $0 \leq z \leq y_0$. Due to $f(x_1) = 0$ one has $f(w) = 0$, hence $f(z) = 1$. Moreover, $\|z\| \leq N\|y_0\| = C$ ∎

Theorem 3.3. *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property. Furthermore, let the cone K be closed, normal, non-flat and b-generating and let f be an extremal of K'. If $g \in X'$ is such that $f \perp g$ and $g(x) \geq 0$ for each $x \in f^{-1}(0) \cap K$, then $g \in K'$.*

Proof. Proposition 2.3 ensures that (X', K') is a vector lattice. Let f be an extremal of K' and $g \in X'$ such that $g \neq 0$ and $f \perp g$. For $x > 0$ we get

$$0 = (f \wedge |g|)(x) = \inf\big\{f(x') + |g|(x - x') : x' \in [0, x]\big\}.$$

Hence for every $n \in \mathbb{N}$ there exists some $x_n \in [0, x]$ such that $f(x_n) + |g|(x - x_n) \leq \frac{1}{n}$. This implies $f(x_n) \leq \frac{1}{n}$ and also

$$|g(x) - g(x_n)| = |g(x - x_n)| \leq |g|(x - x_n) \leq \frac{1}{n}. \qquad (2)$$

If $f(x_n) = 0$, then the premise ensures $g(x_n) \geq 0$. If $f(x_n) > 0$, we obtain a lower bound for $g(x_n)$ as follows: For the extremal f of K' let C be the constant from Lemma 3.2. Since $\frac{1}{f(x_n)} x_n \in K$ and $f(\frac{1}{f(x_n)} x_n) = 1$, we get an element $z_n \in f^{-1}(1)$ such that $z_n \leq \frac{1}{f(x_n)} x_n$ and $\|z_n\| \leq C$. Then $w_n = x_n - f(x_n) z_n$ lies in $f^{-1}(0) \cap K$ and one has

$$\|x_n - w_n\| = f(x_n)\|z_n\| \leq f(x_n) C \leq \frac{C}{n}.$$

The premise ensures $g(w_n) \geq 0$. Since

$$|g(x_n) - g(w_n)| \leq \|g\| \, \|x_n - w_n\| \leq \|g\| \frac{C}{n}$$

we conclude

$$g(x_n) \geq -\frac{\|g\| C}{n}. \tag{3}$$

Now we prove the assertion by way of contradiction. Suppose that there exists a vector $x > 0$ such that $g(x) < 0$. Put $n > \frac{\|g\|C+1}{-g(x)}$. Then $-g(x) > \frac{\|g\|C}{n} + \frac{1}{n}$. For the corresponding x_n inequality (3) shows

$$g(x_n) - g(x) > \frac{-\|g\|C}{n} + \frac{\|g\|C}{n} + \frac{1}{n} = \frac{1}{n}$$

which contradicts (2) ∎

Now we come to the

Proof of Theorem 3.1. We already mentioned that condition (ii) implies condition (i). Now assume that A is a positive-off-diagonal operator. We have to show $\|A\|x + Ax \in K$ for every $x \in K$. Since there exists a total set M of extremals of K' it suffices to show $f(\|A\|x + Ax) \geq 0$ for each $f \in M$. Fix some $f \in M$. Since (X', K') is a Dedekind complete vector lattice, from Proposition 2.1 follows that B_f is a projection band in X', i.e. $X' = B_f \oplus B_f^d$. Since (X', K') is Archimedean and f is an extremal of K', Proposition 2.2 yields $B_f = \{\lambda f : \lambda \in \mathbb{R}\}$. This allows us to represent the element $A^* f$ as $A^* f = f_1 + f_2$, where $f_1 = \lambda f$ and $f_2 \perp f$. If we show both

(a) f_2 is positive

(b) $|\lambda| \leq \|A\|$,

then we can conclude

$$\begin{aligned} f(\|A\|x + Ax) &= \|A\| f(x) + (A^* f)(x) \\ &= \|A\| f(x) + \lambda f(x) + f_2(x) \\ &= (\|A\| + \lambda) f(x) + f_2(x) \\ &\geq 0. \end{aligned}$$

Property (a): According to Theorem 3.3 it suffices to show $f_2(x) \geq 0$ for any $x \in K$ with $f(x) = 0$. In this case $f_1(x) = 0$ and $f(Ax) \geq 0$ since A is a positive-off-diagonal operator. Hence

$$f_2(x) = (A^* f)(x) - f_1(x) = f(Ax) \geq 0.$$

Consequently, $f_2 \in K'$.

