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On the Set of Reachable States in the Problem of Controllability of Rotating Timoshenko Beams

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Abstract. This work continues the authors' previous investigation on the controllability problem of a slowly rotating Timoshenko beam. We obtain conditions of exact controllability under the assumption that the parameter γ appearing in the model equation is rational. Our result rests on a generalization of the theorem by Ullrich on the Riesz basis property of exponential divided differences.

Keywords: *Controllability, Timoshenko beam, divided differences*

AMS subject classification: 93B05

1. Introduction

In this work we continue the investigation of a linearized model [1] of a rotating Timoshenko beam begun in [7 - 11]. Following [11: Chapter 3] we consider the model given by two dimension-free equations

$$\left.\begin{aligned} \ddot{\omega}(x,t) - \omega''(x,t) + \xi'(x,t) &= -\ddot{\theta}(t)(r+x) \\ \ddot{\xi}(x,t) - \gamma^2\xi''(x,t) + \xi(x,t) + \omega'(x,t) &= \ddot{\theta}(t) \end{aligned}\right\} \tag{1.1}$$

for $x \in (0,1)$ and $t > 0$, where $\omega(x,t)$ denotes the deflection of the center line of the beam and $\xi(x,t)$ the rotation angle of the cross section area at the location $x \in [0,1]$ and time $t \geq 0$, respectively, $\dot{\omega} = \omega_t$, $\dot{\xi} = \xi_t$ and $\omega' = \omega_x$,

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$\xi' = \xi_x$, θ is the rotation angle of the motor disk, $\dot\theta = \frac{d\theta}{dt}$, r is the radius of the disk, $\gamma^2 = \frac{EA}{K}$ with K the shear modulus, E the Young's modulus and A the cross section area. In addition, we have boundary conditions of the form

$$\left.\begin{aligned} \omega(0,t) = \xi(0,t) = 0 \\ \omega'(1,t) + \xi(1,t) = 0 \\ \xi'(1,t) = 0 \end{aligned}\right\} \qquad (t \geq 0). \tag{1.2}$$

Our goal is to describe exactly the set of the states of the system (the beam plus the disk) which can be reached from the position of rest

$$\omega(x,0) = \dot\omega(x,0) = \xi(x,0) = \dot\xi(x,0) = \theta(0) = \dot\theta(0) = 0 \qquad (x \in [0,1]) \tag{1.3}$$

at the time T, for some large enough $T > 0$. Such a problem has been solved in [8] for the case $\gamma^2 = 1$, using a certain modification of Ullrich's theorem [12]. In the present work we give a solution in the case of an arbitrary rational $\gamma^2 > 1$. In Section 2 we recall the operator model equation derived in [11] as well as spectral analysis given in that work. On this basis we formulate our controllability problem as a special non-harmonic moment problem. Conditions of solvability of this problem are obtained in Section 3 analyzing the Riesz basis properties of the corresponding system of exponentials. In particular, we make use of some theorem from [1]. In the final Section 4 we give a solution of the controllability problem in terms of coefficients of the model operator. Our main result is the exact description of the reachability set for the time $T > 2\frac{1+\gamma}{\gamma}$. Note that in the case $\gamma^2 = 1$ our result coincides with the one given in [8].

2. A moment problem describing the conditions of controllability

We consider the rotation of a Timoshenko beam in horizontal plane whose left end is clamped into the disk of a driving motor. Let r be the radius of the disk and let $\theta = \theta(t)$ be the rotation angle as a function of the time $t \geq 0$. If $\omega(x,t)$ denotes the deflection of the center line of the beam at the location $x \in [0,1]$ (the length of the beam is assumed to be 1) and the time $t \geq 0$ and $\xi(x,t)$ denotes the rotation angle of the cross section area at x and t and if we assume the rotation to be slow, then ω and ξ are governed by (1.1) - (1.2). In this paper we investigate the following problem of controllability:

Given a time $T > 0$ and a position $\left(\omega_T, \xi_T, \dot\omega_T, \dot\xi_T, \theta_T, \dot\theta_T\right)$ of the beam where $\omega_T, \xi_T, \dot\omega_T, \dot\xi_T$ are chosen in suitable function spaces and $\theta_T, \dot\theta_T$ are given real numbers, find

$$\theta \in H_0^2(0,T) = \left\{\theta \in H^2(0,T) \,|\, \theta(0) = \dot\theta(0) = 0\right\}$$

such that

$$\begin{aligned} \theta(T) &= \theta_T \\ \dot{\theta}(T) &= \dot{\theta}_T \end{aligned} \tag{2.1}$$

and the weak solution (ω, ξ) of problem (1.1) - (1.3) satisfies the end conditions

$$\begin{aligned} \omega(\cdot, T) &= \omega_T \\ \xi(\cdot, T) &= \xi_T \\ \dot{\omega}(\cdot, T) &= \dot{\omega}_T \\ \dot{\xi}(\cdot, T) &= \dot{\xi}_T. \end{aligned} \tag{2.2}$$

Let $H = L^2((0,1), \mathbb{R}^2)$. Then we define a linear operator $A\colon D(A) \to H$ by

$$A\binom{y}{z} = \binom{-y'' - z'}{-\gamma^2 z'' + z + y'}$$

for $\binom{y}{z} \in D(A)$ where

$$D(A) = \left\{ \binom{y}{z} \in H^2((0,1), \mathbb{R}^2) \;\middle|\; \begin{aligned} y(0) = z(0) &= 0 \\ y'(1) + z(1) &= 0 \\ z'(1) &= 0 \end{aligned} \right\}.$$

