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Deriving Harmonic Functions in Higher Dimensional Spaces

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Abstract. For a harmonic function, by replacing its variables with norms of vectors in some multi-dimensional spaces, we may induce a new function in a higher dimensional space. We show that, after applying to it a certain power of the Laplacian, we obtain a new harmonic function in the higher dimensional space. We show that Poisson and Cauchy kernels and Newton potentials, and even heat kernels are all deducible using this method based on their forms in the lowest dimensional spaces. Fueter's theorem and its generalizations are deducible as well from our results. The latter has been used to singular integral and Fourier multiplier theory on the unit spheres and their Lipschitz perturbations of higher dimensional Euclidean spaces.

Keywords: *Harmonic functions, Cauchy-Riemann operator, Clifford monogenic functions, singular integrals, Fourier multipliers*

AMS subject classification: Primary 30G35, 31A05, secondary 42B20, 42B35

1. Introduction

If f^0 is a holomorphic function in an open set of the upper half complex plane and

$$f^0(z) = u(s,t) + iv(s,t) \qquad (z = s + it),$$

then Fueter's Theorem [3] asserts that in the corresponding region there holds

$$D\Delta\Big(u(q_0, |\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} v(q_0, |\underline{q}|)\Big) = 0$$

with

$\underline{q} = q_1 i + q_2 j + q_3 k$
$D = D_0 + \underline{D}$, $\underline{D} = D_1 i + D_2 j + D_3 k$
$\Delta = D_0^2 + D_1^2 + D_2^2 + D_3^2$

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where i, j, k are the basic elements of the Hamilton quaternionic space and $D_i = \frac{\partial}{\partial q_i}$ $(i = 0, 1, 2, 3)$. The quaternionic space may be identified with R_1^n for $n = 3$, where

$$R_1^n = \left\{x = x_0 + \underline{x} : x_0 \in \mathbb{R} \text{ and } \underline{x} \in \mathbb{R}^n\right\}$$

and

$$R^n = \left\{\underline{x} = x_1 e_1 + \ldots x_n e_n : x_i \in \mathbb{R} \ \ (i = 1, \ldots, n)\right\}$$

where $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i, j = 1, \ldots, n$ with $i < j$ [1].

Fueter's Theorem was extended by Sce in 1957 to R_1^n for n being odd positive integers [8]. He proved that, under the same assumptions on f, there holds

$$D\Delta^{\frac{n-1}{2}}\left(u(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|)\right) = 0$$

where

$D = D_0 + \underline{D}$, $\underline{D} = D_1 e_1 + \ldots + D_n e_n$
$\Delta = D_0^2 + D_1^2 + \ldots + D_n^2$

with $D_i = \frac{\partial}{\partial x_i}$ for $i = 0, 1, \ldots, n$. Using Fourier transformation, Qian extended the results to R_1^n for n being arbitrary even positive integers [6].

In a recent paper Sommen proved the following result: If n is an odd positive integer, then

$$D\Delta^{k+\frac{n-1}{2}}\left(\left(u(x_0, \underline{x}) + \frac{\underline{x}}{|\underline{x}|} v(x_0, |\underline{x}|)\right) P_k(\underline{x})\right) = 0$$

where P_k is any polynomial in $\underline{x}$ of homogeneity k, left-monogenic with respect to the Dirac operator $\underline{D}$, viz. $\underline{D}P_k(\underline{x}) = 0$ [9]. When $k = 0$, this reduces to Sce's result.

Extension of Fueter's theorem starting from holomorphic functions of one complex variable is made complete by including the non-integer powers (the space dimension $n + 1$ is odd) of the Laplacian in [4].

The present paper deals with harmonic functions using elementary knowledge of Clifford analysis. The writing plan is as follows. Section 2 contains statements of our new results. The results are in terms of identities in relation to multiple powers of the Laplacian on induced functions from lower dimensional harmonic functions. The identities in a great extent simplify the computation. As consequence, they conclude the hamonicity of the induced functions after being applied certain powers of the Laplacian. Section 3 is devoted to proofs of the theorems. In Section 4 we deal with applications. We show that almost all commonly used kernels and generalizations of Fueter's Theorem are deducible from our theorems.

In below, the letters c or c_p denote general constants, the latter stressing the dependence on the parameter p, which may be different from occurrence to occurrence.

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2. Statement of the theorems

Denote

$$R^p = \Big\{ \underline{x} = x_1 e_1 + \ldots + x_p e_p : \, x_i \in \mathbb{R} \;\; (i = 1, \ldots, p) \Big\}$$
$$R^q = \Big\{ \underline{y} = y_1 e_{p+1} + \ldots + y_q e_{p+q} : \, y_i \in \mathbb{R} \;\; (i = 1, \ldots, q) \Big\}$$

where $p, q \in \mathbb{N}$ (the set of all positive integers), $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$ and $i, j = 1, \ldots, m$ with $p + q = m$. Here we use the standard notation of [1].

