Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 2, 261–273

On Harmonic Potential Fields and the Structure of Monogenic Functions

F. Brackx and R. Delanghe

Abstract. In specific open domains of Euclidean space, a correspondence is established between a monogenic function and a sequence of harmonic potential fields, leading to the construction of a unique vector-valued conjugate harmonic homogeneous polynomial to a given real-valued solid harmonic.

Keywords: *Monogenic function, solid harmonic, harmonic potential field*

AMS subject classification: 30G35

1. Introduction

Consider a holomorphic function $F(z) = u(x, y) + iv(x, y)$ in an open region Ω of the complex plane. As is well known, its real and imaginary parts are real-valued harmonic functions in Ω, satisfying the so-called Cauchy-Riemann system

$$\left.\begin{aligned} \partial_x u &= \partial_y v \\ \partial_y u &= -\partial_x v \end{aligned}\right\}.$$

The vector field $F^*(z) = \big(u(x, y), -v(x, y)\big)$ associated with this holomorphic function $F(z)$ then satisfies the system

$$\left.\begin{aligned} \operatorname{div} F^* &= 0 \\ \operatorname{curl} F^* &= 0 \end{aligned}\right\}. \tag{1.1}$$

If Ω is simply connected, then this vector field F^* can be realized as the gradient of a real-valued harmonic potential $h(x, y)$ in Ω:

$$F^* = \operatorname{grad} h,$$

which in terms of the original holomorphic function F reads

$$F = \partial h,$$

$\partial = \partial_x - i\partial_y$ being the conjugate of the Cauchy-Riemann operator $\overline{\partial} = \partial_x + i\partial_y$.

Both authors: Ghent University, Dept. Math. Anal., Galglaan 2, B-9000 Gent, Belgium
fb@cage.rug.ac.be and rd@cage.rug.ac.be

ISSN 0232-2064 / $ 2.50

Now consider in an open region Ω of $(m+1)$-dimensional Euclidean space a vector-valued function $F^* = (u_0, u_1, \ldots, u_m)$ for which

$$\left.\begin{aligned} \sum_{i=0}^{m} \partial_{x_i} u_i = 0 \\ \partial_{x_j} u_i - \partial_{x_i} u_j = 0 \quad (0 \le i < j \le m) \end{aligned}\right\} \tag{1.2}$$

or, equivalently, in a more compact form:

$$\left.\begin{aligned} \operatorname{div} F^* = 0 \\ \operatorname{curl} F^* = 0 \end{aligned}\right\}. \tag{1.3}$$

This $(m+1)$-tuple F^* is said to be a system of conjugate harmonic functions in Ω (see [8]) and system (1.3) is called the Riesz system, after M. Riesz who obtained it in the following way (see [7]).

The $(m+1)$-dimensional real quadratic vector space $\mathbb{R}^{0,m+1}$ of signature $(0, m+1)$ with orthonormal basis $(e_0, e_1, \ldots, e_m)$ generates the universal Clifford algebra $\mathbb{R}_{0,m+1}$ which contains the scalars $\mathbb{R}$ and the vectors $\mathbb{R}^{0,m+1}$ (see, e.g., [4]). Multiplication is non-commutative and governed by the rules

$$\begin{aligned} e_i^2 = -1 \\ e_i e_j + e_j e_i = 0 \quad (i \neq j) \end{aligned} \qquad (0 \le i, j \le m).$$

Making use of the Dirac operator $\partial_x = \sum_{i=0}^m e_i \partial_{x_i}$ and the Cauchy-Riemann operator $D_x = -e_0 \partial_x = \partial_{x_0} + \sum_{j=1}^m \varepsilon_j \partial_{x_j}$, where $\varepsilon_j = -e_0 e_j \quad (j = 1, \ldots, m)$, M. Riesz observed that the function

$$F = u_0 - \sum_{j=1}^{m} \varepsilon_j u_j \tag{1.4}$$

satisfies $D_x F = 0$ in Ω if and only if the associated vector-valued function $F^* = (u_o, u_1, \ldots, u_m)$ satisfies system (1.3) in Ω. If Ω is simply connected, then this F^* may be realized as the gradient of a real-valued harmonic potential H in Ω, i.e. $F^* = \operatorname{grad} H$.

In this paper we generalize the above results in the framework of Clifford analysis. Clifford analysis offers a function theory which is a higher dimensional analogue of the theory of holomorphic functions of one complex variable (see, e.g., [2, 4]). Central notion is that of a monogenic function, i.e. a null solution of the above mentioned Dirac operator. Our main result (Theorem 3.1) states that if the open region $\Omega \subset \mathbb{R}^{m+1}$ satisfies a specific geometric condition, and the function F, taking values in the Clifford algebra $\mathbb{R}_{0,m+1}$, is monogenic in Ω, then there exists a harmonic potential H in Ω such that $F = \overline{D}_x H$, $\overline{D}_x$ being the conjugate of the Cauchy-Riemann operator D_x. In the particular case where H is real-valued, the function F turns out to be of form (1.4), thus reobtaining the notion of a system of conjugate harmonic functions in the sense of Stein-Weiß. This structure theorem is moreover applied to the so-called spherical monogenics, i.e. homogeneous monogenic polynomials which offer a refinement of the notion of solid spherical harmonics.