With this operator, equations (1.1) can be rewritten in the form

$$\begin{pmatrix} \ddot{\omega}(\cdot, t) \\ \ddot{\xi}(\cdot, t) \end{pmatrix} + A \begin{pmatrix} \omega(\cdot, t) \\ \xi(\cdot, t) \end{pmatrix} = \begin{pmatrix} f_1(\cdot, t) \\ f_2(\cdot, t) \end{pmatrix} \tag{2.3}$$

for $t > 0$ where

$$\begin{aligned} f_1(x,t) &= -\ddot{\theta}(t)(r + x) \\ f_2(x,t) &= \ddot{\theta}(t) \end{aligned}$$

for $x \in (0,1)$ and $t > 0$. It is shown in [11] that $A\colon D(A) \to H$ is positive, self-adjoint and has an orthonormal sequence of eigenelements $\binom{y_j}{z_j} \in D(A)$ $(j \in \mathbb{N})$ and a corresponding sequence of eigenvalues $\lambda_j \in \mathbb{R}$ such that $1 < \lambda_j \uparrow \infty$ as $j \to \infty$. It is further shown that for large n the eigenvalues of A are of the form

$$\lambda_n = \begin{cases} \frac{1}{4}\big(\gamma(2k-1)\pi + \varepsilon_{2k-1}\big)^2 & \text{for } n = 2k-1 \\ \frac{1}{4}\big((2k-1)\pi + \varepsilon_{2k}\big)^2 & \text{for } n = 2k \end{cases} \tag{2.4}$$

where $\varepsilon_{2k-1}, \varepsilon_{2k} > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

The unique weak solution of equation (2.3) corresponding to initial conditions (1.3) is then given by

$$\begin{pmatrix}\omega(x,t)\\ \xi(x,t)\end{pmatrix} = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \int_0^t \sin\sqrt{\lambda_j}\,(t-s)\Big\langle \begin{pmatrix} f_1(\cdot,t)\\ f_2(\cdot,t)\end{pmatrix}, \begin{pmatrix} y_j\\ z_j\end{pmatrix}\Big\rangle_H ds \begin{pmatrix} y_j\\ z_j\end{pmatrix}$$

for $x \in [0,1]$ and $t \geq 0$ and its time derivative reads

$$\begin{pmatrix}\dot\omega(x,t)\\ \dot\xi(x,t)\end{pmatrix} = \sum_{j=1}^{\infty} \int_0^t \cos\sqrt{\lambda_j}\,(t-s)\Big\langle \begin{pmatrix} f_1(\cdot,t)\\ f_2(\cdot,t)\end{pmatrix}, \begin{pmatrix} y_j\\ z_j\end{pmatrix}\Big\rangle_H ds \begin{pmatrix} y_j\\ z_j\end{pmatrix}.$$

From these representations we infer that end conditions (2.2) are equivalent to

$$\left.\begin{aligned} a_j \int_0^T \sin\sqrt{\lambda_j}\,(T-t)\ddot\theta(t)\,dt &= \sqrt{\lambda_j}\Big\langle \begin{pmatrix}\omega_T\\ \xi_T\end{pmatrix}, \begin{pmatrix} y_j\\ z_j\end{pmatrix}\Big\rangle_H \\ a_j \int_0^T \cos\sqrt{\lambda_j}(T-t)\ddot\theta(t)\,dt &= \Big\langle \begin{pmatrix}\omega_T\\ \xi_T\end{pmatrix}, \begin{pmatrix} y_j\\ z_j\end{pmatrix}\Big\rangle_H \end{aligned}\right\} \quad (j \in \mathbb{N}) \tag{2.5}$$

where

$$a_j = \Big\langle b, \begin{pmatrix} y_j\\ z_j\end{pmatrix}\Big\rangle_H = -\int_0^1 (r+x)y_j(x)\,dx + \int_0^1 z_j(x)\,dx.$$

Later on we will assume that $a_j \neq 0 \quad (j \in \mathbb{N})$. Following [11] we can make a short analysis of this assumption. The values of the disk radius such that the equality $a_j = 0$ is valid for some eigenvector $\binom{y_j}{z_j}$ are called singular ones. Obviously, in the case when the disk radius is of a singular value there exists a fundamental frequency of the beam which is invariable under the influence of the control, i.e. system (2.3) is not controllable. However the following statement holds

Remark 2.1 (see [11]). There exists at most countable set of singular values of the disk radius.

Example. In the case $\gamma^2 = 1$ the singular values of r are of the form (see [7 - 8])

$$r_n = \frac{\sigma_1^{(n)} \sin\sigma_1^{(n)} - \sigma_3^{(n)}\sin\sigma_3^{(n)}}{\sqrt{\lambda_n}\,(\cos\sigma_3^{(n)} + \cos\sigma_1^{(n)})}$$

where $\sigma_1^{(n)} = \sqrt{\lambda_n - \sqrt{\lambda_n}}$ and $\sigma_3^{(n)} = \sqrt{\lambda_n + \sqrt{\lambda_n}}$ with λ_n an eigenvalue of A. Let us define $u_n = \sigma_3^{(n)} - \sigma_1^{(n)}$ and $v_n = \sigma_3^{(n)} + \sigma_1^{(n)}$. It was shown in [10 - 11] that

$$u_n^2(1+\cos u_n) = v_n^2(1+\cos v_n).$$

That gives

$$r_n = -\frac{\sigma_3^{(n)}}{\lambda_n} \frac{\sin \frac{u_n}{2} \pm \sin \frac{v_n}{2}}{\cos \frac{u_n}{2} \pm \cos \frac{v_n}{2}}.$$