We have the following

Theorem 1. *Let $h = h(s,t)$ be a harmonic function in the variables s and t. Then*

$$\Delta^k \big(h(|\underline{x}|, |\underline{y}|) \big) = \sum_{l=k}^{0} \frac{k!}{l!\,(k-l)!} \, d_p(l) d_q(k-l) D_s(l) D_t(k-l) h(s,t) \tag{1}$$

where the Laplacian Δ for $p+q$ variables is

$$\Delta = (\partial_{x_1}^2 + \ldots + \partial_{x_p}^2) + (\partial_{y_1}^2 + \ldots + \partial_{y_q}^2),$$

$s = |\underline{x}|$ and $t = |\underline{y}|$, $d_p(l) = (p-1) \cdots (p-2l+1)$ with $d_p(0) = 1$, $D_s(l) = (\frac{1}{s}\partial_s)^l$, and $d_q(l)$ and $D_t(l)$ are defined similarly for $l = 0, 1, \ldots, k$.

More generally, denote

$$\underline{x}^{(r)} = x_1^{(r)} e_1^{(r)} + \ldots + x_{p_r}^{(r)} e_{p_r}^{(r)} \in R^{p_r}$$

where $r = 1, \ldots, d$, $\sum_{r=1}^{d} p_r = m$ and $e_i^{(r)} e_{i'}^{(r')} = -e_{i'}^{(r')} e_i^{(r)}$ whenever $(r,i) \neq (r', i')$ $(i = 1, \ldots, p_r;\ i' = 1, \ldots, p_{r'})$.

Theorem 2. *Let $h = h(s_1, \ldots, s_d)$ be a harmonic function in the d variables $s_1, \ldots, s_d$. Then*

$$\Delta^k h\big(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|\big) = \sum \frac{k!}{l_1! \cdots l_d!} \prod_{r=1}^{d} d_{p_r}(l_r) \prod_{r=1}^{d} D_{s_r}(l_r)\, h(s_1, \ldots, s_d) \tag{2}$$

where the Laplacian Δ is for all m variables and the summation is over all possible $l_1, \ldots, l_d \in \mathbb{N}_0$ such that $\sum_{r=1}^{d} l_r = k$.

Theorem 2 is a generalization of Theorem 1. It has the following application.

Theorem 3. *If, in addition to the assumptions in Theorem 2, we assume that p_r $(r = 1, \ldots, d)$ are odd and $m = \sum_{r=1}^{d} p_r$ is even, then*

$$\Delta^{\frac{m}{2}} h\big(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|\big) = 0.$$

It would be worthwhile noting that the conclusion of Theorem 3 does not hold in general if some p_r's are even. For instance, when $h(s,t) = s+t$ and $p = q = 2$, we have $\Delta^{\frac{p+q}{2}} h(|\underline{x}|, |\underline{y}|) = \frac{1}{s^3} + \frac{1}{t^3} \neq 0$.

Variations of Theorem 3 corresponding to the total dimension m being even are suggested by the discussions in Section 4.

We say that a Clifford-valued function f in

$$R^d = \{\underline{s} = s_1 j_1 + \ldots + s_d j_d\},$$

where $j_r^2 = -1$ and $j_r j_{r'} = -j_{r'} j_r$ for $1 \leq r, r' \leq d$ with $r \neq r'$, is *left-monogenic* with respect to the Dirac operator $\underline{\partial}_{\underline{s}} = \partial_{s_1} j_1 + \ldots + \partial_d j_d$, if $\underline{\partial}_{\underline{s}} f = 0$ (see [1]). If a monogenic function f is vector-valued, then locally one can always find a scalar-valued harmonic function h such that $f = \underline{\partial}_{\underline{s}} h = \sum_{r=1}^d h_r j_r$ with $h_r = \partial_r h$ (cf. conjugate harmonic functions systems in [10]).

We have the following corollary of Theorem 3.

Corollary 1. *Let $f(\underline{s})$ be left-monogenic with respect to $\underline{\partial}_{\underline{s}}$. Then, in accordance with the expansion $f(\underline{s}) = \sum_{r=1}^d h_r j_r$ as cited above and under the assumptions in Theorem 3, the function*

$$\Delta^{\frac{m}{2}-1}\left(\sum_{r=1}^{d} h_r\left(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|\right) \frac{\underline{x}^{(r)}}{|\underline{x}^{(r)}|} \right)$$

is left-monogenic with respect to $\underline{\partial} = \sum_{r=1}^d \underline{+\partial}_r$ and $\underline{\partial}_r = \sum_{i=1}^{p_r} \partial_{x_i^{(r)}} e_i^{(r)}$, that is

$$\underline{\partial}\left(\Delta^{\frac{m}{2}-1}\left(\sum_{r=1}^{d} h_r\left(|\underline{x}^{(1)}|, \ldots, |\underline{x}^{(d)}|\right) \frac{\underline{x}^{(r)}}{|\underline{x}^{(r)}|} \right)\right) = 0.$$

It is easy to see that Fueter's original result can be made (through the commonly used correspondence between $\mathbb{R}^{n+1}$ and R_1^n, see the notation in [1]) to correspond to the case $d = 2, p_1 = 1$ and $p_2 = 3$. And Sce's result can be made to correspond to the case $d = 2, p_1 = 1$ and p_2 any odd positive integer, in the Corollary. We will re-produce these results from an alternative approach in Section 4.

3. Proofs of the theorems

The proofs of Theorems 1 and 2 rely on the following technical lemma.

