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Criteria for Membership of the Mean Lipschitz Spaces

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Abstract. Our aim is to characterize the elements in certain function spaces by means of the Cesáro means and/or partial sums of their Fourier series. Firstly, we seek to extend known results for the Besov spaces B^s_{pq} $(1 \le p, q < \infty)$ to the case where $q = \infty$. Secondly, we consider the Mean Lipschitz spaces $\Lambda(p, s)$. We confine attention to the values $1 \le p < \infty$ and $0 < s \le 1$ for the parameters. For $s < 1$, the spaces $B^s_{p\infty}$ and $\Lambda(p, s)$ coincide. For the case $p = 1$ certain counter-examples are provided; some positive results are also given. We then treat the case $s = 1$ and consider the spaces $B^1_{p\infty}$ and $\Lambda(p, 1)$ separately. Analogues of some known results for the spaces Λ_s are given.

Keywords: *Besov space, Mean Lipschitz space*

AMS subject classification: 46E35, 30H05

1. Introduction

This paper has its origins in a couple of sources. Firstly it seeks to extend the results obtained in [5] for the Besov spaces B^s_{pq} $(1 \le p, q < \infty)$ to the case where $q = \infty$. Most of these results carry over in a straightforward manner. The second source was the interesting and comprehensive article [2] on the Mean Lipschitz spaces $\Lambda(p, s)$, and the desire to extend in one respect results obtained there characterizing the members of these classes by their Fourier partial sums. Since the spaces $B^s_{p\infty}$ and $\Lambda(p, s)$ coincide except when $s = 1$ (and, more generally, for s a positive integer), it is appropriate to consider both classes together and to compare the cases in which they differ.

We shall confine attention to the values $1 \le p < \infty$ and $0 < s \le 1$ for the parameters. Exceptions to the general case are to be expected at the endpoints $p = 1$ and $s = 1$. The B^s_{pq} spaces are distinguished by the fact that they contain for every f its conjugate function; they are self conjugate in the terminology of [2], whereas $\Lambda(1, 1)$ and $\Lambda(\infty, 1) = \Lambda_1$ are not (see [2]). The spaces $\Lambda(p, s)$ decrease as either p or s increases while the other index remains fixed. Also, $B^s_{pq_1} \subset B^s_{pq_2}$ if $q_1 < q_2$. For $ps > 1$ we know that $B^s_{p\infty} \subset \lambda_\beta$ for every β with $0 < \beta < ps - 1$ (see, e.g., [8: Section 3.4]). However, if $ps = 1$, then $B^s_{p\infty}$ and $\Lambda(p, s)$ contain some non-continuous functions [2, 8]. In particular, $\log(1 - e^{it}) \in B^1_{1\infty}$ although not to $\Lambda(1, 1)$, while the space $\Lambda(1, 1)$ can be identified with the space of functions of bounded variation [4].

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The paper is organised as follows. In Section 1 we consider the case $1 < p < \infty$. In Theorem 4, we characterize $B^s_{p\infty}$ using the Cesáro means and partial sums of the Fourier series of a member function. Here we are restricted to the range $0 < s < 1$. In Section 2, we consider the case $p = 1$ and provide certain counter-examples; some positive results are also given which are restricted to the Cesáro means. In Section 3 the case $s = 1$ is treated. For $1 \le q < \infty$, we show how Theorem 4 has to be modified and the result for the partial sums is better than that for the Cesáro means. For $q = \infty$, we regard $B^1_{p\infty}$ as a generalization of the Zygmund class Λ_* and we prove an analogue of a well known result for this class. The next section considers the Mean Lipschitz spaces $\Lambda(p, 1)$. For $1 < p < \infty$, we obtain parallels of earlier results. Lastly, we consider the case $\Lambda(1, 1)$ which may be viewed as a generalization of Λ_1, and again are able to prove an analogue of a well known result for this class.

2. Preliminaries

Let D denote the unit disc, ∂D the unit circle, and $L^p = L^p(\partial D)$ the usual Lebesgue space when $0 < p < \infty$. For $p \ge 1$ we denote the norm of a function $f \in L^p$ by $\|f\|_p$.

Let

$$\Delta_t f(e^{ix}) = f(e^{i(x+t)}) - f(e^{ix})$$
$$\Delta_t^m = \Delta_t(\Delta_t^{m-1}).$$

For $0 < s \le 1$, the Lipschitz class Λ_s is the space of 2π-periodic functions on $[-\pi, \pi]$ for which $|\Delta_t f(e^{ix})| = O(|t|^s)$ uniformly in x. A generalization is the mean Lipschitz class $\Lambda(p, s)$ consisting of all functions f for which $\|\Delta_t f\|_p = O(|t|^s)$ for $t > 0$; $\Lambda(p, s)$ reduces to Λ_s when $p = \infty$. The Zygmund class Λ_* consists of all continuous 2π-periodic functions f such that $|\Delta_t^2 f(e^{ix})| = O(|t|)$ uniformly in x.

Suppose now that f is analytic in D. If $0 \le r < 1$, let

$$M_p(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{it})|^p \, dt\right)^{\frac{1}{p}} \qquad (0 < p < \infty)$$

denote the integral mean of f of order p. It is well known that $M_p(f, r)$ is an increasing function of r on $[0, 1)$ and that the class of functions f for which $\sup_{r<1} M_p(f, r) < \infty$ is the familiar Hardy space H^p [3]. For $1 \le p < \infty$ and $0 < s \le 1$, $f \in \Lambda(p, s)$ if and only if $M_p(f', r) = O\big((1 - r)^{s-1}\big)$. Let P denote the projection defined on the trigonometric polynomials by

$$Pe^{int} = \begin{cases} 0 & \text{if } n < 0 \\ e^{int} & \text{if } n \ge 0 \end{cases}$$

and elsewhere by linearity. By the theorem of M. Riesz, P extends to a bounded operator from L^p to H^p for $1 < p < \infty$, and $PL^p = H^p$ [3].