It is also proven that there exist $n \in \mathbb{N}$ such that $a_n = 0$ if and only if $\pi + 2n\pi < v_n < 2\pi + 2n\pi$. From those two facts one can derive that[1]

$$\lim_{n\to\infty} r_n = \begin{cases} \frac{\tan\frac{1}{4}+1}{\tan\frac{1}{4}-1} & \text{for } n = 2k-1 \\ -\frac{\tan\frac{1}{4}+1}{\tan\frac{1}{4}-1} & \text{for } n = 2k. \end{cases}$$

Summarizing we get that beyond of any neighborhood of the point $-\frac{\tan\frac{1}{4}+1}{\tan\frac{1}{4}-1}$ there exist only a finite number of singular values of disk radius.

From this moment we will assume that r is a non-singular one. If we define

$$\left.\begin{aligned} d_j^1 &= \frac{\sqrt{\lambda_j}}{a_j}\left\langle \begin{pmatrix} \omega_T \\ \xi_T \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H \\ d_j^2 &= \frac{1}{a_j}\left\langle \begin{pmatrix} \dot\omega_T \\ \dot\xi_T \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle_H \end{aligned}\right\} \qquad (j \in \mathbb{N})$$

and put $u(t) = \ddot\theta(T-t)$ for $t \in [0,T]$, then (2.5) can be rewritten in the form

$$\left.\begin{aligned} \int_0^T \sin\sqrt{\lambda_j}\, t u(t)\, dt &= d_j^1 \\ \int_0^T \cos\sqrt{\lambda_j}\, t u(t)\, dt &= d_j^2 \end{aligned}\right\} \qquad (j \in \mathbb{N}). \tag{2.6}$$

End conditions (2.1) turn out to be equivalent to

$$\left.\begin{aligned} \int_0^T t u(t)\, dt &= \theta_T \\ \int_0^T u(t)\, dt &= \dot\theta_T \end{aligned}\right\}. \tag{2.7}$$

Then the problem of controllability is equivalent to finding some $u \in L^2(0,T)$ which satisfies (2.6) - (2.7). If such u has been found, then

$$\theta(t) = \int_0^t (t-s)u(T-s)\, ds \qquad (t \in [0,T])$$

is a solution of the problem of controllability.

[1] Note that the values of $\lim_{n\to\infty} r_n$ correct the ones calculated in [10].

3. The analysis of Riesz basis properties

In this section we examine the Riesz basis properties of functions arising in the moment problem. For the purpose of this paper we assume that in (1.1) $\gamma = \frac{p}{q} \in \mathbb{Q}$ with $p > q$. We consider in detail the case where p, q are odd numbers. We also give briefly a result in the remaining case.

Let us define $d_j = d_j^2 + i d_j^1$. Then system (2.6) is equivalent to

$$\left.\begin{aligned} \int_0^T e^{i\sqrt{\lambda_j}t} u(t)\,dt &= d_j \\ \int_0^T e^{-i\sqrt{\lambda_j}t} u(t)\,dt &= \overline{d_j} \end{aligned}\right\} \qquad (j \in \mathbb{N}).$$

Let us define

$$\begin{aligned} \sqrt{\lambda_j} &= \begin{cases} \gamma\frac{\pi}{2}(2n-1) + \varepsilon_{2n-1} & \text{for } j = 2n-1 \\ \frac{\pi}{2}(2n-1) + \varepsilon_{2n} & \text{for } j = 2n \end{cases} \\ \mu_j &= \begin{cases} \gamma\frac{\pi}{2}(2n-1) & \text{for } j = 2n-1 \\ \frac{\pi}{2}(2n-1) & \text{for } j = 2n \end{cases} \end{aligned} \tag{3.1}$$

for $j \in \mathbb{Z}$, where ε_j are defined in (2.4) for $j \in \mathbb{N}$ and for $j \in \mathbb{Z} \setminus \mathbb{N}$ we put $\varepsilon_j = 0$. Notice that $\sqrt{\lambda_j} = \mu_j + \varepsilon_j$. Also,

$$\mu_{q(2m-1)} = \frac{p}{q}\frac{\pi}{2}q(2m-1) = \frac{\pi}{2}(2pm - p + 1 - 1) = \mu_{2pm-p+1}.$$

We start our analysis with the following

Lemma 3.1. *The system* $\{e^{i\mu_j t}\}_{j\in\mathbb{Z}\setminus(2q\mathbb{Z}-q)} \cup \{te^{i\mu_{q(2j-1)}t}\}_{j\in\mathbb{Z}}$ *is a Riesz basis for* $L^2\big(0, 2\frac{1+\gamma}{\gamma}\big) = L^2\big(0, 2\frac{p+q}{p}\big)$.