Lemma 1. *For any $l \in \mathbb{N}$, let c_l^j $(j = 1, \ldots, l)$ be l real numbers, where $c_l^l = 1$. Then the following conditions are equivalent for the numbers c_l^j:*

(i) $D_s(l) = \left(\frac{1}{s}\partial_s\right)^l = c_l^1 \frac{1}{s^{2l-1}} \partial_s + c_l^2 \frac{1}{s^{2l-2}} \partial_s^2 + \ldots + c_l^l \frac{1}{s^l} \partial_s^l$.

(ii) $c_l^1 = (-1)^{l-1}(2l-3)!!$, $c_l^j + c_l^{j+1}[-(2l-j-1)] = c_{l+1}^{j+1}$ *and* $c_l^{l+1} = 0$.

(iii) $c_l^1 = (-1)^{l-1}(2l-3)!!$ *and* $[(2l-j)]c_l^j = j c_{l+1}^{j+1}$.

(iv) $c_l^j = (-1)^{l-j} \frac{(2l-j-1)!\,(2l-2j-1)!!}{(j-1)!\,(2l-2j)!}$.

Proof. Assuming (i) holds, by applying the operator $\frac{1}{s}\partial_s$ to the expansion of $D_s(l)$ in (i) and comparing the obtained coefficients with those in the expansion for $D_s(l+1) = [\frac{1}{s}\partial_s]^{l+1}$, we obtain the recurrence relation in (ii). Mathematical induction based on (ii) will deduce (i). Conditions (i) and (ii) therefore are equivalent. It is a direct computation to verify that (iii) and (iv) are equivalent, too. Further, it is easy to verify that c_l^j given by (iv) is the only sequence satisfying (ii). The proof is complete ∎

Now we give the

Prove of Theorem 1. We will use the notation

$$\partial = \partial_{\underline{x}} + \partial_{\underline{y}}$$

where $\partial_{\underline{x}} = \sum_{i=1}^p \partial_{x_i} e_i$ and $\partial_{\underline{y}} = \sum_{i=1}^q \partial_{y_i} e_{p+i}$. Therefore, $\Delta = -\partial^2$. We will use the decompositions

$$\partial_{\underline{x}} = \underline{\omega_{x}}\partial_s + \underline{\omega_{x}}\frac{1}{s}\Gamma_{\underline{x}} \qquad \text{and} \qquad \partial_{\underline{y}} = \underline{\omega_{y}}\partial_t + \underline{\omega_{y}}\frac{1}{t}\Gamma_{\underline{y}}$$

where $s = |\underline{x}|$ and $t = |\underline{y}|$, $\underline{x} = s\underline{\omega_{x}}$ and $\underline{y} = t\underline{\omega_{y}}$, and $\Gamma_{\underline{x}}$ and $\Gamma_{\underline{y}}$ are the spherical Dirac operators in the spaces $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively (see [1]).

We first show that for any scalar-valued function $h(s,t)$ with $s = |\underline{x}|$ and $t = |\underline{y}|$ we have

$$\Delta h(\underline{x},\underline{y}) = \Big[\Big(\partial_s^2 + \frac{p-1}{s}\partial_s\Big) + \Big(\partial_t^2 + \frac{q-1}{t}\partial_t\Big)\Big] h(s,t). \tag{3}$$

Indeed, first we have

$$\begin{aligned}\partial h(|\underline{x}|,|\underline{y}|) &= (\partial_{\underline{x}} + \partial_{\underline{y}}) h(|\underline{x}|,|\underline{y}|)\\ &= \Big[\Big(\underline{\omega_{x}}\partial_s + \underline{\omega_{x}}\frac{1}{s}\Gamma_{\underline{x}}\Big) + \Big(\underline{\omega_{y}}\partial_t + \underline{\omega_{y}}\frac{1}{t}\Gamma_{\underline{y}}\Big)\Big] h(|\underline{x}|,|\underline{y}|)\\ &= \big(\underline{\omega_{x}}\partial_s + \underline{\omega_{y}}\partial_t\big) h(|\underline{x}|,|\underline{y}|).\end{aligned}$$

Further, since $\Delta = -\partial^2$ we have

$$\Delta h(|\underline{x}|,|\underline{y}|) = -\Big[\Big(\underline{\omega_{x}}\partial_s + \underline{\omega_{x}}\frac{1}{s}\Gamma_{\underline{x}}\Big) + \Big(\underline{\omega_{y}}\partial_t + \underline{\omega_{y}}\frac{1}{t}\Gamma_{\underline{y}}\Big)\Big] \big(\underline{\omega_{x}}\partial_s + \underline{\omega_{y}}\partial_t\big) h(|\underline{x}|,|\underline{y}|).$$

On the right-hand-side of the above, $\Gamma_{\underline{x}}$ and $\Gamma_{\underline{y}}$ do not vanish only on $\underline{\omega_{x}}$ and $\underline{\omega_{y}}$, respectively. By taking into account the relations

$$\begin{aligned}\Gamma_{\underline{x}}\underline{\omega_{x}} &= (p-1)\underline{\omega_{x}}\\ \Gamma_{\underline{y}}\underline{\omega_{y}} &= (q-1)\underline{\omega_{y}}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\underline{\omega_{x}}\underline{\omega_{y}} + \underline{\omega_{y}}\underline{\omega_{x}} &= 0\\ \underline{\omega_{x}}^2 = \underline{\omega_{y}}^2 &= -1\end{aligned}$$

we obtain (3).