Given $f(e^{it}) \sim \sum_{-\infty}^{\infty} a_n e^{int}$, we write

$$s_n(f)(e^{it}) = \sum_{k=-n}^{n} a_k e^{ikt} \quad (n \ge 0)$$

$$\sigma_n(f)(e^{it}) = \sum_{k=-(n-1)}^{n-1} \Big(1 - \frac{|k|}{n}\Big) a_k e^{ikt} \quad (n \ge 1)$$

for the partial sums and Cesàro means of the Fourier series for f, respectively. Let

$$K_0(t) = 0$$

$$K_n(t) = \sum_{k=-(n-1)}^{n-1} \Big(1 - \frac{|k|}{n}\Big) e^{ikt} \quad (n \geq 1)$$

be Fejér's kernel and

$$V_0(t) = 0$$

$$V_{2m}(t) = 2K_{2m}(t) - K_m(t) \quad (m \geq 1)$$

$$V_{2m+1}(t) = V_{2m}(t) \quad (m \geq 1)$$

de la Vallée Poussin's kernel. It is clear that the Fourier coefficients of V_{2m} satisfy

$$\widehat{V}_{2m}(k) = \begin{cases} 1 & \text{if } |k| \leq m \\ \frac{2m-|k|}{m} & \text{if } m \leq |k| \leq 2m \\ 0 & \text{if } |k| \geq 2m. \end{cases} \tag{1}$$

We also define the kernels W_n for integral n. Namely, if $n \geq 1$, then W_n is a trigonometric polynomial such that $\widehat{W}_n$ is a linear function on the intervals $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$, $\widehat{W}_n(2^n) = 1$ and $\widehat{W}_n = 0$ outside $(2^{n-1}, 2^{n+1})$. If $n < 0$, then $W_n = \overline{W}_{-n}$. We let $W_0(z) = \bar{z} + 1 + z$.

We write $V_n(f)$ for

$$(V_n * f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) V_n(x-t)\, dt,$$

the convolution of f with V_n. These kernels can be used to characterize the Besov spaces B^s_{pq}. For $1 \leq p < \infty, s > 0$ and an arbitrary integer $m > s$, we define the Besov space B^s_{pq} by

$$B^s_{pq} = \left\{ f \in L^p : \int_{-\pi}^{\pi} \frac{\|\Delta_t^m f\|_p^q}{|t|^{1+sq}}\, dt < \infty \right\} \quad (1 \leq q < \infty)$$

$$B^s_{p\infty} = \left\{ f \in L^p : \sup_{t>0} |t|^{-s} \|\Delta_t^m f\|_p < \infty \right\} \quad (q = \infty).$$

It is well known that this definition is independent of the choice of m (see, e.g., [9: Chapter 5]). The proof there is for the real line but, as noted in [7], the corresponding statements can be proved in a similar way for functions on the unit circle.

An alternative description of the Besov spaces can be given in terms of the kernels W_n as the class of all functions in L^p such that

$$\|f\| = \left(\sum_{-\infty}^{\infty} 2^{|n|sq} \|W_n(f)\|_p^q \right)^{\frac{1}{q}} < \infty \quad (1 \leq q < \infty)$$

$$\|f\| = \sup_n 2^{|n|s} \|W_n(f)\|_p \quad (q = \infty)$$

where the norm on the left is the Besov-space norm of f and is equivalent to the first norm [7]. For a discussion of these spaces see [6, 8 - 10]. It is clear that the Riesz projection is a bounded operator from B^s_{pq} to itself. Let A^s_{pq} denote the subspace of B^s_{pq} consisting of analytic functions. We may characterize $A^s_{p\infty}$ as follows. The analytic function $f \in A^s_{p\infty}$ if and only if

$$\|f\|_A = \|f\|_p + \sup_{0<r<1} (1-r)^{m-s} M_p(f^{(m)}; r) < \infty.$$

We observe that $B^s_{p\infty} = \Lambda(p,s)$ for $0 < s < 1$ but the second space is a proper subspace of the first for $s = 1$. For example, with $p > 1$, the function $f(z) = (1-z)^{1-\frac{1}{p}}$ belongs to $A^1_{p\infty}$ but not to $\Lambda(p,1)$.

Let $\mathcal{P}_n$ denote the class of trigonometric polynomials of degree not exceeding n, and let

$$d_p(f, \mathcal{P}_n) = \inf\left\{\|f-g\|_p : g \in \mathcal{P}_n\right\}.$$

The following theorem is well known [6 - 8]:

Theorem A. *For $1 \le p < \infty$ and $s > 0$ the following statements are equivalent:*

- **(i)** $f \in A^s_{p\infty}$
- **(ii)** $d_p(f, \mathcal{P}_n) = O(n^{-s})$
- **(iii)** $\|f - V_n(f)\|_p = O(n^{-s})$
- **(iv)** $\|f - V_{2^n}(f)\|_p = O(2^{-ns})$
- **(v)** $\|W_n(f)\|_p = O(2^{-ns})$.

Suppose f is an analytic function. For each statement above we interpret the theorem as saying there is a corresponding norm on f and all these norms are equivalent. Thus with (ii) we associate the norm $\|f\|_p + \sup_n n^s d_p(f, \mathcal{P}_n)$. This remark applies also to Theorems B, 1, 2, 4, 6, 8, 9 below (and to the corresponding theorems in [5]). The constants associated with any pair of norms can be found by a perusal of the proof of the equivalences concerned.

The following known lemma from [5] states that under certain conditions a converse of Bernstein's Theorem [11] on polynomials holds .

Lemma 1. *Suppose that f is a polynomial whose Fourier coefficients are supported on $[n, 2n]$. For every $p, 1 \le p \le \infty$, there exists a constant $C_p > 0$ such that $\|f'\|_p \ge C_p n \|f\|_p$.*

In the theorems that follow C denotes a positive constant though not always the same one. All sums without limits are taken from 1 to ∞.

We have the following extension of Theorem A, the proof of which follows easily from that in [5].

Theorem B. *For $p \ge 1$ and $s > 0$, $f \in A^s_{p\infty}$ if and only if $\|V_{2n}(f) - V_n(f)\|_p = O(\frac{1}{n^s})$.*

3. Results for the spaces $B^s_{p\infty}$

3.1 The case $1 < p < \infty$. The first two theorems below follow in a straightforward manner from the corresponding proofs in [5]. Accordingly we simply state the results; we also state Lemma 2 from [5] which is used in the proof of Theorem 2 and later in Theorem 2′ below.