2. Conjugate harmonic functions in $\mathbb{R}^{m+1}$

Let $\mathbb{R}^{0,m+1}$ be the real vector space $\mathbb{R}^{m+1}$ $(m \geq 1)$ endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$ of signature $(0, m+1)$, and let $(e_0, e_1, \ldots, e_m)$ be an associated orthonormal basis:

$$\mathcal{B}(e_i, e_j) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (1 \leq i, j \leq m).$$

Furthermore, let $\mathbb{R}_{0,m+1}$ be the universal real Clifford algebra constructed over $\mathbb{R}^{0,m+1}$. The multiplication in $\mathbb{R}_{0,m+1}$ is governed by the rules

$$\begin{aligned} e_i^2 &= -1 \\ e_i e_j + e_j e_i &= 0 \quad (i \neq j) \end{aligned} \qquad (0 \leq i, j \leq m).$$

A basis for the Clifford algebra $\mathbb{R}_{0,m+1}$ is given by $\{e_A\}_{A \subset \{0,1,\ldots,m\}}$ where, for $A = \{i_1, \ldots, i_h\}$ with $0 \leq i_1 < i_2 < \ldots < i_h \leq m$, $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$ and $e_\phi = 1$ is the identity element.

Conjugation in $\mathbb{R}_{0,m+1}$ is defined as the anti-involution for which

$$\bar{e}_i = -e_i \qquad (0 \leq i \leq m).$$

The space of real numbers is identified with the subspace $\mathbb{R}e_\phi = \mathbb{R}1$ of $\mathbb{R}_{0,m+1}$, while the space of vectors $\mathbb{R}^{m+1}$ is identified with the subspace $\mathbb{R}^{0,m+1}$ by the correspondence

$$(x_0, x_1, \ldots, x_m) \to x = \sum_{i=0}^{m} x_i e_i.$$

Note that in $\mathbb{R}_{0,m+1}$

$$x^2 = -\sum_{i=0}^{m} x_i^2 = -|x|^2.$$

Now let Ω be open in $\mathbb{R}^{m+1}$. We consider functions F defined in Ω and taking values in $\mathbb{R}_{0,m+1}$:

$$F(x) = \sum_A F_A(x)\, e_A \qquad (x \in \Omega).$$

Such a function F is called $\mathcal{C}_1$ in Ω, notation $F \in \mathcal{C}_1(\Omega; \mathbb{R}_{0,m+1})$, if all of its components F_A are in $\mathcal{C}_1(\Omega; \mathbb{R})$.

A central notion in so-called Clifford analysis (see, e.g., [2, 4]) is that of monogenicity, which is introduced by means of the Dirac operator

$$\partial_x = \sum_{i=0}^{m} e_i \partial_{x_i}$$

in $\mathbb{R}^{m+1}$.

Definition 2.1. A function $F \in \mathcal{C}_1(\Omega; \mathbb{R}_{0,m+1})$ is (left) *monogenic* in Ω if $\partial_x F = 0$ in Ω.

As the Dirac operator splits the Laplace operator in $\mathbb{R}^{m+1}$: $\Delta_x = -\partial_x^2$, it follows that a monogenic function in Ω is harmonic in Ω, and so are all of its components.

Now consider the m-dimensional real quadratic space $\mathbb{R}^{0,m}$ of signature $(0,m)$, with orthonormal basis $(e_1, \ldots, e_m)$. It generates, within $\mathbb{R}_{0,m+1}$, the universal real Clifford algebra $\mathbb{R}_{0,m}$. We shall identify $\mathbb{R}^m$ with $\mathbb{R}^{0,m}$ by the correspondence

$$(x_1, \ldots, x_m) \to \underline{x} = \sum_{j=1}^{m} x_j e_j.$$

Naturally we still have

$$\underline{x}^2 = -\sum_{j=1}^{m} x_j^2 = -|\underline{x}|^2,$$

but also $x = x_0 e_0 + \underline{x}$ and $x^2 = -x_0^2 + \underline{x}^2$. We will also use the Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

in $\mathbb{R}^m$, for which $\Delta_{\underline{x}} = -\partial_{\underline{x}}^2$. Introducing for $\widetilde{\Omega}$ open in $\mathbb{R}^m$

$$\ker \partial_{\underline{x}} = \left\{ f \in C_\infty(\widetilde{\Omega}; \mathbb{R}_{0,m}) : \partial_{\underline{x}} f = 0 \text{ in } \widetilde{\Omega} \right\}$$
$$\ker \Delta_{\underline{x}} = \left\{ g \in C_\infty(\widetilde{\Omega}; \mathbb{R}_{0,m}) : \Delta_{\underline{x}} g = 0 \text{ in } \widetilde{\Omega} \right\},$$

the surjectivity of the Dirac operator

$$\partial_{\underline{x}} : \mathcal{C}_\infty(\widetilde{\Omega}; \mathbb{R}_{0,m}) \to \mathcal{C}_\infty(\widetilde{\Omega}; \mathbb{R}_{0,m})$$

(see [2]) leads to

Lemma 2.1. *The relation* $\partial_{\underline{x}}(\ker \Delta_{\underline{x}}) = \ker \partial_{\underline{x}}$ *is true.*

We already studied (see [3]) conjugate harmonic functions in $\mathbb{R}^{m+1}$ in the framework of Clifford analysis. A basic idea therein is the splitting of the Clifford algebra $\mathbb{R}_{0,m+1}$ as

$$\mathbb{R}_{0,m+1} = \mathbb{R}_{0,m} \oplus \overline{e}_0 \mathbb{R}_{0,m}$$

and the corresponding splitting of the functions considered

$$F = U + \overline{e}_0 V$$

where U and V are $\mathbb{R}_{0,m}$-valued. It is clear that if F is monogenic in Ω, then U and V are harmonic in Ω. And, as $\partial_x = e_0 \partial_{x_0} + \partial_{\underline{x}}$, we also have

Theorem 2.1. *A function $F = U + \overline{e}_0 V$ is monogenic in Ω if and only if the $\mathbb{R}_{0,m}$-valued functions U and V satisfy the system*

$$\left.\begin{array}{l} \partial_{x_0} U + \partial_{\underline{x}} V = 0 \\ \partial_{\underline{x}} U + \partial_{x_0} V = 0 \end{array}\right\} . \tag{2.1}$$

Clearly, system (2.1) generalizes the Cauchy-Riemann system for holomorphic functions in the complex plane. This leads to the following

Definition 2.1. A pair (U, V) of $\mathbb{R}_{0,m}$-valued harmonic functions in Ω such that $F = U + \overline{e}_0 V$ is monogenic in Ω is called a *pair of conjugate harmonic functions* in Ω.