We use mathematical induction on k. When $k=1$, since h is harmonic, owing to relation (3) we have

$$\begin{aligned}\Delta h(|\underline{x}|,|\underline{y}|) &= \Big[(p-1)\Big(\frac{1}{s}\partial_s\Big)+(q-1)\Big(\frac{1}{t}\partial_t\Big)\Big]h(s,t)\\ &= \big[d_p(1)D_s(1)+d_q(1)D_t(1)\big]h(s,t)\end{aligned}$$

as desired. Now we proceed to show that in the case (1) holds for an integer k, then (1) also holds for $k+1$. We apply the operator Δ to the l-th general term of (1), namely

$$T(k;l)=\frac{k!}{l!\,(k-l)!}\,d_p(l)d_q(k-l)D_s(l)D_t(k-l)h(s,t).$$

We write

$$\Delta T(k;l)=L(s)T(k;l)+L(t)T(k;l)$$

where

$$L(s)=\partial_s^2+\frac{p-1}{s}\partial_s \qquad \text{and} \qquad L(t)=\partial_t^2+\frac{q-1}{t}\partial_t.$$

We now compute $L(s)T(k;l)$ (that of $L(t)T(k;l)$ is similar). Indeed, temporarily putting aside the coefficient and the differential operator $D_t(k-l)$, using the expansion of $D_s(l)$ in Lemma 1, we have

$$\begin{aligned}\partial_s D_s(l)h(s,t) &= sD_s(l+1)h(s,t)\\ &= \Big(c_{l+1}^1\frac{1}{s^{2l}}\partial_s+c_{l+1}^2\frac{1}{s^{2l-1}}\partial_s^2+\ldots+c_{l+1}^{l+1}\frac{1}{s^l}\partial_s^{l+1}\Big)h(s,t).\end{aligned}$$

Successively,

$$\begin{aligned}\partial_s^2 D_s(l)h(s,t) &= \partial_s\Big(c_{l+1}^1\frac{1}{s^{2l}}\partial_s+c_{l+1}^2\frac{1}{s^{2l-1}}\partial_s^2+\ldots+c_{l+1}^{l+1}\frac{1}{s^l}\partial_s^{l+1}\Big)h(s,t)\\ &= \Big(c_{l+1}^1[-(2l)]\frac{1}{s^{2l+1}}\partial_s+c_{l+1}^1\frac{1}{s^{2l}}\partial_s^2\\ &\quad+c_{l+1}^2[-(2l-1)]\frac{1}{s^{2l}}\partial_s^2+c_{l+1}^2\frac{1}{s^{2l-1}}\partial_s^3\\ &\quad\vdots\\ &\quad+c_{l+1}^{l+1}[-l]\frac{1}{s[l+1]}\partial_s^{l+1}+c_{l+1}^{l+1}\frac{1}{s^l}\partial_s^{l+2}\Big)h(s,t)\\ &= \Big(c_{l+1}^1[-(2l)]\frac{1}{s^{2l+1}}\partial_s+\big(c_{l+1}^1+c_{l+1}^2[-(2l-1)]\big)\frac{1}{s^{2l}}\partial_s^2\\ &\quad+\big(c_{l+1}^2+c_{l+1}^3[-(2l-2)]\big)\frac{1}{s^{2l-1}}\partial_s^3\\ &\quad\vdots\\ &\quad+\big(c_{l+1}^l+c_{l+1}^{l+1}[-l]\big)\frac{1}{s^{l+1}}\partial_s^{l+1}+c_{l+1}^{l+1}\frac{1}{s^l}\partial_s^{l+2}\Big)h(s,t).\end{aligned}$$

Owing to the relation $c_{l+1}^2 = -c_{l+1}^1$ obtained from Lemma 1, we can verify that the coefficient of the second term of the last obtained expression is

$$c_{l+1}^1 + c_{l+1}^2[-(2l-1)] = (-2l)c_{l+1}^2.$$

Relations (ii) and (iii) of Lemma 1, read as

$$c_l^{j-1} + c_l^j[-(2l-j)] = c_{l+1}^j \quad \text{and} \quad [(2l-j)]c_l^j = jc_{l+1}^{j+1} \qquad (j = 2, \ldots, l),$$

imply

$$c_{l+1}^j + jc_{l+1}^{j+1} = c_l^{j-1}.$$

This decides that in the last obtained expression the coefficients of the terms after the second are

$$c_{l+1}^j + c_{l+1}^{j+1}[-(2l-j)] = (-2l)c_{l+1}^{j+1} + (c_{l+1}^j + jc_{l+1}^{j+1}) = (-2l)c_{l+1}^{j+1} + c_l^{j-1}.$$

We hence arrive

$$\begin{aligned}
&\partial_s^2 D_s(l) D_t(k-l) h(s,t) \\
&\quad = (-2l)\Big(c_{l+1}^1 \frac{1}{s^{2l+1}}\partial_s + c_{l+1}^2 \frac{1}{s^{2l}}\partial_s^2 + c_{l+1}^3 \frac{1}{s^{2l-1}}\partial_s^3 \\
&\qquad + \ldots + c_{l+1}^{l+1}\frac{1}{s^{[l+1]}}\partial_s^{l+1}\Big) D_t(k-l)h(s,t) \\
&\qquad + \Big(c_l^1 \frac{1}{s^{2l-1}}\partial_s + c_l^2 \frac{1}{s^{2l-2}}\partial_s^2 + \ldots + c_l^l \frac{1}{s^l}\partial_s^l\Big) D_t(k-l)\partial_s^2 h(s,t) \\
&\quad = (-2l) D_s(l+1) D_t(k-l) h(s,t) + D_s(l) D_t(k-l)\partial_s^2 h(s,t).
\end{aligned}$$