Lemma 2. *Let p_n be a polynomial of degree n. For $1 \le p < \infty$ and $1 - \frac{1}{2n} \le r < 1$, we have $\|p_n\|_p \le 2M_p(p_n; r)$.*

Our first theorem is not unexpected. The norm associated with statement (iii) for instance is $\sup_n \|s_n(f)\|_A$.

Theorem 1. *For $1 \le p < \infty$ and $s > 0$, the following statements are equivalent:*

(i) $f \in A^s_{p\infty}$

(ii) $\|\sigma_n(f)\|_A = O(1)$

(iii) $\|s_n(f)\|_A = O(1)$.

Let $r_n = 1 - \frac{1}{2n}$.

Theorem 2. *For $1 < p < \infty$ and $0 < s < m$, the following statements are equivalent:*

(i) $f \in A^s_{p\infty}$

(ii) $M_p(f^{(m)}, r_n) = O(n^{m-s})$

(iii) $\|s_n^{(m)}(f)\|_p = O(n^{m-s})$

(iv) $\|\sigma_n^{(m)}(f)\|_p = O(n^{m-s})$.

Remark. The norm associated with statement (iii) for instance is given by $\|f\|_p + \sup_n n^{s-m}\|s_n^{(m)}\|_p$.

Corollary 3. *For $1 < p < \infty$ and $0 < s < 1$, $f \in A^s_{p\infty}$ if and only if*

$$\|s_n(f) - \sigma_{n+1}(f)\|_p = O\big(\tfrac{1}{n+1}\big)^s.$$

Proof. Since

$$s_n(f)(z) - \sigma_{n+1}(f)(z) = \frac{zs_n'(f)(z)}{n+1}, \tag{2}$$

we can apply Theorem 2 with $m = 1$ ∎

The proof of the next theorem is given in full.

Theorem 4. *Suppose $1 < p < \infty$ and $0 < s < 1$. The following statements are equivalent:*

(i) $f \in A^s_{p\infty}$

(ii) $\|\sigma_{2n}(f) - \sigma_n(f)\|_p = O(\frac{1}{n^s})$

(iii) $\|f - \sigma_n(f)\|_p = O(\frac{1}{n^s})$

(iv) $\|f - s_n(f)\|_p = O(\frac{1}{n^s})$

(v) $\|s_{2n}(f) - s_n(f)\|_p = O(\frac{1}{n^s})$.

Proof. The order of proof is (i) $\Leftrightarrow$ (ii), (i) $\Leftrightarrow$ (iii), (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$m (i).

Step (i) $\Rightarrow$ (ii): Writing $f(z) = \sum_0^\infty a_k z^k$ and $\zeta = e^{it}$, we have

$$\begin{aligned} \sigma_{2n}(f)(\zeta) - \sigma_n(f)(\zeta) &= \frac{1}{2n}\sum_{k=0}^{n-1} k a_k \zeta^k + \sum_{k=n}^{2n-1}\Big(1 - \frac{k}{2n}\Big) a_k \zeta^k \\ &= \frac{1}{2n}\zeta s'_{n-1}(f)(\zeta) + \frac{1}{2} R_n(f)(\zeta). \end{aligned} \tag{3}$$

It is easy to check from (1) that

$$R_n(f)(\zeta) = \zeta^n P\big\{\zeta^{-n}\big(V_{2n}(f)(\zeta) - V_n(f)(\zeta)\big)\big\}$$

and therefore there exists a constant $C_p > 0$ such that for all $n \geq 1$

$$\|R_n(f)\|_p \leq C_p \|V_{2n}(f) - V_n(f)\|_p. \tag{4}$$

Since

$$\|\sigma_{2n}(f) - \sigma_n(f)\|_p \leq \tfrac{1}{2n}\|s'_{n-1}(f)\|_p + \tfrac{1}{2}\|R_n(f)\|_p,$$

it follows now from Theorem 2 and Theorem B that (ii) holds.

Step (ii) $\Rightarrow$ (i): The sequence $\{\sigma_{2^n}(f)\}$ is a Cauchy sequence in L^p and therefore converges to a limit which is f. Also,

$$\|f - \sigma_{2^k}(f)\|_p \leq \sum_{j=k}^{\infty} \big\|\sigma_{2^{j+1}}(f) - \sigma_{2^j}(f)\big\|_p \leq C_p \sum_{j=k}^{\infty} \frac{1}{2^{js}} \leq C_p \frac{1}{2^{ks}}.$$

It follows that $d_p(f, \mathcal{P}_n) = O(n^{-s})$ holds, initially for dyadic integers, and then for all n. Theorem A completes the proof.

Step (i) $\Rightarrow$ (iii): We write

$$\begin{aligned} f - \sigma_{2n}(f) &= f - \big(2\sigma_{2n}(f) - \sigma_n(f)\big) + \sigma_{2n}(f) - \sigma_n(f) \\ &= \big(f - V_{2n}(f)\big) + \big(\sigma_{2n}(f) - \sigma_n(f)\big) \end{aligned}$$

and therefore

$$\|f - \sigma_{2n}(f)\|_p \leq \|f - V_{2n}(f)\|_p + \|\sigma_{2n}(f) - \sigma_n(f)\|_p.$$

Invoking Theorem A and the fact that (i) $\Rightarrow$ (ii) gives statement (iii) for even n. For the case of odd integers we write

$$f - \sigma_{2n+1}(f) = \big(f - \sigma_{4n+2}(f)\big) + \big(\sigma_{4n+2}(f) - \sigma_{2n+1}(f)\big).$$

Using what we have already proved we conclude that statement (iii) holds.

Step (iii) $\Rightarrow$ (i): It is evident that (iii) $\Rightarrow$ (ii) and we already know that (ii) $\Rightarrow$ (i).