Proof. Let us rewrite the system in question in the form

$$\big\{e^{i\mu_j t}, \ldots, t^{m_j-1}e^{i\mu_j t}\big\}_{j\in\mathbb{Z}\setminus(2q\mathbb{Z}-q)}$$

where

$$m_j = \begin{cases} 1 & \text{for } j \neq 2pm - p + 1 \\ 2 & \text{for } j = 2pm - p + 1 \end{cases} \qquad (m \in \mathbb{Z}).$$

Consider the sine-type function

$$F(z) = 4e^{i\frac{\gamma+1}{\gamma}z} \sin\big(\tfrac{z}{\gamma} + \tfrac{\pi}{2}\big) \sin\big(z + \tfrac{\pi}{2}\big).$$

Then

$$\begin{aligned} F(z) &= -2e^{i\frac{\gamma+1}{\gamma}z}\Big(\cos\big(\tfrac{1+\gamma}{\gamma}z+\pi\big)-\cos\big(\tfrac{1-\gamma}{\gamma}z\big)\Big) \\ &= -e^{i\frac{\gamma+1}{\gamma}z}\big(e^{i\frac{1+\gamma}{\gamma}z+i\pi}+e^{-i\frac{1+\gamma}{\gamma}z-i\pi}-e^{i\frac{1-\gamma}{\gamma}z}-e^{i\frac{1-\gamma}{\gamma}z}\big) \\ &= e^{i2\frac{\gamma+1}{\gamma}z}+1+e^{i2z}+e^{i\frac{2}{\gamma}z} \end{aligned}$$

and one can easily check that the set of zeroes of the function F is $\{\mu_j\}_{j\in\mathbb{Z}\setminus(2q\mathbb{Z}-q)}$. $\blacksquare$ Moreover, the multiplicity of the root μ_j equals m_j. Then we can apply [1: Theorem II.4.23], which completes the proof $\blacksquare$

Let us denote $I_\gamma = (0, 2\frac{1+\gamma}{\gamma})$. The main result of the section is

Theorem 3.1. *If $1 < \gamma = \frac{p}{q} \in \mathbb{Q}$ with $p, q \in 2\mathbb{N}-1$ and, for some $\delta > 0$, $|\varepsilon_j| < \delta \quad (j \in \mathbb{Z})$, then the system*

$$\int_{I_\gamma} f(t)e^{-i\sqrt{\lambda_j}t}dt = c_j \qquad (j \in \mathbb{Z}) \tag{3.2}$$

with λ_j defined by (3.1) has a solution $f \in L^2(I_\gamma)$ if and only if

$$\sum_{j=-\infty}^{\infty}\left(|c_j|^2 + \left|\frac{c_{2qj-q}-c_{2pj-p+1}}{\sqrt{\lambda_{2qj-q}}-\sqrt{\lambda_{2pj-p+1}}}\right|^2\right) < \infty. \tag{3.3}$$

Moreover, if system (3.2) has a solution, then it is unique.

Our proof will use the following three lemmas from [12].

Lemma 3.2. *Let $\{x_n\}$ be a Riesz basis for a complex Hilbert space H, i.e. there exist constants $A, B > 0$ such that, for any $N \in \mathbb{N}$ and any complex $a_{-N}, \ldots, a_N$, $A^2\sum_{-N}^N |a_n|^2 \le \big\|\sum_{-N}^N a_n x_n\big\|^2 \le B^2\sum_{-N}^N |a_n|^2$. Further, let $0 < A' < A$ and $\{y_n\}$ be a sequence of elements from H such that, for any $N \in \mathbb{N}$ and any complex $a_{-N}, \ldots, a_N$, $\|\sum_{-N}^N a_n y_n\|^2 \le (A')^2\sum_{-N}^N |a_n|^2$. Then $\{x_n + y_n\}$ is complete in H and forms a Riesz basis.*

Lemma 3.3. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that if w_0, w_1 are distinct complex numbers satisfying $|w_0|, |w_1| < \delta$, then for any $z \in \mathbb{C}$ with $|z| \le 3\pi + 1$ the inequalities*

$$|1-e^{iw_0z}|, \ \left|iz - \frac{e^{iw_0z}-e^{iw_1z}}{w_0-w_1}\right| < \varepsilon$$

hold and imply

$$\left.\begin{aligned} &\Big|\frac{\partial}{\partial z}(1-e^{iw_0z})\Big| \\ &\Big|\frac{\partial^2}{\partial z^2}(1-e^{iw_0z})\Big| \end{aligned}\right\} < \varepsilon \qquad \textit{and} \qquad \left.\begin{aligned} &\Big|\frac{\partial}{\partial z}\Big(iz-\frac{e^{iw_0z}-e^{iw_1z}}{w_0-w_1}\Big)\Big| \\ &\Big|\frac{\partial^2}{\partial z^2}\Big(iz-\frac{e^{iw_0z}-e^{iw_1z}}{w_0-w_1}\Big)\Big| \end{aligned}\right\} < \varepsilon.$$

Lemma 3.4. *Let $\varphi_n \in C^2(-(K+2)\pi,(K+2)\pi)$ $(n \in \mathbb{Z})$ satisfy the inequalities $\|\varphi_n\|_\infty, \|\varphi_n'\|_\infty \le \varepsilon, \|\varphi_n''\|_\infty < \varepsilon$ for $n \in \mathbb{Z}$ and some $\varepsilon > 0$. Then, for any complex sequence $\{a_n\}$ and any $N \in \mathbb{N}$,*

$$\int_{I_K} \left| \sum_{-N}^{N} a_n e^{int} \varphi_n(t) \right|^2 dt \le \varepsilon^2 M^2 \sum_{-N}^{N} |a_n|^2$$

where M depends on K only.