This gives that

$$\begin{aligned}
L(s)T(k;l) &= \Big(\partial_s^2 + \frac{p-1}{s}\partial_s\Big) T(k;l) \\
&= \frac{k!}{l!\,(k-l)!} d_p(l+1) d_q(k-l) D_s(l+1) D_t(k-l) h(s,t) \\
&\quad + \frac{k!}{l!\,(k-l)!} d_p(l) d_q(k-l) D_s(l) D_t(k-l) \partial_s^2 h(s,t)
\end{aligned} \tag{4}$$

where we have used the relation $(p-2l-1)d_p(l) = d_p(l+1)$. Similarly,

$$\begin{aligned}
L(t)T(k;l) &= \frac{k!}{l!\,(k-l)!} d_p(l) d_q(k-l+1) D_s(l) D_t(k-l+1) h(s,t) \\
&\quad + \frac{k!}{j!\,(k-l)!} d_p(l) d_q(k-l) D_s(l) D_t(k-l) \partial_t^2 h(s,t).
\end{aligned} \tag{5}$$

Since h is harmonic, in the expansion of $\Delta T(k;l)$ the two second terms of (4) and (5) are canceled out.

Now we show that the first term in the expansion of $L(s)T(k;l)$ can be combined with the first term in the expansion of $L(t)T(k;l+1)$ for $l<k$. In formula (5), with l being replaced by $l+1$, its first term becomes

$$\frac{k!}{(l+1)!\,(k-l-1)!}\,d_p(l+1)d_q(k-l)D_s(l+1)D_t(k-l)h(s,t).$$

Now add this term with the first term of (4). After combining their coefficients, summation gives

$$\begin{aligned}\frac{(k+1)!}{(l+1)!\,\big((k+1)-(l+1)\big)!}\,&d_p(l+1)d_q(k-l)D_s(l+1)D_t(k-l)h(s,t)\\ &=T(k+1;l+1).\end{aligned}$$

There is no such combining for the first term in $L(s)T(k;k)$, that is $T(k+1;k+1)$. In the above pattern, the first term of $L(s)T(k;k-1)$ is combined with the first term of $L(t)T(k;k)$ to form $T(k+1;k)$, the first term of $L(s)T(k;k-2)$ is combined with the first term of $L(t)T(k;k-1)$ to form $T(k+1;k-1)$, and so on. There is no such combining for the first term of $L(t)T(k;0)$, that is $T(k+1;0)$. The proof is complete ∎

Theorem 2 can be proved in the same spirit. With the preparations made in the proof of Theorem 1, its proof will now be briefly cited as follows.

Proof of Theorem 2. First, relation (3) is extended to

$$\Delta h\big(|\underline{x}^{(1)}|,\ldots,|\underline{x}^{(d)}|\big)=\sum_{r=1}^{d}L(s_r)h(s_1,\ldots,s_d)$$

where $L(s_r)=\partial_{s_r}^2+\frac{p_r-1}{s_r}\partial_{s_r}$ $(r=1,\ldots,d)$. To proceed the proof, we use mathematical induction. Since h is harmonic, the desired relation holds for $k=1$. Now denote

$$T(k;l_1,\ldots,l_d)=\frac{k!}{l_1!\cdots l_d!}\prod_{r=1}^{d}d_{p_r}(l_r)\prod_{r=1}^{d}D_{s_r}(l_r)h(s_1,\ldots,s_d).$$

Under the mathematical induction hypothesis on k, we have

$$\Delta^{k+1}=\sum_{r'=1}^{d}L(s_{r'})\sum_{l_1,\ldots,l_d}T(k;l_1,\ldots,l_d). \tag{6}$$

A direct computation as in the proof of Theorem 1 gives

$$\begin{aligned}&L(s_{r'})T(k;l_1,\ldots,l_d)\\ &\quad=\frac{k!}{l_1!\cdots l_d!}\Big(\prod_{r=1}^{r'-1}d_{p_r}(l_r)\Big)d_{p_{r'}}(l_{r'}+1)\Big(\prod_{r=r'+1}^{d}d_{p_r}(l_r)\Big)\\ &\qquad\times\Big(\prod_{r=1}^{r'-1}D_{s_r}(l_r)\Big)D_{s_{r'}}(l_{r'}+1)\Big(\prod_{r=r'+1}^{d}D_{s_r}(l_r)\Big)h(s_1,\ldots,s_d)\\ &\qquad+\frac{k!}{l_1!\cdots l_d!}\prod_{r=1}^{d}d_{p_r}(l_r)\prod_{r=1}^{d}D_{s_r}(l_r)\partial_{s_{r'}}^2h(s_1,\ldots,s_d).\end{aligned}$$

Now, for any fixed index $(l_1, \dots, l_d)$, all the second terms of the expansions

$$L(s_{r'})T(k; l_1, \dots, l_d)$$

corresponding to $r' = 1, \dots, d$ are canceled out owing to the fact that h is harmonic in the d variables. In this way in summation (6) all the second terms of the expansions $L(s_{r'})T(k; l_1, \dots, l_d)$ for $r' = 1, \dots, d$ and $0 \le l_1, \dots, l_d \le k$ with $l_1 + \dots + l_d = k$ are canceled out.