Step (i) $\Rightarrow$ (iv): Since

$$f - s_n(f) = \big(f - \sigma_{n+1}(f)\big) + \big(\sigma_{n+1}(f) - s_n(f)\big),$$

the fact that (i) $\Rightarrow$ (iii) and Corollary 3 together yield (iv). *Step* (iv) $\Rightarrow$ (v) is immediate. *Step* (v) $\Rightarrow$ (i) follows from an argument similar to that in Step (ii) $\Rightarrow$ (i). The proof is complete ∎

Remark. Theorem 4 does not hold for $s \geq 1$. If $s > 1$ and $f(z) = z$ then $\|f - \sigma_n(f)\|_p = 1/n$ and (i) does not imply (iii). Similarly (i) does not imply (ii). To see that the same conclusion holds for $s = 1$, we first state a result for gap series which we shall use several times. If $f(e^{it}) \sim \sum_{-\infty}^{\infty} a_k e^{i2^k t}$ is in L^1, then $f \in L^p$ for all $p \geq 1$ and there exists a constant $C_p > 0$ such that $\|f\|_1 \leq \|f\|_p \leq C_p \|f\|_1$ [1: Chapter 11/Section 5].

Consider now the gap series $f(z) = \sum_1^{\infty} a_k z^{2^k}$. According to Theorem A, $f \in A^1_{p\infty}$ if and only if $\sup_k 2^k \|W_k(f)\|_p < \infty$. Since $W_k(f) = a_k z^{2^k}$ it is clear that $f \in A^1_{p\infty}$ if and only if $\sup_k 2^k |a_k| < \infty$. Consequently the condition is independent of $p \geq 1$. Assume $p > 1$ and that $2^k a_k = 1$ for all k so that $f \in A^1_{p\infty}$. Consider the proof of the first implication in Theorem 4 and apply (4) and Theorem B; it follows that $\|\sigma_{2n}(f) - \sigma_n(f)\|_p = O(\frac{1}{n})$ if and only if $\|s'_{n-1}(f)\|_p = O(1)$. Take $n = 2^k$ and apply the quoted result to the gap series $\|s'_n(f)\|_p$; this yields constants C_1, C_2 independent of f and n, such that $C_1 \|s'_n(f)\|_p \leq \|s'_n(f)\|_2 \leq C_2 \|s'_n(f)\|_p$. But $\|s'_n(f)\|_2 = \{\sum_{j=1}^k (2^j |a_j|)^2\}^{1/2} = k^{1/2}$ which is unbounded as k increases and therefore $\|s'_n(f)\|_p \neq O(1)$. It follows that (i) does not imply (ii). It is evident now that (i) does not imply (iii) either.

We can give an application of Theorem 4 to the rate at which $\|f(re^{ix}) - f(e^{ix})\|_p \to 0$ as $r \to 1$. Let $f \in A^s_{p\infty}$. Using summation by parts, we have

$$f(re^{ix}) - f(e^{ix}) = (1 - r) \sum_0^{\infty} \big(s_n(f)(e^{ix}) - f(e^{ix})\big) r^n$$

which yields

$$\begin{aligned} \|f(re^{ix}) - f(e^{ix})\|_p &\leq (1 - r)\|f - f(0)\|_p + (1 - r) \sum_{n=1}^{\infty} \|s_n - f\|_p r^n \\ &\leq (1 - r)\|f - f(0)\|_p + C(1 - r) \sum_{n=1}^{\infty} \frac{r^n}{n^s} \\ &\leq (1 - r)\|f - f(0)\|_p + C(1 - r)^s. \end{aligned}$$

Therefore $\limsup_{r \to 1} \frac{\|f(re^{ix}) - f(e^{ix})\|_p}{(1-r)^s} < \infty$.

3.2 The case $p = 1$. The results stated for the case $1 < p < \infty$ do not remain true in full for $p = 1$. The problem is that the Riesz theorem does not hold when $p = 1$. While the results for s_n fail, those for σ_n hold with some exceptions. We shall use two lemmas from [5] but first we note that the kernels W_n satisfy $\|W_n\|_1 \leq 3$ for all n, and that $f = \sum_{-\infty}^{\infty} W_n(f)$ if f is a trigonometric polynomial (see [7]).

Lemma 3. *Suppose f is a polynomial with coefficients supported on $[2^k, 2^{k+1}]$. Then $\|f\|_A$ is equivalent to $2^{ks}\|f\|_1$.*

Proof. By hypothesis $f = W_k(f) + W_{k+1}(f)$ and therefore

$$\begin{aligned} \|f\|' &= \sup 2^{ns} \|W_n(f)\|_1 \\ &= \sup \left\{2^{ks}\|W_k(f)\|_1, 2^{(k+1)s}\|W_{k+1}(f)\|_1\right\} \\ &\leq 6.2^{(k+1)s}\|f\|_1. \end{aligned}$$

On the other hand, $\|f\|_1 \le \|W_k(f)\|_1 + \|W_{k+1}(f)\|_1$ and therefore $2^{ks}\|f\|_1 \le 2\|f\|'$. The result now follows from Theorem A ∎

We shall use the symbol $\approx$ to denote the relation " is equivalent to" used above.

Lemma 4. *The following estimates hold for large n:*

(a) $\left\| \sum_{k=0}^{n} (1-\frac{k}{n}) e^{ikx} \right\|_1 \approx \log n$.

(b) $\left\| \sum_{k=1}^{n} k e^{ikx} \right\|_1 \approx n \log n$.

Suppose that $f(z) = \sum_n^{2n} a_k z^k$. For such polynomials f, Lemma 1 implies that $\|f'\|_1 \approx n\|f\|_1$.

We now construct counterexamples to show how some parts of the theorems break down.