Proof of Theorem 3.1. System (3.2) is obviously equivalent to

$$\left.\begin{aligned} \int_{I_\gamma} f(t) e^{-i\sqrt{\lambda_j}t} dt = c_j \\ \int_{I_\gamma} f(t) \frac{e^{-i\sqrt{\lambda_{2qj-q}}t} - e^{-i\sqrt{\lambda_{2pj-p+1}}t}}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}} dt = \frac{c_{2qj-q} - c_{2pj-p+1}}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}} \end{aligned}\right\}.$$

Thus it is enough to show that the system

$$\left\{ e^{i\sqrt{\lambda_j}t}, \; \frac{e^{i\sqrt{\lambda_{2qj-q}}t} - e^{i\sqrt{\lambda_{2pj-p+1}}t}}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}} \right\} \qquad (j \in \mathbb{Z}) \tag{3.4}$$

(as a set of functions of $t \in I_\gamma$) forms a Riesz basis for $L^2(I_\gamma)$. Let us define

$$\begin{aligned} \varphi_n^0(z) &= 1 - e^{i\varepsilon_n z} \\ \varphi_n^1(z) &= iz - \frac{e^{-i\varepsilon_n z} - e^{-i\varepsilon_{\frac{2p-1}{2q}n+p+\frac{1}{2}} z}}{\sqrt{\lambda_n} - \sqrt{\lambda_{\frac{2p-1}{2q}n+p+\frac{1}{2}}}}. \end{aligned}$$

Then

$$\begin{aligned} e^{i\mu_j t}\varphi_j^0(t) &= e^{i\mu_j t} - e^{i\sqrt{\lambda_j}t} \\ e^{i\mu_{2qj-q}t}\varphi_{2qj-q}^1(t) &= ite^{i\mu_{2qj-q}t} - \frac{e^{i\sqrt{\lambda_{2qj-q}}\,t} - e^{i\sqrt{\lambda_{2pj-p+1}}\,t}}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}}. \end{aligned}$$

Using Lemmas 3.3 and 3.4 we obtain that for any $\varepsilon > 0$ there exists $\delta > 0$ such that, if $a_{-N}^0, a_{-N}^1, \ldots, a_N^0, a_N^1$ are any complex numbers, then

$$\begin{aligned} &\int_{I_\gamma} \left| \sum_{-N}^{N} a_j^0 \left(e^{i\mu_j t} - e^{i\sqrt{\lambda_j}t}\right) \right|^2 dt \\ &\quad = \int_{I_\gamma} \left| \sum_{-N}^{N} a_j^0 e^{i\mu_j t} \varphi_j^0(t) \right|^2 dt \end{aligned}$$

$$\leq 2\int_{I_\gamma}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}-1}}^{N} a_j^0 e^{i\pi\gamma nt}e^{-i\frac{\pi}{2}\gamma t}\varphi_j^0(t)\Big|^2 dt + 2\int_{I_\gamma}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}}}^{N} a_j^0 e^{i\pi nt}e^{-i\frac{\pi}{2}t}\varphi_j^0(t)\Big|^2 dt$$

$$= 2\int_{I_\gamma}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}-1}}^{N} a_j^0 e^{i\pi\gamma nt}\varphi_j^0(t)\Big|^2 dt + 2\int_{I_\gamma}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}}}^{N} a_j^0 e^{i\pi nt}\varphi_j^0(t)\Big|^2 dt \qquad (3.5)_a$$

$$= \frac{2}{\pi\gamma}\int_0^{2\pi(1+\gamma)}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}-1}}^{N} a_j^0 e^{int}\varphi_j^0\big(\frac{t}{\pi\gamma}\big)\Big|^2 dt + \frac{2}{\pi}\int_0^{2\pi\frac{1+\gamma}{\gamma}}\Big|\sum_{\substack{j=-N\\ j\in 2\mathbb{Z}}}^{N} a_j^0 e^{int}\varphi_j^0\Big(\frac{t}{\pi}\Big)\Big|^2 dt$$

$$\leq \frac{2}{\pi}\Big(1+\frac{1}{\gamma}\Big)M^2\varepsilon^2\sum_{-N}^{N}|a_j^0|^2$$

and

$$\int_{I_\gamma}\Big|\sum_{-N}^{N} a_j^1\Big(ite^{i\mu_{2qj-q}t} - \frac{e^{i\sqrt{\lambda_{2qj-q}}t} - e^{i\sqrt{\lambda_{2pj-p+1}}t}}{\sqrt{\lambda_{2qj-q}}-\sqrt{\lambda_{2pj-p+1}}}\Big)\Big|^2 dt$$

$$= \int_{I_\gamma}\Big|\sum_{-N}^{N} a_j^1 e^{i\mu_{2qj-q}t}\varphi_{2qj-q}^1(t)\Big|^2 dt$$

$$= \int_{I_\gamma}\Big|\sum_{-N}^{N} a_j^1 e^{i\pi pjt}e^{-ip\frac{\pi}{2}t}\varphi_{2qj-q}^1(t)\Big|^2 dt$$

$$= \int_{I_\gamma}\Big|\sum_{-N}^{N} a_j^1 e^{i\pi pjt}\varphi_{2qj-q}^1(t)\Big|^2 dt \qquad (3.5)_b$$

$$= \int_0^{2\pi(p+q)}\Big|\sum_{-N}^{N} a_j^1 e^{ijt}\varphi_{2qj-q}^1\Big(\frac{t}{\pi p}\Big)\Big|^2 dt$$