Now we show that in summation (6) the first terms, either by themselves or combined together, will form all the terms in the desired expansion corresponding to the index $k+1$. Indeed, if $1 \le l_1, \dots, l_d \le k-1$, then in the expansions $L(s_{r'})T(k; l_1, \dots, l_d)$ with $r' = 1, \dots, d$ there are d different terms to add up to form the term

$$\frac{k!}{l_1! \cdots l_d!}\left(\prod_{r=1}^{r_1-1} d_{p_r}(l_r)\right) d_{p_{r_1}}(l_{r_1}+1)\left(\prod_{r=r_1+1}^{d} d_{p_r}(l_r)\right)$$
$$\times \left(\prod_{r=1}^{r_1-1} D_{s_r}(l_r)\right) D_{s_{r_1}}(l_{r_1}+1)\left(\prod_{r=r_1+1}^{d} D_{s_r}(l_r)\right) h(s_1, \dots, s_d).$$

Apart from one in the expansion $L(s_{r_1})T(k; l_1, \dots, l_d)$, for every $r' < r_1$ there is one from the first term of

$$L(s_{r'})T\Big(k; l_1, \dots, l_{r'-1}, l_{r'} - 1, l_{r'+1}, \dots, l_{r_1-1}, l_{r_1} + 1, l_{r_1+1}, \dots, l_d\Big)$$

and for every $r' > r_1$ one from the first term of

$$L(s_{r'})T\Big(k; l_1, \dots, l_{r_1-1}, l_{r_1} + 1, l_{r_1+1}, \dots, l_{r'-1}, l_{r'} - 1, l_{r'+1}, \dots, l_d\Big).$$

Their coefficients add as

$$\frac{k!}{l! \cdots l_d!} + \sum_{r' \ne r_1} \frac{k!\, l_{r'}}{(l_{r_1}+1)! \prod_{r' \ne r_1} l_{r'}!} = \frac{k!\,(l_{r_1} + 1 + \sum_{r' \ne r_1} l_{r'})}{(l_{r_1}+1)! \prod_{r' \ne r_1} l_{r'}!}$$
$$= \frac{(k+1)!}{(l_{r_1}+1)! \prod_{r' \ne r_1} l_{r'}!}$$

which is the coefficient in the term

$$T\big(k+1; l_1, \dots, l_{r_1-1}, l_{r_1} + 1, l_{r_1+1}, \dots, l_d\big)$$

as desired. This procedure may be made valid for the cases where some l_r are zero. In this way we obtain in summation (6) all the terms in the desired expansion corresponding to the index $k+1$ from all the first terms of the expansions $L(s_{r'})T(k; l_1, \dots, l_d)$ for $r' = 1, \dots, d$ and $0 \le l_1, \dots, l_d \le k$ with $l_1 + \dots + l_d = k$. The proof is complete ∎

Theorem 3 is an easy application of Theorem 2 (Theorem 1 in the case $d = 2$).

Proof of Theorem 3. We show that, in that case, all $T(k; l_1, \dots, l_d) = 0$. In fact, if there is a non-zero $T(k; l_1, \dots, l_d)$, then we have $p_r - 2l_r + 1 \ge 2$ for $r = 1, \dots, d$. Since $\sum_{r=1}^{d} p_r = m$ and $\sum_{r=1}^{d} l_r = k = \frac{m}{2}$, adding up the above inequalities together produces the false relation $d \ge 2d$. This shows that the assumption $T(k; l_1, \dots, l_d) \ne 0$ is invalid. The proof is complete. ∎

4. Applications

Higher-dimensional Poisson kernels and Newton potentials are easily deducible from their forms in the lowest-dimensional cases based on the above proved theorems. In accordance with Theorem 3, the cases in which p_r being odd with an even sum are directly ready to be treated. Let $h(s,t)$ be a harmonic function in two variables and consider the replacement $s \to |\underline{x}|$, where $\underline{x} \in \mathbb{R}^p$ with p an odd integer. Owing to Theorem 3, we have

$$\Delta^{\frac{p+1}{2}} h(|\underline{x}|, t) = 0$$

and so

$$\Delta^{\frac{p-1}{2}} h(|\underline{x}|, t)$$

is a harmonic function. Now in Theorem 1 let $k = \frac{p-1}{2}$. Since for $q = 1$ and $k - l \neq 0$ the coefficients $d_q(k-l)$ are always zero, all non-zero terms in the sum should correspond to $k - l = 0$, that is $l = k = \frac{p-1}{2}$. Therefore,

$$\begin{aligned}\Delta^{\frac{p-1}{2}} h(|\underline{x}|, t) &= d_p\Big(\frac{p-1}{2}\Big)\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h(s,t)|_{s=|\underline{x}|}\\ &= (p-1)!!\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h(s,t)|_{s=|\underline{x}|}.\end{aligned}$$

Now apply this to

$$h(s,t) = \frac{t}{s^2+t^2},$$

that is harmonic and, apart from a constant multiple, the Poisson kernel in dimension 1. Recursively, for any odd p, we have

$$\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h(s,t) = c_p \frac{t}{(s^2+t^2)^{\frac{p+1}{2}}}.$$

Replacing s by $|\underline{x}|$, apart from a multiple constant it is the Poisson kernel in $\mathbb{R}^p$. The harmonicity follows from Theorem 3.