Notice the special case where the monogenic function F in Ω is either $\mathbb{R}_{0,m}$-valued or $\overline{e}_0\mathbb{R}_{0,m}$-valued. In fact, such a function F reduces to either U or to $\overline{e}_0 V$ respectively, where then U and V are $\mathbb{R}_{0,m}$-valued, are independent of x_0 and monogenic in $\widetilde{\Omega}$, $\widetilde{\Omega}$ denoting the projection of Ω on $\mathbb{R}^m$ along the e_0-direction.

Given an $\mathbb{R}_{0,m}$-valued harmonic function U in Ω, the construction of a conjugate harmonic to U in Ω was carried out in [3] for open regions Ω of a specific geometric shape.

Definition 2.2. An open set $\Omega \subset \mathbb{R}^{m+1}$ is called *normal with respect to the e_0-direction* if there exists an $x_0^* \in \mathbb{R}$ such that for all $\underline{x} \in \widetilde{\Omega}$, $\Omega \cap \{t\overline{e}_0 + \underline{x} : t \in \mathbb{R}\}$ is non-empty, connected and contains the point $x_0^* e_0 + \underline{x}$.

From now on we always assume Ω to be normal with respect to the e_0-direction. The following theorem was proved in [3].

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the e_0-direction, let $U(x_0, \underline{x})$ be an $\mathbb{R}_{0,m}$-valued harmonic function in Ω and let $\widetilde{h}$ satisfy in Ω the Poisson equation $\Delta_{\underline{x}}\widetilde{h} = \partial_{x_0} U(x_0^*, \underline{x})$. Define $\widetilde{H}(x_0, \underline{x})$ in Ω by*

$$\widetilde{H}(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U(t, \underline{x})\, dt - \widetilde{h}(\underline{x})$$

Then:

(i) *$\widetilde{H}(x_0, \underline{x})$ is $\mathbb{R}_{0,m}$-valued and harmonic in Ω.*

(ii) *$\widetilde{V}(x_0, \underline{x}) = -\partial_{\underline{x}}\widetilde{H}(x_0, \underline{x})$ is $\mathbb{R}_{0,m}$-valued and conjugate harmonic to U in Ω.*

(iii) *$\widetilde{F}(x_0, \underline{x}) = U(x_0, \underline{x}) + \overline{e}_0\widetilde{V}(x_0, \underline{x}) = \partial_x \overline{e}_0 \widetilde{H}(x_0, \underline{x}) = \overline{D}_x \widetilde{H}(x_0, \underline{x})$ is monogenic in Ω.*

Now we make the following observation. Any conjugate harmonic V to U in Ω is of the form $V(x_0, \underline{x}) = \widetilde{V}(x_0, \underline{x}) + W(\underline{x})$ where $W(\underline{x})$ is $\mathbb{R}_{0,m}$-valued and monogenic in $\widetilde{\Omega}$, i.e. $\partial_{\underline{x}} W(\underline{x}) = 0$. By Lemma 1.1 there exists a harmonic function h in $\widetilde{\Omega}$ such that $W(\underline{x}) = \partial_{\underline{x}} h(\underline{x})$ and hence $V(x_0, \underline{x}) = \widetilde{V}(x_0, \underline{x}) + \partial_{\underline{x}} h(\underline{x})$. So we obtain

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the e_0-direction, let $U(x_0, \underline{x})$ be an $\mathbb{R}_{0,m}$-valued harmonic function in Ω, let $\widetilde{h}$ be an $\mathbb{R}_{0,m}$-valued $\mathcal{C}_\infty$-function in $\widetilde{\Omega}$ satisfying $\Delta_{\underline{x}}\widetilde{h} = \partial_{x_0} U(x_0^*, \underline{x})$, and let h be an $\mathbb{R}_{0,m}$-valued harmonic* *function in $\widetilde{\Omega}$. Then the most general conjugate harmonic $V(x_0, \underline{x})$ to U in Ω thas the form*

$$V(x_0, \underline{x}) = -\partial_{\underline{x}} H(x_0, \underline{x})$$

where the $\mathbb{R}_{0,m}$-valued harmonic function H in Ω is given by

$$H(x_0, \underline{x}) = \int_{x_0^*}^{x_0} U(t, \underline{x}) dt - \widetilde{h}(\underline{x}) - h(\underline{x}) \quad .$$

Remark 2.1. Given the $\mathbb{R}_{0,m}$-valued harmonic function U in Ω, let V be its conjugate harmonic in Ω as defined in the above Theorem 2.3. Then the pair (U, V) of conjugate harmonics in Ω takes the form $(U, V) = (\partial_{x_0} H, -\partial_{\underline{x}} H)$ and the associated monogenic function F in Ω reads $F = U + \overline{e}_0 V = \partial_x(\overline{e}_0 H) = \overline{D}_x H$.

Remark 2.2. The notion of a conjugate harmonic pair (U, V) of $\mathbb{R}_{0,m}$-valued functions was first introduced by Xu in [9]. Theorem 2.3 extends and completes the results in [10, 11] on the general form of a conjugate harmonic V to U. In [10, 11] conjugate harmonics to the Poisson kernel in the open unit ball and the upper half space respectively were explicitly constructed in this setting.