Let $f(z) = \sum_{j=2^k}^{2^{k+1}} a_j z^j$ where

$$a_j = \begin{cases} \frac{j-2^k}{2^{k-1}} & \text{if } 2^k \le j \le 2^k + 2^{k-1} \\ \frac{2^{k+1}-j}{2^{k-1}} & \text{if } 2^k + 2^{k-1} \le j \le 2^{k+1}. \end{cases}$$

With $z = e^{it}$, we can write $f(z) = z^{2^k+2^{k-1}} K_{2^{k-1}}(z)$ and so $\|f\|_1 = 1$. Next we take $n = 2^k + 2^{k-1}$ and we estimate $\|s_n(f)\|_1$. Since

$$s_n(f)(z) = z^{2^k} \sum_{j=2^k}^{n} a_j z^{j-2^k} = z^{2^k} \sum_{m=0}^{2^{k-1}} \frac{m}{2^{k-1}} z^m$$

and since $\|s_n\|_1 = \|ls_n\|_1$ we have

$$\begin{aligned} \|s_n(f)\|_1 &= \left\| \sum_0^{2^{k-1}} \frac{m}{2^{k-1}} z^{-m} \right\|_1 \\ &= \left\| \sum_0^{2^{k-1}} \frac{m}{2^{k-1}} z^{2^{k-1}-m} \right\|_1 \\ &= \left\| \sum_0^{2^{k-1}} (1 - \frac{j}{2^{k-1}}) z^j \right\|_1 \\ &\approx \log n \end{aligned}$$

by Lemma 4/(a). Now the norm associated with statement (iii) of Theorem 1 is $\sup_m \|s_m(f)\|_A$. By Lemma 3, $\|f\|_A \approx 2^{ks}\|f\|_1 = 2^{ks}$ and $\|s_n(f)\|_A \approx 2^{ks}\|s_n(f)\|_1 \approx 2^{ks}\log 2^k$. It follows that there does not exist a constant C such that $\|s_n(f)\|_A \le C\|f\|_A$ for all f and arbitrary n, and so (i) does not imply (iii) in Theorem 1.

Next consider Theorem 2 with f and n as above. Lemma 1 combined with Bernstein's Theorem on polynomials yield a constant C independent of f and n such that

$$Cn\|s_n(f)\|_1 \le \|s_n'(f)\|_1 \le n\|s_n(f)\|_1.$$

Now choose $m > s > 0$ and apply the same reasoning m times to yield

$$C^m n^m \|s_n(f)\|_1 \leq \|s_n^{(m)}(f)\|_1 \leq n^m \|s_n(f)\|_1.$$

Consequently

$$n^{s-m}\|s_n^{(m)}(f)\|_1 \approx n^s \|s_n(f)\|_1 \approx n^s \log n \approx \log n \|f\|_A.$$

Therefore there does not exist a constant C such that $n^{m-s}\|s_n^{(m)}(f)\|_1 \leq C\|(f)\|_A$. This says that (i) does not imply (iii) in Theorem 2 .

Moving on to Theorem 4 with the same f, we let $2n = 2^k + 2^{k-1}$. Then $s_{2n}(f) - s_n(f) = s_{2n}(f)$ and

$$\|s_{2n}(f)\|_1 \approx \log 2n \approx (2n)^{-s} \log 2n \|f\|_A.$$

We conclude that (i) does not imply (v).

3.3 Some positive results. We now state some positive results for the case $p = 1$. First, there is the following substitute for Theorem 1:

Theorem $1'$**.** *Suppose* $s > 0$. *Then:*

(i) $f \in A^s_{1\infty}$ *if and only if* $\|\sigma_n(f)\|_A = O(1)$.

(ii) *If* $\|s_n(f)\|_A = O(1)$, *then* $f \in A^s_{1\infty}$.

Proof. The arguments are exactly the same as in Theorem 1 ∎

We have a substitute for Theorem 2 as follows:

Theorem $2'$**.** *Suppose* $s > 0$ *and* m *is an integer such that* $m > s$. *Then:*

(i) $f \in A^s_{1\infty}$ *if and only if* $\|\sigma_n^{(m)}(f)\|_1 = O(n^{m-s})$.

(ii) *If* $\|s_n^{(m)}(f)\|_1 = O(n^{m-s})$, *then* $f \in A^s_{1\infty}$.

Proof. (i): For the step "if" the argument is the same as in Theorem 2/(iv) $\Rightarrow$ (i). For the step "only if" apply Lemma 2 to the polynomial $\sigma_n(z^m f^{(m)}(z))$ and follow the argument in Theorem 2, noting that $\|\sigma_n(f)\|_1 \leq \|f\|_1$.

(ii): The proof is the same as in Theorem 2/(iv) $\Rightarrow$ (i), replacing σ_n by s_n ∎

The following lemma is an adaptation of [5: Lemma 6] for the case $q = \infty$ and the proof is the same.

Lemma 5. *Suppose* $s > 0$ *and* $f \in A^s_{1\infty}$. *There is a constant* C *independent of* n *and* f *such that* $\|\sigma_{2n+1}(f) - \sigma_{2n}(f)\|_1 \leq C(2n)^{-1} \log 2n$ *for* $n \geq 1$.

We can now prove an analogue of [5: Theorem 8]; a different argument is needed here.

Theorem $4'$. *Suppose that* $0 < s < 1$. *The following statements are equivalent:*

(i) $f \in A^s_{1\infty}$

(ii) $\|\sigma_{2n}(f) - \sigma_n(f)\|_1 = O(n^{-s})$

(iii) $\|f - \sigma_n(f)\|_1 = O(n^{-s})$.

Proof. (ii) $\Rightarrow$ (i): The proof of the same implication in Theorem 4 applies.

(i) $\Rightarrow$ (ii): We start with two identities:

$$\begin{aligned} V_{2n}(f) - \sigma_{2n}(f) &= \sigma_{2n}(f) - \sigma_n(f) \\ V_{2n}(f) - V_n(f) &= 2\big(\sigma_{2n}(f) - \sigma_n(f)\big) + \big(\sigma_n(f) - V_n(f)\big) \end{aligned} \tag{5}$$

so that

$$V_{4n}(f) - V_{2n}(f) = 2\big(\sigma_{4n}(f) - \sigma_{2n}(f)\big) + \big(\sigma_{2n}(f) - V_{2n}(f)\big). \tag{6}$$