$$\leq \frac{1}{\pi p}M^2\varepsilon^2\sum_{-N}^{N}|a_j^1|^2$$

where M depends on γ only. From (3.5) it follows that for any a_j^0 and a_j^1 $(j \in \mathbb{Z})$

$$\int_{I_\gamma}\Big|\sum_{-N}^{N} a_j^0\big(e^{i\mu_j t} - e^{i\sqrt{\lambda_j}\,t}\big) + \sum_{-N}^{N} a_j^1\Big(ite^{i\mu_{2qj-q}t} - \frac{e^{i\sqrt{\lambda_{2qj-q}}\,t} - e^{i\sqrt{\lambda_{2pj-p+1}}\,t}}{\sqrt{\lambda_{2qj-q}}-\sqrt{\lambda_{2pj-p+1}}}\Big)\Big|^2 dt$$

$$\leq \frac{2(p+q)+1}{\pi p}M^2\varepsilon^2\sum_{-N}^{N}\big(|a_j^0|^2 + |a_j^1|^2\big) \qquad (3.6)$$

where M depends on γ only. Since the set $\{e^{i\mu_j t}\}_{j\in\mathbb{Z}\setminus(2q\mathbb{Z}-q)}\cup\{te^{i\mu_{q(2j-1)}t}\}_{j\in\mathbb{Z}}$ forms a Riesz basis for $L^2(I_\gamma)$ by Lemma 3.1, then due to Lemma 3.2 there exists a constant $A>0$ such that, for any complex a_j^0 and a_j^1 $(j\in\mathbb{Z})$,

$$A^2\sum_{-N}^{N}\left(|a_j^0|^2+|a_j^1|^2\right)\le\int_{I_\gamma}\left|\sum_{-N}^{N}a_j^0e^{i\mu_j t}+\sum_{-N}^{N}a_j^1ite^{i\mu_{2qj-q}t}\right|^2dt.$$

If M is as in (3.5), pick $\varepsilon>0$ so that $\frac{2(p+q)+1}{\pi p}M^2\varepsilon^2<A^2$. Now choose δ so that (3.5) holds with this ε, for $|\varepsilon_j|<\delta$. Then (3.6) shows that the hypotheses of Lemma 3.2 are satisfied, so that set (3.4) is a Riesz basis for $L^2(I_\gamma)$, which completes the proof ∎

Remark to Theorem 3.1. The above theorem is a theorem of Ullrich-type [12]. Recently there has appeared a number of works considering some generalizations of Ullrich theorem and its application for specific parameter distributed systems [2 - 5]. Apparently the most essential progress in this direction is made in [2], where a fairly powerful theorem of Ullrich-type is obtained. In this context let us notice that in our case this theorem cannot be applied directly, because a function which one can naturally consider as a generating one for moment problem (2.6) - (2.7) doesn't have zeroes in $\sqrt{\lambda_j}$, but in some close points μ_j. Moreover, we do not know the exact values of $\sqrt{\lambda_j}$, only their asymptotic behavior. Theorem 3.1 just overcomes this difficulty in our particular case. It seems that the further progress in this field may be connected with obtaining an Ullrich-type theorem considering the stability of basis property of a family with respect to perturbations of generating functions (as it is done in [1: Theorem II.4.32] for the case of separated exponents).

Remark 3.1. In Theorem 3.1 we can replace the interval $I_\gamma=(0,2\frac{\gamma+1}{\gamma})$ with an arbitrarily chosen interval of length $2\frac{\gamma+1}{\gamma}$.

Remark 3.2. For any $T>2\frac{\gamma+1}{\gamma}$ the system

$$\int_0^T f(t)e^{-i\sqrt{\lambda_j}\,t}dt=c_j\qquad(j\in\mathbb{N})$$

has a solution if and only if condition (3.3) holds. However, this solution is not unique.

The proof of Remarks 3.1 and 3.2 is similar to the one given in [8].

In the case when p is odd and q is even or vice versa, one can easily observe that the sequence μ_j is separable, i.e. $\inf_{n\neq m}|\mu_n-\mu_m|>0$. Then instead of Lemma 3.1 one can prove

Lemma 3.1'. *The system* $\{e^{i\mu_j t}\}_{j\in\mathbb{Z}}$ *is a Riesz basis for* $L^2(0, 2\frac{1+\gamma}{\gamma})$.

As a consequence we obtain the following theorem, which replaces Theorem 3.1.

Theorem 3.1'. *For any* $1 < \gamma = \frac{p}{q} \in \mathbb{Q}$ *such that exactly one of the numbers* p, q *is even there exists* $\delta > 0$ *such that if* $|\varepsilon_j| < \delta$ *for* $j \in \mathbb{Z}$, *then the system*

$$\int_{I_\gamma} f(t)e^{-i\sqrt{\lambda_j}\,t}dt = c_j \qquad (j \in \mathbb{Z})$$

with λ_j *defined by* (3.1) *has a solution* $f \in L^2(I_\gamma)$ *if and only if* $\sum_{j=-\infty}^{\infty} |c_j|^2 < \infty$. *Moreover, if the system has a solution, it is unique.*