Remark 2.3. If the given harmonic function u in Ω is real-valued, then the potential function H, as given in Theorem 2.3, can be chosen to be real-valued; the conjugate harmonic v to u in Ω is then $\mathbb{R}^{0,m}$-valued. This special case of conjugate harmonicity was first introduced by Moisil in [6] and taken up again by Stein and Weiß in [8] (see also [5: Proposition 1.7]).

Let us comment on this classical case. If the real-valued function u and the $\mathbb{R}^{0,m}$-valued function $v = \sum_{j=1}^m v_j e_j$ are conjugate harmonic in Ω, then the $(m+1)$-tuple

$$F^* = (u, -v_1, \ldots, -v_m)$$

of real-valued harmonic functions in Ω satisfies the Riesz system

$$\left.\begin{aligned} \operatorname{div} F^* &= 0 \\ \operatorname{curl} F^* &= 0 \end{aligned}\right\} \tag{2.2}$$

and F^* can be realized in Ω as the gradient of a real-valued harmonic potential h, i.e. $F^* = \operatorname{grad} h$, this potential h coinciding with the function H of Theorem 2.3. Conversely, if $F^* = (u, -v_1, \ldots, -v_m)$ is an $(m+1)$-tuple of real-valued functions in Ω satisfying the Riesz system (2.2), then those functions $u, v_1, \ldots, v_m$ are harmonic in Ω and, moreover, the function $F = u + \overline{e}_0 \big(\sum_{j=1}^m v_j e_j \big)$ is monogenic in Ω.

Such an $(m+1)$-tuple F^* was called by Stein and Weiß a system of conjugate harmonic functions in Ω. They played a fundamental role in the characterization of H^p-spaces in the upper half space $\mathbb{R}_+^{m+1} = \{(x_0, \underline{x}) \in \mathbb{R}^{m+1} : x_0 > 0\}$.

From now on we call F^* a *real-valued Stein-Weiß field.*

Remark 2.4. When applying the results of the special case mentioned in the preceeding Remark 2.3 to the two-dimentional case ($m = 1$), we obtain that for a given real-valued harmonic function $u(x_0, x_1)$ in an open set $\Omega \subset \mathbb{R}^2$, which is normal with respect to the e_0-direction, there exists a conjugate harmonic $v(x_0, x_1)$ to u in Ω of the form $v(x_0, x_1) = e_1 v_1(x_0, x_1)$, the function v_1 being real-valued. This conjugate harmonic is determined up to a function of the form $e_1 w_1(x_1)$, the function w_1 being a real-valued null solution to the operator $e_1 \partial_{x_1}$, in other words a real constant.

This is nothing else but the classical pair of conjugate harmonic functions in the plane, yielding the holomorphic function

$$F(x_0, x_1) = u(x_0, x_1) + \varepsilon_1 v_1(x_0, x_1)$$

where $\varepsilon_1 = \overline{e}_0 e_1$ satisfies $\varepsilon_1^2 = -1$, the Cauchy-Riemann operator being identified with

$$\partial_{x_0} + \varepsilon_1 \partial_{x_1} = \overline{e}_0 (e_0 \partial_{x_0} + e_1 \partial_{x_1}).$$

The ordered pair $F^*(x_0, x_1) = (u_1, -v_1)$ satisfies the Riesz system

$$\left.\begin{aligned} \operatorname{div} F^* &= 0 \\ \operatorname{curl} F^* &= 0 \end{aligned}\right\}$$

which coincides with the Cauchy-Riemann system. There exists a real-valued harmonic potential $H(x_0, x_1)$ in Ω such that

$$F^*(x_0, x_1) = \operatorname{grad} H(x_0, x_1),$$

whence the holomorphic function $F(x_0, x_1)$ takes the form

$$\begin{aligned} F(x_0, x_1) &= \partial_{x_0} H + \overline{e}_0 e_1 (-\partial_{x_1} H) \\ &= (e_0 \partial_{x_0} + e_1 \partial_{x_1})(\overline{e}_0 H) \\ &= (\partial_{x_0} - \varepsilon_1 \partial_{x_1}) H(x_0, x_1) \end{aligned}$$

with now the conjugate Cauchy-Riemann operator appearing.

3. A structure theorem for monogenic functions

Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the e_0-direction, let F be monogenic in Ω, and consider its decomposition $F = U + \overline{e}_0 V$, where U and V are $\mathbb{R}_{0,m}$-valued. Then U and V are harmonic in Ω. For the harmonic function U in Ω, construct the $\mathbb{R}_{0,m}$-valued harmonic function $\widetilde{H}$ in Ω as in Theorem 2.2, and put $\widetilde{V} = -\partial_{\underline{x}} \widetilde{H}$ in Ω. Then we know that $\widetilde{F} = U + \overline{e}_0 \widetilde{V}$ is monogenic in Ω, whence

$$F - \widetilde{F} = \overline{e}_0 (V - \widetilde{V})$$

is $\overline{e}_0\mathbb{R}_{0,m}$-valued and monogenic in Ω. It follows (see Section 2) that $V-\widetilde{V}$ is independent of x_0 and monogenic in $\widetilde{\Omega}$: $\partial_{\underline{x}}(V-\widetilde{V})=0$. Hence, by Lemma 1.1, there exists an $\mathbb{R}_{0,m}$-valued harmonic function $h(\underline{x})$ in $\widetilde{\Omega}$ such that $V-\widetilde{V}=\partial_{\underline{x}}h$ or $V=-\partial_{\underline{x}}(\widetilde{H}-h)$ in $\widetilde{\Omega}$ while still $U=\partial_{x_0}(\widetilde{H}-h)$ since h is independent of x_0.