Next, we take

$$h(s,t) = \log\sqrt{s^2+t^2},$$

that is harmonic and essentially the Newton potential in the plane. Then, for all odd p,

$$\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h(s,t) = c_p \frac{1}{(s^2+t^2)^{\frac{p-1}{2}}} = c_p \frac{1}{(|\underline{x}|^2+t^2)^{n-2}}$$

that is the n-dimensional Newton potential with $n = p+1$. The harmonicity is a consequence of Theorem 3.

For p being even we proceed as follows. First we show that if $h(s,t)$ is a two-dimensional harmonic function in a certain domain, then

$$h_1(s, t_1, t_2) = \Delta^{\frac{1}{2}} h\big(s, \sqrt{t_1^2+t_2^2}\big)$$

(where the fractional power of the Laplacian for three variables is taken to be of the distribution sense) is a three-dimensional harmonic function in the corresponding domain. In other words,

$$\Delta h_1(s,t_1,t_2) = \Delta^{\frac{3}{2}} h\big(s, \sqrt{t_1^2+t_2^2}\big) = 0.$$

To show this we first note that

$$g_1(s,t) = \Big(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\Big)h(s,t)$$

is holomorphic in a certain domain of the complex plane. Secondly, we notice the relation (see [2])

$$\begin{aligned}\Delta^{\frac{1}{2}}\bigg[\Big(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\Big)h(s,t)\bigg|_{t=\sqrt{t_1^2+t_2^2},\, i=\frac{t_1e_1+t_2e_2}{\sqrt{t_1^2+t_2^2}}}\bigg] \\ = \Big(\frac{\partial}{\partial s} - \frac{\partial}{\partial t_1}e_1 - \frac{\partial}{\partial t_2}e_2\Big)h_1(s,t_1,t_2).\end{aligned} \tag{7}$$

This enables us to invoke the generalization of Fueter's theorem to fractional powers of the Laplacian [6]: The left-hand-side of (7) is a monogenic function in R_1^2. The latter amounts to say that h_1 is harmonic in three variables.

Note that the conclusion still holds if h_1 is obtained through a different substitution:

$$h_1(s,t_1,t_2) = \Delta^{\frac{1}{2}} h\big(\sqrt{s^2+t_1^2}, t_2\big).$$

The above argument shows that the result in Theorem 3 still holds for some cases in which the total dimension m is odd.

Now use Theorem 2 to any (not necessarily obtained from the above procedure) three-dimensional harmonic function $h_1(s,t_1,t_2)$ for $d=3, p_1=p$ with p being any odd integer, $p_2=p_3=1$ and $k=\frac{p-1}{2}$, where we replace s by $|\underline{x}|$ with $\underline{x}\in\mathbb{R}^p$. Taking into account that $d_1(i)=0$ unless $i=0$, we have

$$\begin{aligned}\Delta^{\frac{p-1}{2}} h_1(|\underline{x}|,t_1,t_2) &= d_p\Big(\frac{p-1}{2}\Big)\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h_1(s,t_1,t_2)|_{s=|\underline{x}|} \\ &= (p-1)!!\,\Big(\frac{1}{s}\partial_s\Big)^{\frac{p-1}{2}} h_1(s,t_1,t_2)|_{s=|\underline{x}|}.\end{aligned} \tag{8}$$

Now for

$$h(s,t) = \frac{t}{s^2+t^2} \qquad \text{and} \qquad h(s,t) = \log\sqrt{s^2+t^2},$$

replacing s by $\sqrt{s^2+t_1^2}$ and t by t_2, we have, respectively,

$$h_1(s,t_1,t_2) = \Delta^{\frac{1}{2}}\Big(\frac{t_2}{s^2+t_1^2+t_2^2}\Big) = c\,\frac{t_2}{(s^2+t_1^2+t_2^2)^{\frac{3}{2}}}$$

and

$$h_1(s,t_1,t_2) = \Delta^{\frac{1}{2}} \log \sqrt{s^2+t_1^2+t_2^2} = c\,\frac{1}{(s^2+t_1^2+t_2^2)^{\frac{1}{2}}}$$

being harmonic in three variables as generally asserted above. Those being harmonic can also be verified directly. Then using formula (8), we obtain respectively

$$\Delta^{\frac{p-1}{2}} h_1(|\underline{x}|,t_1,t_2) = c_p\,\frac{t_2}{(|\underline{x}|^2+t_1^2+t_2^2)^{\frac{p+2}{2}}}$$

and

$$\Delta^{\frac{p-1}{2}} h_1(|\underline{x}|,t_1,t_2) = c_p\,\frac{1}{(|\underline{x}|^2+t_1^2+t_2^2)^{\frac{p}{2}}}.$$

In the first case, by combining $\underline{x} \in \mathbb{R}^p$ with t_1 to form a $(p+1)$-dimensional vector, we obtain the Poisson kernel

$$\frac{t_2}{(|\underline{y}|^2+t_2^2)^{\frac{n+1}{2}}},$$

where $n = p+1$, and in the second case, by combining $\underline{x} \in \mathbb{R}^p$ with t_1 and t_2 to form a $(p+2)$-dimensional vector, we have the Newton potential

$$\frac{1}{|\underline{y}|^{n-2}}$$

where $n = p+2$. Applying the Laplacian of $\mathbb{R}^{p+2}$ to (8), since $d_p(\frac{p+1}{2}) = 0$, in using Theorem 2 all the coefficients are zero. Thus Poisson kernels and Newton potentials are harmonic.