Property (b): From inequality (1) we conclude

$$|\lambda| \, \|f\| = \|\lambda f\| \leq \|\lambda f + f_2\| = \|A^* f\| \leq \|A\| \, \|f\|$$

and hence $|\lambda| \leq \|A\|$ ∎

If K is closed, then K' is total [7: Section II.4]. If, additionally, there exists an interior point u of K, then $F_u = \{f \in K' : f(u) = 1\}$ is a $\sigma(X', X)$-compact base of K' [8: Theorem II.3.2] and the set of extreme points of F_u is a total set of extremals of K'. Hence the following conclusion is obvious.

Corollary 3.4. *Let $(X, K, \|\cdot\|)$ be an ordered normed space that satisfies the Riesz decomposition property and let K be a closed normal b-generating cone with non-empty interior. Then for any positive-off-diagonal operator $A \in \mathcal{L}(X)$ one has $\|A\| I + A \geq 0$.*

4. Examples

First we present an example which shows that a positive-off-diagonal operator A in general can not be represented as $A = sI + B$ with a positive operator B and a number $s \in \mathbb{R}$, even if A operates on a finite-dimensional space.

Example 4.1. We consider the ordered normed space $(\mathbb{R}^3, K, \|\cdot\|)$, where

$$K = \left\{ t(x_1, x_2, 1) \colon x_1^2 + x_2^2 \leq 1 \;\text{ and }\; t \geq 0 \right\}$$

is a circular cone (see Figure 1) and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^3$. The cone K is closed, normal and has a non-empty interior. K is not b-generating and does not satisfy the Riesz decomposition property (see, e.g., [2]). Note that $K' = K$. Consider the operator given by the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $x \in K$ and $y \in K'$ such that $\langle x, y \rangle = 0$ and assume $x = (x_1, x_2, 1)$. Then $x_1^2 + x_2^2 = 1$ and $y = (-x_1, -x_2, 1)$. Hence $\langle Ax, y \rangle = x_1^2 \geq 0$, i.e. A is a positive-off-diagonal operator with respect to K. Note that A is the generator of a semigroup of positive operators. Moreover, there is no number s such that $sI + A$ is positive. Indeed, for $v = (0, -1, 1) \in K$ one has $(sI + A)v = (-1, -s + 1, s - 1) \notin K$ for every $s \in \mathbb{R}$.

Figure 1: Illustration of Example 4.1

A similar example can be found, e.g., in [6]. Example 4.1 provides an operator with the additional property

$$(-A)^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \geq 0.$$

Indeed, if $y = (y_1, y_2, 1)$ with $y_1^2 + y_2^2 \leq 1$ and $x = (-A)^{-1}y = (x_1, x_2, x_3)$, then $x_3 = 1$ and

$$\begin{aligned} x_1^2 + x_2^2 &= \frac{1}{9}(y_1 + y_2)^2 + \frac{1}{9}(-y_1 + 2y_2)^2 \\ &= \frac{1}{9}(2y_1^2 + 5y_2^2 - 2y_1y_2) \\ &\leq 1 \\ &= x_3, \end{aligned}$$

hence $x \in K$.

Referring to the remark in Section 1 on M-operators in the space $(\mathbb{R}^n, \mathbb{R}^n_+, \|\cdot\|)$, the operator $C = -A$ presents an example such that $-C$ is positive-off-diagonal, $C^{-1} \geq 0$ but C can not be represented as an M-operator (see also Figure 1).

In the following example we consider an ordered normed space that is not a vector lattice, but satisfies all assumptions of Theorem 3.1.

Example 4.2. Let

$$\begin{aligned} X &= \{x \in C[0,4] \colon x(2) = x(1) + x(3)\} \\ K &= \{x \in X \colon x(t) \geq 0 \ \text{ for all } t \in [0,4]\}. \end{aligned}$$

The ordered vector space (X, K) satisfies the Riesz decomposition property, it is not a vector lattice, and with the maximum norm it becomes a Banach space where K is closed (see [7: Section V.2]). Furthermore, the cone K is normal. As an order unit we can choose the function e with

$$e(t) = \begin{cases} 1 & \text{for } t \in [0,1] \cup [3,4] \\ t & \text{for } t \in [1,2] \\ -t+4 & \text{for } t \in [2,3] \end{cases}$$

(note that $e \in \operatorname{int}(K)$). For any $x \in X$ one has $x = y - z$, where $y = \|x\| e$ and $z = \|x\| e - x$ are positive. Furthermore, $\|y\| \leq 2\|x\|$ and $\|z\| \leq 3\|x\|$, hence we get the constant of non-flatness $\kappa = 3$.