Decompose the $\mathbb{R}_{0,m}$-valued harmonic function $H=\widetilde{H}-h$ in Ω into its components:

$$H=\sum_{B\subset\{1,\ldots,m\}} H_B\, e_B,$$

and put for each $B\subset\{1,\ldots,m\}$

$$F_B^*=(F_{B,0}^*,\, F_{B,1}^*,\ldots,F_{B,m}^*)=\operatorname{grad} H_B.$$

It is clear that for each B the $(m+1)$-tuple F_B^* satisfies the Riesz system

$$\left.\begin{aligned}\operatorname{div} F_B^*&=0\\ \operatorname{curl} F_B^*&=0\end{aligned}\right\},$$

in other words, for each B, F_B^* is a real-valued Stein-Weiß field. For the initial monogenic function F in Ω we find

$$\begin{aligned}F&=U+\overline{e}_0V\\ &=\partial_{x_0}H+\overline{e}_0(-\partial_{\underline{x}}H)\\ &=\sum_B\left(\partial_{x_0}H_B+e_0\sum_{j=1}^m e_j\partial_{x_j}H_B\right)e_B\\ &=\sum_B\left(F_{B,0}^*+e_0\sum_{j=1}^m e_jF_{B,j}^*\right)e_B.\end{aligned}$$

Conversely, given a sequence $(F_B^*)_{B\subset\{1,\ldots,m\}}$, with $F_B^*=(F_{B,0}^*,\ldots,F_{B,m}^*)$, of real-valued Stein-Weiß fields in Ω, the function

$$F=\sum_B\left(F_{B,0}^*+e_0\sum_{j=1}^m e_jF_{B,j}^*\right)e_B$$

is monogenic in Ω. Putting $F_i=\sum_B f_{B,i}e_B$ $(i=0,1,\ldots,m)$, the $(m+1)$-tuple F^* of $\mathbb{R}_{0,m}$-valued harmonic functions in Ω given by

$$F^*=(F_0,F_1,\ldots,F_m)$$

satisfies the system

$$\left.\begin{aligned}\operatorname{div} F^*&=0\\ \operatorname{curl} F^*&=0\end{aligned}\right\}.$$

We call F^* an $\mathbb{R}_{0,m}$*-valued Stein-Weiß field.*

Combining these results with Theorem 2.3, we obtain the following structure theorem for monogenic functions.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the e_0-direction and let $F \in \mathcal{C}_1(\Omega; \mathbb{R}_{0,m+1})$. Then the following assertions are equivalent :*

(i) *F is monogenic in Ω.*

(ii) *There exists an $\mathbb{R}_{0,m}$-valued harmonic potential H in Ω such that $F = \overline{D}_x H$.*

(iii) *There exists an $\mathbb{R}_{0,m}$-valued Stein-Weiß field $F^* = (F_0, F_1, \ldots, F_m)$ such that $F = F_0 + e_0 \sum_{j=1}^{m} e_j F_j$.*

Remark 3.1. Let $\Omega \subset \mathbb{R}^{m+1}$ be open and normal with respect to the e_0-direction and let $F = U + \overline{e}_0 V$ be monogenic in Ω. Recall the construction made at the beginning of this section to end up with $F = U + \overline{e}_0 \widetilde{V} + \overline{e}_0 (V - \widetilde{V})$ or

$$F = \partial_x (\overline{e}_0 \widetilde{H}) + \overline{e}_0 W \tag{3.1}$$

where $W = V - \widetilde{V}$ is an $\mathbb{R}_{0,m}$-valued function, independent of x_0, which is monogenic in $\widetilde{\Omega}$.

If $\widetilde{\Omega}$ satisfies some supplementary geometric conditions, expression (3.1) leads to a further decomposition of the monogenic function F.

For each $j \in \{1, \ldots, m\}$ fixed, consider the real quadratic space $\mathbb{R}^{0,m+1-(j-1)}$ with orthonormal basis $(e_{j-1}, \ldots, e_m)$ which, within $\mathbb{R}_{0,m+1}$, generates the Clifford algebra $\mathbb{R}_{0,m+1-(j-1)}^{\{j-1,\ldots,m\}}$. It gives rise to the splitting

$$\mathbb{R}_{0,m+1-(j-1)}^{\{j-1,\ldots,m\}} = \mathbb{R}_{0,m+1-j}^{\{j,\ldots,m\}} \oplus \overline{e}_{j-1} \mathbb{R}_{0,m+1-j}^{\{j,\ldots,m\}}.$$

We assume $\Omega^{(0)} = \Omega$, open in $\mathbb{R}^{m+1}$, to be such that for $j = 1, \ldots, m$ its projection $\Omega^{(j-1)}$ on

$$\mathbb{R}^{m+1-(j-1)} = \left\{ x^{(j-1)} = (x_{j-1}, \ldots, x_m) : \, x_i \in \mathbb{R} \quad (i = j-1, \ldots, m) \right\}$$

is normal with respect to the e_{j-1}-direction. In each of $\mathbb{R}^{m+1-(j-1)}$ we introduce the corresponding Dirac operator

$$\partial_{x^{(j-1)}} = \sum_{k=j-1}^{m} e_k \partial_{x_k}.$$

Successive application of (3.1) then yields a sequence $\widetilde{H}^{(0)} = \widetilde{H}$, $\widetilde{H}^{(j-1)}$ $(j = 2, \ldots, m)$ of $\mathbb{R}_{0,m+1-j}^{\{j,\ldots,m\}}$-valued harmonic potentials in $\Omega^{(j-1)}$ such that

$$F(x_0, \underline{x}) = \partial_x (\overline{e}_0 \widetilde{H}^{(0)}) + \sum_{j=2}^{m} \overline{e}_0 \overline{e}_1 \cdots \overline{e}_{j-2} \, \partial_{x^{(j-1)}} \, \overline{e}_{j-1} \widetilde{H}^{(j-1)}(x^{(j-1)}). \tag{3.2}$$

Note that the function $\partial_{x^{(j-1)}} \overline{e}_{j-1} \widetilde{H}^{(j-1)}$ determines an $\mathbb{R}_{0,m-(j-1)}$ -valued Stein-Weiß field in $\Omega^{(j-1)}$, for each $j = 1, \ldots, m$. Moreover, as straightforward computations show, at the final stage an $\mathbb{R}_{0,2}^{\{m-1,m\}}$-valued monogenic function $F^{(m-1)}(x_{m-1}, x_m)$ in $\Omega^{(m-1)}$ is found, which can be decomposed in terms of a classical pair of conjugate harmonic functions in the plane (see also Remark 2.4). Decomposition (3.2) is valid, at least locally, for any monogenic function.