Now choose an integer n. Suppose that $2^r \le n < 2^{r+1}$ and write n in binary form as

$$n = q_0 + q_1 2 + \ldots + q_{r-1}2^{r-1} + 2^r$$

where each $q_k = 0$ or $q_k = 1$. Let $p_0 = 1$, and for $1 \le k \le r$ we let

$$p_k = q_{r-k} + q_{r-k+1}2 + \ldots + q_{r-1}2^{k-1} + 2^k.$$

Thus $p_r = n$ and it can be checked that $p_k = q_{r-k} + 2p_{k-1}$. From (5) and (6) we have

$$\sigma_{4n}(f) - \sigma_{2n}(f) = \frac{1}{2}\big(V_{4n}(f) - V_{2n}(f) + \sigma_{2n}(f) - \sigma_n(f)\big)$$

while

$$\begin{aligned} &\sigma_{4n+2}(f) - \sigma_{2n+1}(f) \\ &\quad = \tfrac{1}{2}\big(V_{4n+2}(f) - V_{2n+1}(f)\big) + \tfrac{1}{2}\big(\sigma_{2n}(f) - \sigma_n(f)\big) + \tfrac{1}{2}\big(\sigma_{2n+1}(f) - \sigma_{2n}(f)\big). \end{aligned}$$

Taking norms gives the obvious inequalities. By Theorem B, there is an M such that

$$\big\|V_{2m}(f) - V_m(f)\big\|_1 \le M\frac{1}{m^s}$$

for all m. From these inequalities and from Lemma 5 we deduce

$$\big\|\sigma_{2p_k}(f) - \sigma_{p_k}(f)\big\|_1 \le \frac{M}{2p_k^s} + \frac{1}{2}\big\|\sigma_{p_k}(f) - \sigma_{p_{k-1}}(f)\big\|_1 + \frac{C\log p_k}{2p_k}.$$

On replacing k by r above and iterating the procedure, we obtain

$$\begin{aligned} &\big\|\sigma_{2p_r}(f) - \sigma_{p_r}(f)\big\|_1 \\ &\quad \le M\Big(\frac{1}{2p_r^s} + \frac{1}{2^2 p_{r-1}^s} + \ldots \frac{1}{2^r p_1^s}\Big) + C\Big(\frac{\log p_r}{2p_r} + \frac{\log p_{r-1}}{2^2 p_{r-1}} + \ldots + \frac{\log p_1}{2^r p_1}\Big) \\ &\quad \le M\Big(\frac{1}{2^{rs+1}} + \frac{1}{2^{(r-1)s+2}} + \ldots + \frac{1}{2^{r+1}}\Big) + C\Big(\frac{1}{2^{rs+1}} + \frac{1}{2^{(r-1)s+2}} + \ldots + \frac{1}{2^{r+1}}\Big) \\ &\quad \le \frac{C}{2^{rs+1}}\Big(1 + \frac{1}{2^{1-s}} + \ldots + \frac{1}{2^{r(1-s)}}\Big) \\ &\quad \le \frac{C}{n^s} \end{aligned}$$

and statement (ii) holds. For step (ii) $\Rightarrow$ (iii) we refer to the proof of Theorem 4/(i) $\Rightarrow$ (iii). Step (iii) $\Rightarrow$ (i) follows from Theorem A. The proof of the theorem is complete ∎

Theorem 5. *Suppose that* $0 < s$ *and any one of the following conditions is satisfied:*

(i) $\sup_n n^s \|f - \sigma_n(f)\|_1 < \infty$

(ii) $\sup_n n^s \|\sigma_{2n}(f) - \sigma_n(f)\|_1 < \infty$

(iii) $\sup_n n^s \|f - s_n(f)\|_1 < \infty$

(iv) $\sup_n n^s \|s_{2n}(f) - s_n(f)\|_1 < \infty$.

Then $f \in A^s_{1\infty}$.

Proof. Suppose condition (ii) or (iv) holds. A similar argument to that in (ii) $\Rightarrow$ (i) of Theorem 4 applies. Theorem A applies to conditions (i) and (iii). The proof is complete ∎

3.4 The case $s = 1$. This case is special as indicated earlier. Indeed, in [5] this case was not considered at all. It was observed there that the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) of Theorem 4 in [5] do not hold. We first want to fill in this gap in [5] before continuing. It turns out that $\sum n^{q-1}\|f - \sigma_n(f)\|_p^q$ is infinite for all non-constant f but there is a better result for the partial sums.

Theorem 6. *Suppose* $1 < p < \infty$ *and* $1 \le q < \infty$. *The following statements are equivalent:*

(i) $f \in A^1_{pq}$

(ii) $\sum n^{q-1}\big\|f - s_n(f)\big\|_p^q < \infty$

(iii) $\sum n^{q-1}\big\|s_{2n}(f) - s_n(f)\big\|_p^q < \infty$.

Proof. (i) $\Rightarrow$ (ii): For each n we can, by a compactness argument, find a polynomial $T = T_n(f)$ such that $d_p(f, \mathcal{P}_n) = \|f - T\|_p$. Write $f = T + U$ so that $\|U\|_p = d_p(f, \mathcal{P}_n)$. Let $\{u_k\}$ denote the partial sums of U. For $k \ge n$ we have $s_k(f) = T + u_k$. Therefore

$$\begin{aligned} \|f - s_k(f)\|_p &\le \|f - T\|_p + \|u_k\|_p \\ &\le d_p(f, \mathcal{P}_n) + C_p\|U\| \\ &\le (1 + C_p)d_p(f, \mathcal{P}_n). \end{aligned} \tag{7}$$

Letting $k = n$ above, we have

$$\sum n^{q-1}\|f - s_n(f)\|_p^q \le C_p \sum n^{q-1} d_p(f, \mathcal{P}_n)^q < \infty$$

by [5: Theorem A].

The implication (ii) $\Rightarrow$ (iii) is immediate. The proof of the implication (iii) $\Rightarrow$ (i) is the same as in that of (v) $\Rightarrow$ (i) of [5: Theorem 4]. This completes the proof of the theorem ∎

Remark. Suppose that $f \in A^1_{pq}$ is non-constant. Since

$$f - \sigma_{n+1}(f) = \big(f - s_n(f)\big) + \big(s_n(f) - \sigma_{n+1}(f)\big)$$

it follows from Theorem 6 that

$$\sum n^{q-1}\|f - \sigma_{n+1}(f)\|_p^q \qquad \text{and} \qquad \sum n^{q-1}\|s_n(f) - \sigma_{n+1}(f)\|_p^q$$

both converge or diverge together. Since

$$s_n(f)(z) - \sigma_{n+1}(f)(z) = \frac{zs_n'(f)(z)}{n+1}$$

we are led to consider the sum $\frac{1}{n}\sum \|s_n'(f)\|_p^q$. But this is divergent if f is not a constant.