4. The set of reachable states

Let us again consider the case of $p, q \in 2\mathbb{N} - 1$, let $T = 2\frac{\gamma+1}{\gamma}$ and define

$$\begin{aligned} c_j &= \overline{d_j} \quad \text{for } j \in \mathbb{N} \\ c_{-j} &= d_j \quad \text{for } j \in 2\mathbb{N} - 1 \\ c_{-j+2} &= d_j \quad \text{for } j \in 2\mathbb{N}. \end{aligned}$$

According to Theorem 3.1 there exists a unique complex function $u \in L^2(0, 2\frac{\gamma+1}{\gamma})$ satisfying

$$\int_0^{2\frac{\gamma+1}{\gamma}} u(t)e^{-i\sqrt{\lambda_j}\,t}dt = c_j \qquad (j \in \mathbb{Z}) \tag{4.1}$$

if and only if condition (3.3) holds. Note that the moment equalities are equivalent to

$$\left.\begin{aligned} \int_0^{2\frac{\gamma+1}{\gamma}} e^{i\sqrt{\lambda_j}\,t}u(t)dt = d_j \\ \int_0^{2\frac{\gamma+1}{\gamma}} e^{-i\sqrt{\lambda_j}\,t}u(t)dt = \overline{d_j} \end{aligned}\right\} \qquad (j \in \mathbb{N}) \tag{4.2}$$

and condition (3.3) is equivalent to the condition

$$\sum_{j=1}^{\infty} |d_j^1|^2 + |d_j^2|^2 + {\frac{d_{2qj-q}^1 - d_{2pj-p+1}^1}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}}}^2 + {\frac{d_{2qj-q}^2 - d_{2pj-p+1}^2}{\sqrt{\lambda_{2qj-q}} - \sqrt{\lambda_{2pj-p+1}}}}^2 < \infty. \tag{4.3}$$

Now we prove

Lemma 4.1. *The only solution of system* (4.2) *is a real function.*

Proof. Let us denote $u(t) = \operatorname{Re} u(t) + i\operatorname{Im} u(t)$. We have

$$\left.\begin{aligned} \int_0^{2^{\frac{\gamma+1}{\gamma}}} e^{-i\sqrt{\lambda_j}\,t}\big(\operatorname{Re} u(t) + i\operatorname{Im} u(t)\big)dt = \overline{d_j} \\ \int_0^{2^{\frac{\gamma+1}{\gamma}}} e^{-i\sqrt{\lambda_j}\,t}\big(\operatorname{Re} u(t) - i\operatorname{Im} u(t)\big)dt = \overline{d_j} \end{aligned}\right\} \qquad (j \in \mathbb{N}).$$

This implies

$$\int_0^{2^{\frac{\gamma+1}{\gamma}}} e^{-i\sqrt{\lambda_j}\,t}\operatorname{Im} u(t)\,dt = 0 \qquad (j \in \mathbb{N}).$$

According to Theorem 3.1, a solution of system (3.2) is unique, which implies $\operatorname{Im} u(t) \equiv 0$. The proof is completed ∎

Corollary 4.1. *Every solution of system* (4.1) *is a solution of system* (4.2)*. Conversely, every solution of system* (4.2) *is real and is a solution of system* (4.1)*.*

Applying Theorem 3.1 we obtain

Assertion 4.1. *Let* $T = 2^{\frac{\gamma+1}{\gamma}}$*. Then moment problem* (2.6) − (2.7) *has a solution* (*unique*) *if and only if condition* (4.3) *holds and the unique solution of system* (4.2) *satisfies* (2.7)*.*

Let $T > 2^{\frac{\gamma+1}{\gamma}}$ and let moment problem (2.6) have a solution $u \in L^2(0,T)$. This implies that u is a solution of

$$\int_0^T u(t)e^{-i\sqrt{\lambda_j}\,t}dt = c_j \qquad (j \in \mathbb{N}).$$

Then taking into account Remark 3.2 we conclude that condition (3.3) holds. That yields condition (4.3).

Conversely, let condition (4.3) hold. Denote by $u \in L^2(0, 2^{\frac{\gamma+1}{\gamma}})$ the unique solution of system (4.2) and let us define $u \equiv 0$ on the interval $(2^{\frac{\gamma+1}{\gamma}}, T)$. Then $u \in L^2(0,T)$ and the condition

$$\left.\begin{aligned} \int_0^T e^{i\sqrt{\lambda_j}\,t}u(t)\,dt = d_j \\ \int_0^T e^{-i\sqrt{\lambda_j}\,t}u(t)\,dt = \overline{d_j} \end{aligned}\right\} \qquad (j \in \mathbb{N})$$

holds which is equivalent to

$$\left.\begin{aligned} \int_0^T \sin(\sqrt{\lambda_j}\,t)u(t)\,dt &= d_j^1 \\ \int_0^T \cos(\sqrt{\lambda_j}\,t)u(t)\,dt &= d_j^2 \end{aligned}\right\} \qquad (j \in \mathbb{N}).$$

Let

$$V = \left\{1, \cos\sqrt{\lambda_j}\,t, \sin\sqrt{\lambda_j}\,t \,\middle|\, t \in [0,T],\, j \in \mathbb{N}\right\}.$$