4. Homogeneous harmonic polynomials

The results on conjugate harmonics and on the structure of monogenic functions, as established in the previous sections, is now applied to the special case of homogeneous harmonic polynomials in $\mathbb{R}^{m+1}$. As $\mathbb{R}^{m+1}$ is trivially normal with respect to each e_j-direction ($j = 0, 1, \ldots, m$), no supplementary geometric conditions are necessary. In the sequel $2 \leq k \in \mathbb{N}$ will be fixed.

4.1 The general case. Let $U_k(x_0, \underline{x})$ be an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree k in $\mathbb{R}^{m+1}$. We aim at establishing the general form of an $\mathbb{R}_{0,m}$-valued homogeneous polynomial $V_k(x_0, \underline{x})$ of degree k in $\mathbb{R}^{m+1}$ which is conjugate harmonic to $U_k(x_0, \underline{x})$. In view of the results obtained in Section 2 and choosing $x_0^* = 0$, we have to solve in $\mathbb{R}^m$ the equation

$$\Delta_{\underline{x}} \widetilde{h}(\underline{x}) = \widetilde{h}_{k-1}(\underline{x}) \tag{4.1}$$

where $\widetilde{h}_{k-1}(\underline{x}) = \partial_{x_0} U_k(0, \underline{x})$ is an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree $(k-1)$ in the variable $\underline{x} \in \mathbb{R}^m$. According to the Fisher decomposition of homogeneous polynomials (see, e.g., [1]), the function

$$\widetilde{h}_{k+1}(\underline{x}) = \frac{1}{2(2-m-2k)}\, \underline{x}^2 \widetilde{h}_{k-1}(\underline{x}) \tag{4.2}$$

is the unique $\mathbb{R}_{0,m}$-valued homogeneous polynomial of degree $(k+1)$ of this particular form, which satisfies equation (4.1). It follows that the function

$$\widetilde{V}_k(x_0, \underline{x}) = -\partial_{\underline{x}}\, \widetilde{H}_{k+1}(x_0, \underline{x}) \tag{4.3}$$

where

$$\widetilde{H}_{k+1}(x_0, \underline{x}) = \int_0^{x_0} U_k(t, \underline{x})\, dt - \frac{1}{2(2-m-2k)}\, \underline{x}^2 \widetilde{h}_{k-1}(\underline{x}), \tag{4.4}$$

is the unique $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree k of this particular form which is conjugate harmonic to U_k in $\mathbb{R}^{m+1}$.

Obviously, any $\mathbb{R}_{0,m}$-valued homogeneous conjugate harmonic of degree k to U_k is given by

$$V_k(x_0, \underline{x}) = -\partial_{\underline{x}}\, H_{k+1}(x_0, \underline{x}) \tag{4.5}$$

where

$$H_{k+1}(x_0, \underline{x}) = \widetilde{H}_{k+1}(x_0, \underline{x}) - h_{k+1}(\underline{x}), \tag{4.6}$$

$h_{k+1}(\underline{x})$ being an arbitrary $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial of degree $(k+1)$ in $\mathbb{R}^m$.

Finally, denoting by $M^+(k; \mathbb{R}_{0,m+1})$ the space of $\mathbb{R}_{0,m+1}$-valued homogeneous monogenic polynomials of degree k in $\mathbb{R}^{m+1}$ (see [4]), in view of Theorem 3.1 the following structure theorem is obtained.

Theorem 4.1. *Let $P_k \in M^+(k; \mathbb{R}_{0,m+1})$. Then there exists an $\mathbb{R}_{0,m}$-valued homogeneous harmonic polynomial H_{k+1} of degree $(k+1)$ in $\mathbb{R}^{m+1}$ such that $P_k = \overline{D}_x H_{k+1}$.*

4.2 The scalar case. Now assume that $u_k(x_0, \underline{x})$ is a real-valued homogeneous harmonic polynomial of degree k in $\mathbb{R}^{m+1}$, i.e. u_k is a solid harmonic of degree k. Then, according to (4.2) and (4.4), $\widetilde{h}_{k+1}(\underline{x})$ and $\widetilde{H}_{k+1}(x_0, \underline{x})$ are also real-valued, whence, in view of (4.3), $\widetilde{v}_k(x_0, \underline{x})$ is $\mathbb{R}^{0,m}$-valued. So we have at once

Theorem 4.2. *Given a real-valued solid harmonic $u_k(x_0, \underline{x})$ of degree k in $\mathbb{R}^{m+1}$, there exists a unique $\mathbb{R}^{0,m}$-valued homogeneous harmonic polynomial $\tilde{v}_k(x_0, \underline{x})$ of degree k conjugate to $u_k(x_0, \underline{x})$ of the particular form*

$$\widetilde{v}_k(x_0, \underline{x}) = -\partial_{\underline{x}} \widetilde{H}_{k+1}(x_0, \underline{x}) \tag{4.7}$$

where

$$\widetilde{H}_{k+1}(x_0, \underline{x}) = \int_0^{x_0} u_k(t, \underline{x})\, dt - \frac{1}{2(2-m-2k)}\, \underline{x}^2 \partial_{x_0} u_k(0, \underline{x}) \tag{4.8}$$

is a real-valued homogeneous harmonic polynomial of degree $(k+1)$ in $\mathbb{R}^{m+1}$.