We return to the case $q = \infty$. The implication (i) $\Rightarrow$ (ii) of Theorem 4 does not hold when $s = 1$. Before discussing this in more detail we mention first a result for the Zygmund class $\Lambda_* = B^1_{\infty\infty}$. It is well known (see [11: Theorem 3.15]) that if $f \in \Lambda_*$, then $\|\sigma_n(f) - f\|_\infty = O(\frac{\log n}{n})$. The converse is false.

We prove a generalization of the result above for the class $A^1_{p\infty}$. Accordingly, we have in place of Theorem 4 when $s = 1$:

Theorem 7.

(a) *Suppose* $1 < p < \infty$. *Then* $f \in A^1_{p\infty}$ *if and only if* $\|f - s_n(f)\|_p = O(\frac{1}{n})$.

(b) *Suppose* $1 \le p < \infty$. *If* $f \in A^1_{p\infty}$, *then* $\|f - \sigma_n(f)\|_p = O(\frac{\log n}{n})$.

Proof. (a) If $f \in A^1_{p\infty}$, then (7) and Theorem A give the result. The converse is obvious.

(b) The proof of the known result above for the Zygmund class adapts readily. We follow the argument there replacing the sup norm by the p-norm at the appropriate point ∎

The converse in (b) is false. To see this consider the gap series $f(z) = \sum \frac{\sqrt{k}}{2^k} z^{2^k}$. For each m,

$$\begin{aligned} f(z) - \sigma_{2^m}(f)(z) &= \sum \frac{\sqrt{k}}{2^k} z^{2^k} - \sum_{k=1}^{m-1}\Big(1 - \frac{2^k}{2^m}\Big)\frac{\sqrt{k}}{2^k} z^{2^k} \\ &= \sum_{k=1}^{m-1} \frac{\sqrt{k}}{2^m} z^{2^k} + \sum_{m}^{\infty} \frac{\sqrt{k}}{2^k} z^{2^k}. \end{aligned}$$

Taking norms we get

$$\begin{aligned} \|f - \sigma_{2^m}\|_p &\approx \|f - \sigma_{2^m}\|_2 \\ &= \Big(\sum_{k=1}^{m-1} \frac{k}{2^{2m}} + \sum_{k=m}^{\infty} \frac{k}{2^{2k}}\Big)^{\frac{1}{2}} \approx \sqrt{\frac{m^2}{2^{2m}}} = \frac{m}{2^m} \approx \frac{\log 2^m}{2^m}. \end{aligned}$$

A similar estimate holds for $2^m \le n < 2^{m+1}$. It follows that $\|f - \sigma_n(f)\|_p = O(\frac{\log n}{n})$ for all n. However, it is clear that $W_n(f)(z) = \frac{n^{1/2} z^{2^n}}{2^n}$ from which $2^n\|W_n(f)\|_p = n^{\frac{1}{2}}$. From Theorem A we conclude that $f \notin A^1_{p\infty}$.

4. Results for the spaces $\Lambda(p,1)$

4.1 The spaces $\Lambda(p,1)$. We know that the $\Lambda(p,1)$ spaces differ from the $B^1_{p\infty}$ spaces considered so far and we want to take a look at these now. They fall into two cases

(a) $1 < p < \infty$ and

(b) $p = 1$.

For the first case we know that the Riesz projection is a bounded operator on $\Lambda(p,1)$ (see [4: p. 621] and [2]), and so it suffices to consider the subspace of analytic functions.

There are the following analogues of Theorems 2 and 4.

Theorem 8. *Suppose* $1 < p < \infty$. *The following statements are equivalent for an analytic function* f*:*

(i) $f \in \Lambda(p,1)$

(ii) $\|s_n'(f)\|_p = O(1)$

(iii) $\|\sigma_n'(f)\|_p = O(1)$.

Proof. Since (i) says that $f' \in H^p$, this result is well known ∎

We have a satisfactory analogue of Theorem 4:

Theorem 9. *Suppose* $1 < p < \infty$ *and that* f *is analytic. The following statements are equivalent:*

(i) $f \in \Lambda(p,1)$

(ii) $\|\sigma_{2n}(f) - \sigma_n(f)\|_p = O(\frac{1}{n})$

(iii) $\|f - \sigma_n(f)\|_p = O(\frac{1}{n})$.

Proof. (i) $\Rightarrow$ (ii): From the proof of Theorem 4 we have

$$\sigma_{2n}(f)(\zeta) - \sigma_n(f)(\zeta) = \tfrac{1}{2n}\zeta s_{n-1}'(f)(\zeta) + \frac{1}{2}R_n(f)(\zeta) \tag{8}$$

with

$$R_n(f)(\zeta) = \zeta^n P\big\{\zeta^{-n}\big(V_{2n}(f)(\zeta) - V_n(f)(\zeta)\big)\big\}$$

and therefore there exists a constant $C_p > 0$ such that for all $n \geq 1$

$$\|R_n(f)\|_p \leq C_p \|V_{2n}(f) - V_n(f)\|_p. \tag{9}$$

Since $f \in A^1_{p\infty}$, it follows now from Theorem 8 and Theorem B that (ii) holds.

(ii) $\Rightarrow$ (i): We first show that $\|V_{2n}(f) - V_n(f)\|_p = O(\frac{1}{n})$. From (5) and (6) we have $\|V_{4n}(f) - V_{2n}(f)\|_p = O(\frac{1}{n})$ and it remains to consider $V_{4m+2}(f) - V_{2m+1}(f)$. On writing

$$V_{4m+2}(f) - V_{2m+1}(f) = \big(V_{4m+2}(f) - V_{4m}(f)\big) + \big(V_{4m}(f) - V_{2m}(f)\big)$$

we need only to consider the first term on the right-hand side. An examination of the Fourier coefficients of V_{4m} in (1) shows that

$$V_{4m+2}(f) - V_{4m}(f) = \frac{1}{2m(2m+1)}\big[s_{4m}'(f) - s_{2m}'(f)\big] + \frac{a_{4m+1}}{2m+1}e^{i(4m+1)t}.$$

We note that $f \in L^p$ by (ii). Applying Bernstein's theorem to the first term on the right-hand side yields

$$\left\|V_{4m+2}(f) - V_{2m+1}(f)\right\|_p \le \frac{\|s_{4m}(f) - s_{2m}(f)\|_p}{m} + o\Big(\frac{1}{2m+1}\Big) = o\Big(\frac{1}{m}\Big)$$

as $m \to \infty$. We now have $\|R_n(f)\|_p = O(\frac{1}{n})$ from (9). From (8) we have $\|s_n'(f)\|_p = O(1)$ and $f \in \Lambda(p,1)$ by Theorem 8.