A set of extremals of K' is the collection of the evaluation maps ε_t (i.e. $\varepsilon_t(x) = x(t)$ for each $x \in X$) determined by the points $t \in [0,2) \cup (2,4]$. This set is total.

Finally, K is b-generating. To see this fix $s \in [0,2) \cup (2,4]$ and $x \in \varepsilon_s^{-1}(0) = \{x \in X \colon x(s) = 0\}$. The element x belongs to the vector lattice $C[0,4]$, where x can be represented as $x = x^+ - x^-$ with the non-negative functions $x^+(t) = \max\{0, x(t)\}$ and $x^-(t) = \max\{0, -x(t)\}$. Note that $x^+(s) = x^-(s) = 0$. In order to show that $\varepsilon_s^{-1}(0) \cap K$ is generating in $\varepsilon_s^{-1}(0)$ consider the following two cases:

Case (a): If $x(1)$ and $x(3)$ have the same sign, say $x(1) \geq 0$ and $x(3) \geq 0$, then x^+ and x^- belong to the subspace $X \subset C[0,4]$. Indeed, one has $x^+(1) = x(1), x^+(3) = x(3)$ and $x(2) = x(1) + x(3) \geq 0$, hence $x^+(2) = x(2) = x^+(1) + x^+(3)$ and therefore $x^+ \in X$. Because of $x^-(1) = x^-(3) = x^-(2) = 0$ one has $x^- \in X$. The case $x(1) \leq 0, x(3) \leq 0$ can be considered analogously.

Case (b): If $x(1)$ and $x(3)$ have different signs, say $x(1) > 0$ and $x(3) < 0$, then x^+ and x^- may not belong to X. However, we can still find another representation $x = x_1 - x_2$ such that $0 \geq x_1, x_2 \in X$ with $x_1(s) = x_2(s) = 0$. Let $0 \leq w \in C[0,4]$ with $w(1) = w(3) = w(s) = 0$ and $w(2) = x(1) - x^+(2)$.

Note that $w(2) \geq 0$. Indeed, if $x(2) < 0$, then $x^+(2) = 0 \leq x(1)$. If $x(2) \geq 0$, then $x^+(2) = x(2) = x(1)+x(3) \leq x(1)$. Put now $x_1 = x^+ + w$ and $x_2 = x_1 - x$. Then $x_1 \geq 0$ and $x_2 = x^+ + w - (x^+ - x^-) = w + x^- \geq 0$. Obviously, $x_1(s) = x_2(s) = 0$. We show that $x_1, x_2 \in X$. For x_1 this follows from

$$\begin{aligned} x_1(1) + x_1(3) &= x^+(1) + w(1) + x^+(3) + w(3) \\ &= x(1) \\ &= x(1) - x^+(2) + x^+(2) \\ &= w(2) + x^+(2) \\ &= x_1(2), \end{aligned}$$

i.e. $x_1 \in X$. For x_2 we proceed as follows:

$$\begin{aligned} x_2(1) + x_2(3) &= x_1(1) - x(1) + x_1(3) - x(3) \\ &= x_1(1) + x_1(3) - (x(1) + x(3)) \\ &= x_1(2) - x(2) \\ &= x_2(2), \end{aligned}$$

therefore $x_2 \in X$.

Note that the functional ε_2 is not an extremal of K', and in the subspace $\varepsilon_2^{-1}(0)$ the cone $\varepsilon_2^{-1}(0) \cap K$ is not generating. Consider for example $x \in \varepsilon_2^{-1}(0)$, where $x(1) \neq 0$, and assume $x = x_1 - x_2$ for some $x_1, x_2 \in \varepsilon_2^{-1}(0) \cap K$. Then $0 = x_1(2) = x_1(1) + x_1(3) \geq 0$ implies, in particular, $x_1(1) = 0$. Analogously, $x_2(1) = 0$. Finally, $x(1) = x_1(1) - x_2(1) = 0$ yields a contradiction.

The space X in Example 4.2 satisfies all assumptions of Theorem 3.1 (and of Corollary 3.4, respectively), hence every positive-off-diagonal operator on X is an operator of the kind $sI + B$ with positive B and $s \in \mathbb{R}$.

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