Remark 4.1. Taking in (4.6) $h_{k+1}(\underline{x})$ real-valued, then (4.5) - (4.6) express the general form of an $\mathbb{R}^{0,m}$-valued homogeneous polynomial $v_k(x_0, \underline{x})$ of degree k conjugate to the real-valued $u_k(x_0, \underline{x})$.

Remark 4.2. Writing $\widetilde{v}_k$, defined by (4.7), as $\widetilde{v}_k = \sum_{j=1}^m e_j \widetilde{v}_{k,j}$, then, according to the general theory outlined in the previous sections, $P_k = u_k + \overline{e}_0 \widetilde{v}_k$ is an $\mathbb{R} \oplus \overline{e}_0 \mathbb{R}^{0,m}$-valued homogeneous monogenic polynomial of degree k in $\mathbb{R}^{m+1}$. Equivalently, the $(m+1)$-tuple $F_k^* = (u_k, -v_{k,1}, \ldots, -v_{k,m})$ is a real-valued Stein-Weiß field, which can be realized as the gradient of a real-valued solid harmonic of degree $(k+1)$, which is precisely the function $\widetilde{H}_{k+1}$ given by (4.8).

Remark 4.3. The conjugate harmonic $\widetilde{v}_k$ to u_k given by (4.7) - (4.8) may be rewritten as

$$\widetilde{v}_k(x_0, \underline{x}) = \widetilde{v}_k^{(1)}(x_0, \underline{x}) + \underline{x}\, w_{k-1}^{(1)}(\underline{x}) + \underline{x}^2 w_{k-2}^{(2)}(\underline{x}) \tag{4.9}$$

where

(i) $\widetilde{v}_k^{(1)}(x_0, \underline{x}) = -\int_0^{x_0} \partial_{\underline{x}}\, u_k(t, \underline{x}) dt$ is an $\mathbb{R}^{0,m}$-valued homogeneous harmonic polynomial of degree k in $\mathbb{R}^{m+1}$

(ii) $w_{k-1}^{(1)}(\underline{x}) = \frac{1}{2k+m-2}\, \partial_{x_0} u_k(0, \underline{x})$ is a real-valued homogeneous harmonic polynomial of degree $(k-1)$ in $\mathbb{R}^m$

(iii) $w_{k-2}^{(2)}(\underline{x}) = -\frac{1}{2}\, \partial_{\underline{x}} w_{k-1}^{(1)}(\underline{x})$ is an $\mathbb{R}^{0,m}$-valued homogeneous harmonic polynomial of degree $k-2$ in $\mathbb{R}^m$.

Remark 4.4. When applied to the specific case where the dimension $m = 1$, Theorem 4.2 yields the existence of a unique homogeneous conjugate harmonic polynomial $\widetilde{v}_k(x_0, x_1)$ to the real-valued homogeneous harmonic polynomial $u_k(x_0, x_1)$ in $\mathbb{R}^2$. This conjugate harmonic is $\mathbb{R}^{0,1}$-valued and of the particular form

$$\widetilde{v}_k(x_0, x_1) = -e_1 \partial_{x_1} \widetilde{H}_{k+1}(x_0, x_1)$$

where

$$\widetilde{H}_{k+1}(x_0,x_1) = \int_0^{x_0} u_k(t,x_1)\,dt + \frac{1}{2(1-2k)}\,\partial_{x_0} u_k(0,x_1).$$

The corresponding homogeneous monogenic polynomial in $\mathbb{R}^2$ takes the form

$$P_k(x_0,x_1) = (\partial_{x_0} - \varepsilon_1 \partial_{x_1})\widetilde{H}_{k+1}(x_0,x_1)$$

which is nothing else but a homogeneous holomorphic polynomial in the plane, where, as already mentioned in Remark 2.4, $\varepsilon_1 = \overline{e}_0\, e_1$ takes over the role of the imaginary unit i in $\mathbb{C}$.

4.3 The polynomials $V_{\underline{\alpha}}$. A basis for the right $\mathbb{R}_{0,m+1}$-module $M^+(k;\mathbb{R}_{0,m+1})$ is given by

$$\left\{V_{\underline{\alpha}}(x) : \underline{\alpha} = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m,\ |\underline{\alpha}| = k\right\}$$

where $V_{\underline{\alpha}}$ is the so-called Cauchy-Kowalewskaia extension of the real-valued polynomial $\underline{x}^{\underline{\alpha}} = x_1^{\alpha_1}\cdots x_m^{\alpha_m}$ in $\mathbb{R}^m$ (see [4]):

$$V_{\underline{\alpha}}(x_0,\underline{x}) = \sum_{j=0}^{|\underline{\alpha}|} (-1)^j\,\frac{x_0^j}{j!}\,(\overline{e}_0\,\partial_{\underline{x}})^j\,\underline{x}^{\underline{\alpha}}. \tag{4.10}$$

Then each $P_k \in M^+(k;\mathbb{R}_{0,m+1})$ may be expressed as

$$P_k(x) = \sum_{|\alpha|=k} V_{\underline{\alpha}}(x)\,a_{\underline{\alpha}}$$

where $a_{\underline{\alpha}} = \frac{1}{\underline{\alpha}!}\,\partial^{\underline{\alpha}} P_k$. Expression (4.10) shows that $V_{\underline{\alpha}}$ is $\mathbb{R} \oplus \overline{e}_0\,\mathbb{R}^{0,m}$-valued. Its real part is given by

$$u_{\underline{\alpha}}(x_0,\underline{x}) = \sum_{l=0}^{[|\underline{\alpha}|/2]} \frac{x_0^{2l}}{(2l)!}\,(-\Delta_{\underline{x}})^l\,\underline{x}^{\underline{\alpha}}. \tag{4.11}$$