From these examples we also see that the theorems obtained may be used in a great extent to simplify computations in relation to multiple powers of the Laplacian on radial functions.

In the same spirit, as alternative proofs, the relation

$$\Big(\frac{\partial}{\partial s} - i\frac{\partial}{\partial t}\Big)h(s,t)\Big|_{t=\sqrt{t_1^2+t_2^2},\, i=\frac{t_1e_1+t_2e_2}{\sqrt{t_1^2+t_2^2}}} = \Big(\frac{\partial}{\partial s} - \frac{\partial}{\partial t_1}e_1 - \frac{\partial}{\partial t_2}e_2\Big)h(s,\sqrt{t_1^2+t_2^2})$$

implies Sce's generalization of Fueter's Theorem to odd integers [8], and relation (7) with the auxiliary harmonic function h_1 in three real variables implies the generalization of the Fueter's Theorem to even integers ([6] or [7]). In below we briefly illustrate an aspect of use of this device in harmonic analysis.

Let f^0 be holomorphic in a domain in the upper-half plane, and $f^0(z) = u(s,t) + iv(s,t)$ $(z = s+it)$, where u and v are real-valued. Denote

$$\vec{f^0}(x) = \Big(u(x_0,|\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}v(x_0,|\underline{x}|)\Big)$$

where $x = x_0 + \underline{x} \in R_1^n$. Further, denote by τ the mapping

$$\tau(f^0) = \kappa_n \Delta^{\frac{n-1}{2}} \vec{f^0}$$

where the Laplacian Δ is for all $n+1$ variables: for n being odd $\Delta^{\frac{n-1}{2}}$ is a pointwise differential operator and for n being even it is defined through the Fourier multiple $c_n|\xi|^{n-1}$, and κ_n is the normalizing constant that makes $\tau((\cdot)^{-1})(x) = \frac{\overline{x}}{|x|^{n+1}}$ [7]. It is stressed that τ maps the Cauchy kernel $f^0(z) = \frac{1}{z}$ in the complex plane to the Cauchy kernels $E(x) = \frac{\overline{x}}{|x|^{n+1}}$ in R_1^n.

The *basic monomial functions* in R_1^n are defined to be

$$P^{(-k)} = \tau(\cdot)^{(-k)}, \quad P^{(k-1)} = I(P^{(-k)}) \qquad (k \geq 1)$$

where I is the Kelvin inversion: $(If)(x) = E(x)f(x^{(-1)})$. Now let f be a function left-monogenic in an annulus centered at the origin. Then f has a Laurent series expansion. As a sum of projections onto multi-dimensional spaces a Laurent series expansions has a complicated form. With help of the basic monomial functions the Laurent expansion may be written

$$f(x) = \sum_{k=-\infty}^{\infty} \frac{1}{\Omega_n} \int_{\Sigma} P^{(k)}(y^{(-1)}x)E(y)n(y)f(y)\,ds(y)$$

where Ω_n is the surface area of the unit sphere in R_1^n, Σ is any Lipschitz surface laying in the annulus, $n(y)$ is the outward unit normal of Σ at y, and $ds(y)$ is the surface Lebesgue area measure. The expression is in exactly the same form as that in the complex plane. By virtue of this in a great extent one may deal with Laurent series in the same way as to those in the complex plane. It has been seen to be particularly convenient with studies of partial sums of Fourier series, as well as Fourier multipliers on spheres. This approach together with the related methods was crucial in producing the operator algebra theory of singular integral operators with monogenic kernels and the corresponding Fourier multiplier operators on the unit sphere and its Lipschitz perturbations, first in the quaternionic space and then in general Euclidean spaces [5, 7]. The operator algebra is equivalent to the Cauchy-Dunford bounded holomorphic functional calculus of the spherical Cauchy-Riemann operator on the spaces. By means of this approach, some technical problems on the sphere may be reduced to the corresponding ones on the unit circle in the complex plane.

To the end, it would be interesting to note that if we apply the same operator $(\frac{1}{s}\frac{\partial}{\partial s})^{\frac{p-1}{2}}$ with p odd to the non-harmonic functions

$$h(s,t_2) = \frac{1}{t_2}e^{\frac{s^2}{4t_2^2}} \qquad \text{and} \qquad h(s,t_1,t_2) = \frac{1}{t_2^2}e^{\frac{(s^2+t_1^2)}{4t_2^2}}$$

in replacing s by $|\underline{x}|$ in the result, we obtain

$$c_n \frac{1}{t_2^n} e^{\frac{|\underline{x}|^2}{4t_2^2}}$$

where $\underline{x} \in \mathbb{R}^n$ for $n = p$ or $n = p+1$, respectively. Replacing t_2 by $t^{\frac{1}{2}}$ we obtain, apart from a constant multiple, the heat kernel

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|\underline{x}|^2}{4t}}$$

in $\mathbb{R}^n$.

We also note that the same recursive procedure has been used to produce the commonly used fundamental solutions of Helmholtz operators.

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