We make use of

Lemma 4.2 (see [8]). *The system $V \cup \{t\}$ is minimal in $L^2(0,T)$.*

Minimality of $V \cup \{t\}$ implies that there exist functions $u_1, u_2 \in L^2(0,T)$ such that

$$\int_0^T u_1(t)\,dt = 0, \quad \int_0^T tu_1(t)\,dt = 1, \quad \int_0^T u_2(t)\,dt = 1, \quad \int_0^T tu_2(t)\,dt = 0$$

and

$$\begin{aligned} \int_0^T u_1(t)\sin\sqrt{\lambda_j}\,t\,dt &= \int_0^T u_1(t)\cos\sqrt{\lambda_j}\,t\,dt \\ &= \int_0^T u_2(t)\sin\sqrt{\lambda_j}\,t\,dt \\ &= \int_0^T u_2(t)\cos\sqrt{\lambda_j}\,t\,dt \\ &= 0. \end{aligned}$$

Define $\tilde{\theta}_T = \int_0^T tu(t)\,dt$ and $\tilde{\dot{\theta}}_T = \int_0^T u(t)\,dt$ and put

$$u^*(t) = u(t) + (\theta_T - \tilde{\theta}_T)u_1(t) + (\dot{\theta}_T - \tilde{\dot{\theta}}_T)u_2(t)$$

for $t \in [0,T]$. Then $u^* \in L^2(0,T)$ and

$$\left.\begin{aligned} \int_0^T u^*(t)\,dt &= \dot{\theta}_T \\ \int_0^T tu^*(t)\,dt &= \theta_T \\ \int_0^T \sin(\sqrt{\lambda_j}\,t)u^*(t)\,dt &= d_j^1 \\ \int_0^T \cos(\sqrt{\lambda_j}\,t)u^*(t)\,dt &= d_j^2 \end{aligned}\right\} \qquad (j \in \mathbb{N}).$$

Hence $u^* \in L^2(0,T)$ is a solution of moment problem (2.6) - (2.7). So we proved

Assertion 4.2. *Let* $T > 2\frac{\gamma+1}{\gamma}$. *Then moment problem* (2.6) – (2.7) *is solvable if and only if condition* (4.3) *holds.*

Summarizing Assertions 4.1 and 4.2 we obtain

Theorem 4.1. *Assume that* $1 < \gamma = \frac{p}{q} \in \mathbb{Q}$ *with* $p, q \in 2\mathbb{N} - 1$ *and let* $T > 2\frac{\gamma+1}{\gamma}$. *The state*

$$\left(\omega_T, \xi_T, \dot{\omega}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T\right)$$

is null-reachable by virtue of system (2.3) *if and only if condition* (4.3) *holds. If* $T = 2\frac{\gamma+1}{\gamma}$, *then this state is null-reachable by virtue of system* (2.3) *if and only if condition* (4.3) *and end conditions* (2.7) *hold.*

Finally, consider the case when $\gamma = \frac{p}{q} \in \mathbb{Q}$, where exactly one of the numbers p, q is even. Now applying Theorem 3.1' instead of Theorem 3.1 one can prove the following

Theorem 4.2. *Assume that* $1 < \gamma = \frac{p}{q} \in \mathbb{Q}$, *where exactly one of numbers* p, q *is even, and let* $T > 2\frac{\gamma+1}{\gamma}$. *The state*

$$\left(\omega_T, \xi_T, \dot{\omega}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T\right)$$

is null-reachable by virtue of system (2.3) *if and only if the condition*

$$\sum_{j=1}^{\infty} \left(|d_j^1|^2 + |d_j^2|^2\right) < \infty \tag{4.4}$$

holds. If $T = 2\frac{\gamma+1}{\gamma}$, *then this state is null-reachable by virtue of system* (2.3) *if and only if condition* (4.4) *and end condition* (2.7) *hold.*

Note that this result gives a more precise estimation of the time of controllability compared with [6].

Final Remark. It should be interesting to obtain conditions of controllability of the beam in the case where γ is irrational. To this end it seems to be natural to make a passage to the limit as rational γ's tend to some irrational number. It turns out that such a passage is hard to be made because the change of γ means not only the change of eigenvalues but also the change of eigenvectors. That, in turn, leads to a change of the right-hand side in the moment problem. Another way for examining the case of irrational γ can be found in [2] where some generalizations of Ullrich's theorem are given. Although, as we have noticed before, this theorem cannot be applied to our case, one can expect the following analogue of Theorems 4.1 and 4.2:

Supposition 4.3. *Assume that* $1 < \gamma \in \mathbb{R} \setminus \mathbb{Q}$. *Let* $T > 2\frac{\gamma+1}{\gamma}, r < \frac{\pi}{4}$ *and let* $\{\tilde{\lambda}\}$ *be a set of eigenvalues ordered in a way to form an increasing sequence. The state*

$$\left(\omega_T, \xi_T, \dot{\omega}_T, \dot{\xi}_T, \theta_T, \dot{\theta}_T\right)$$

is null-reachable by virtue of system (2.3) *if and only if*

$$\sum_{j=1}^{\infty} \left(|d_j^1|^2 + |d_j^2|^2 + \chi_{r,j} \frac{\left|d_{j+1}^1 - d_j^1|^2 + |d_{j+1}^2 - d_j^2\right|^2}{\left|\sqrt{\tilde{\lambda}_{j+1}} - \sqrt{\tilde{\lambda}_j}\right|^2} \right) < \infty$$

where $\chi_{r,j} = \begin{cases} 1 & \text{if } \sqrt{\tilde{\lambda}_{j+1}} - \sqrt{\tilde{\lambda}_j} < r \\ 0 & \text{if } \sqrt{\tilde{\lambda}_{j+1}} - \sqrt{\tilde{\lambda}_j} \geq r. \end{cases}$

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