Its unique $\mathbb{R}^{0,m}$-valued conjugate harmonic $\widetilde{v}_{\underline{\alpha}}(x_0,\underline{x})$ determined by (4.7) - (4.8) then reads

$$\widetilde{v}_{\underline{\alpha}}(x_0,\underline{x}) = -\partial_{\underline{x}}\,\widetilde{H}_{\underline{\alpha}}(x_0,\underline{x})$$

where

$$\widetilde{H}_{\underline{\alpha}}(x_0,\underline{x}) = \int_0^{x_0} u_{\underline{\alpha}}(t,\underline{x})\,dt = \sum_{l=0}^{[|\underline{\alpha}|/2]} \frac{x_0^{2l+1}}{(2l+1)!}\,(-\Delta_{\underline{x}})^l\,\underline{x}^{\underline{\alpha}}$$

since $\partial_{x_0} u_{\underline{\alpha}}(x_0,\underline{x})|_{x_0=0} = 0$. Consequently,

$$V^*_{\underline{\alpha}}(x_0,\underline{x}) = u_{\underline{\alpha}}(x_0,\underline{x}) + \overline{e}_0\widetilde{v}_{\underline{\alpha}}(x_0,\underline{x}) = \partial_x\,\overline{e}_0\widetilde{H}_{\underline{\alpha}}(x_0,\underline{x}) = \overline{D}_x\widetilde{H}_{\underline{\alpha}}(x_0,\underline{x})$$

belongs to $M^+(k;\mathbb{R}_{0,m+1})$. Now taking the restriction to $\mathbb{R}^m$, identified with $\{x_0 = 0\}$, yields $V^*_{\underline{\alpha}}(0,\underline{x}) = x^{\underline{\alpha}} = V_{\underline{\alpha}}(0,\underline{x})$. The uniqueness of the Cauchy-Kowalewskaia extension then implies that $V^*_{\underline{\alpha}}(x) = V_{\underline{\alpha}}(x)$.

We thus have proved

Theorem 4.3. *Let* $\underline{\alpha} = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ *with* $|\underline{\alpha}| = k$ *and let* $V_{\underline{\alpha}}(x_0, \underline{x})$ *be the associated basis polynomial in* $M^+(k; \mathbb{R}_{0,m+1})$ *given by*

$$V_{\underline{\alpha}}(x_0, \underline{x}) = \sum_{j=0}^{|\underline{\alpha}|} (-1)^j \frac{x_0^j}{j!} (\overline{e}_0 \partial_{\underline{x}})^j \underline{x}^{\underline{\alpha}}.$$

Then this basis polynomial can be expressed as

$$V_{\underline{\alpha}}(x_0, \underline{x}) = \overline{D}_x \widetilde{H}_{\underline{\alpha}}(x_0, \underline{x})$$

where

$$\widetilde{H}_{\underline{\alpha}}(x_0, \underline{x}) = \sum_{l=0}^{[|\underline{\alpha}|/2]} \frac{x_0^{2l+1}}{(2l+1)!} (-\Delta_{\underline{x}})^l \underline{x}^{\underline{\alpha}}$$

is a real-valued homogeneous harmonic polynomial of degree $k+1$ *in* $\mathbb{R}^{m+1}$.

Acknowledgement. The authors wish to thank the referee for his valuable comments and suggestions.

References

[1] Brackx, F., Constales, D., Ronveaux, A. and H. Serras: *On the harmonic and monogenic decomposition of polynomials.* J. Symbolic Computation 8 (1989), 297 – 304.

[2] Brackx, F., Delanghe, R. and F. Sommen: *Clifford Analysis.* London: Pitman 1982.

[3] Brackx, F., Delanghe, R. and F. Sommen: *On conjugate harmonic functions in Euclidean space.* Math. Methods Appl. Sci. 25 (2002), 1553 – 1562.

[4] Delanghe, R., Sommen, F. and V. Souček: *Clifford Algebra and Spinor-Valued Functions.* Dordrecht: Kluwer 1992.

[5] Mitrea, M.: *Clifford Wavelets, Singular Integrals and Hardy Spaces.* Berlin: Springer-Verlag 1994.

[6] Moisil, Gr.: *Sur la généralisation des fonctions conjuguées.* Rend. Acad. Naz. dei Lincei 14 (1931), 401 – 408.

[7] Riesz, M.: *Clifford numbers and spinors* (Inst. Phys. Sci. & Techn. Lect. Ser.: Vol. 38). Maryland 1958.

[8] Stein, E. M. and G. Weiß: *On the theory of harmonic functions of several variables.* Part I: *The theory of* H^p *spaces.* Acta Math. 103 (1960), 25 – 62.

[9] Xu, Z.: *Boundary Value Problems and Function Theory for Spin-Invariant Differential Operators.* Ph.D thesis. Gent 1989.

[10] Xu, Z., Chen, J. and W. Zhang: *A harmonic conjugate of the Poisson kernel and a boundary value problem for monogenic functions in the unit ball of* $\mathbb{R}^n (n \geq 2)$. Simon Stevin 64 (1990), 187 – 201.

[11] Xu, Z. and C. Zhou: *On boundary value problems of Riemann-Hilbert type for monogenic functions in a half space of* $\mathbb{R}^m (m \geq 2)$. Complex Variables: Theory & Appl. 22 (1993), 181 – 194.

Received 02.05.2002; in revised form 09.01.2003