(i) $\Rightarrow$ (iii): Since $f \in A^1_{p\infty}$, the proof of the same implication in Theorem 4 applies. (iii) $\Rightarrow$ (i): Well, (iii) $\Rightarrow$ (ii) and we know that (ii) $\Rightarrow$ (i). This completes the proof ∎

Remark. By means of Theorem 8 and a similar proof, *(i)* $\Rightarrow$ *(iv)* of Theorem 4 holds here, but the opposite implication does not hold. To see this consider the function given by the gap series $f(z) = \sum \frac{z^{2^k}}{2^k}$. Then

$$\|f - s_{2^n}\|_p \approx \|f - s_{2^n}\|_2 = \left\{\sum_{n+1}^{\infty} \frac{1}{2^{2k}}\right\}^{\frac{1}{2}} \approx \frac{1}{2^n}.$$

Using Theorem A/(iv), we readily see that $f \in A^1_{p\infty}$; but since $zf'(z) = \sum z^{2^n}$ it follows that $f' \notin H^q$ for all $q \ge 1$ and the result follows.

4.2 The class $\mathbf{\Lambda(1,1)}$. We now turn to the class $\Lambda(1,1)$ any element of which is equal a.e. to a function of bounded variation on $[-\pi,\pi]$ [4: Lemma 9]. For a general function f we let $\tilde{f}$ denote its conjugate function. As stated in [4], this class does not contain the conjugate function of each of its members and therefore is not a Besov space. Consequently, we do not expect that results analogous to those previously obtained hold for this class.

First we state a well known result (see [11: Chapter 3/Theorem 13.34]) for the class Λ_1:

$$\|\sigma_n(f) - f\|_\infty = O\big(\tfrac{1}{n}\big) \quad \text{ if and only if } \quad \tilde{f} \in \Lambda_1.$$

We can extend this to its analogue $\Lambda(1,1)$ and the proof is based on the original result:

Theorem 10. $\tilde{f} \in \Lambda(1,1)$ *if and only if* $\|\sigma_n(f) - f\|_1 = O(\frac{1}{n})$.

Proof. If $\|\sigma_n(f) - f\|_1 \le K\frac{1}{n}$ for all n, then the same result holds a fortiori with $\frac{1}{n}$ replaced by $\frac{1}{n^s}$ for all $s < 1$. It follows from Theorem 4′ that $\tilde{f} \in \Lambda(1,s)$; we note that $\Lambda(1,1) \subset \Lambda(1,s)$. Write $f = \sigma_n(f) + g_n$ so that $\|g_n\|_1 \le K\frac{1}{n}$. For clarity we temporarily write T_n for $\sigma_n(f)$. Then $T_n = \sigma_n(T_n) + \sigma_n(g_n)$ giving

$$\|T_n - \sigma_n(T_n)\|_1 = \|\sigma_n(g_n)\|_1 \le \|g_n\|_1 \le K\frac{1}{n}.$$

In order to see that $T_n - \sigma_n(T_n) = \frac{\widetilde{T_n'}}{n}$, suppose that $T_n = \sum_{j=-n+1}^{n-1} b_j e^{ijt}$. Then

$$\widetilde{T}_n = -i\sum_{j=1}^{n-1} b_j e^{ijt} + i\sum_{j=-n+1}^{-1} b_j e^{ijt}.$$

Differentiating with respect to t we get

$$\widetilde{T}_n'(t) = \sum_{j=1}^{n-1} jb_j e^{ijt} - \sum_{j=-n+1}^{-1} jb_j e^{ijt}.$$

But

$$\begin{aligned} T_n - \sigma_n(T_n) &= \sum_{j=-n+1}^{n-1} \frac{|j|}{n} b_j e^{ijt} \\ &= \frac{1}{n}\Big\{ \sum_{j=1}^{n-1} jb_j e^{ijt} - \sum_{j=-n+1}^{-1} jb_j e^{ijt} \Big\} \\ &= \frac{\widetilde{T}_n'}{n} \end{aligned}$$

as claimed. Putting these facts together it is clear that $\|\widetilde{T}_n'\|_1 = \|\tilde{\sigma}_n'(f)\|_1 = \|\sigma_n'(\tilde{f})\|_1 \leq K$ for all n. This means that if we define a sequence of measures by $d\mu_n = \sigma_n'(\tilde{f})\,dt$, the sequence has a weak star cluster point μ for which $\|\mu\| \leq K$. For each integer m we have

$$\hat{\mu}(m) = \begin{cases} i\,m\hat{f}(m) & \text{if } m \geq 0 \\ -i\,m\hat{f}(m) & \text{if } m < 0. \end{cases}$$

We note that $\hat{\mu}(0) = 0$. We claim that $\tilde{f}$ is equal a.e. to a function in $BV[-\pi,\pi]$. For, let us define a function g on $-\pi \leq x \leq \pi$ by $g(x) = \mu[-\pi, x) = \int_{-\pi}^{x} d\mu$. For each integer $m \neq 0$,

$$\begin{aligned} \int_{-\pi}^{\pi} g(x) e^{-imx} dx &= \int_{-\pi}^{\pi} e^{-imx}\Big(\int_{-\pi}^{x} d\mu(t)\Big) dx \\ &= \int_{-\pi}^{\pi} \Big(\int_{t}^{\pi} e^{-imx} dx\Big) d\mu(t) \\ &= -2\pi i \frac{\hat{\mu}(m)}{m}. \end{aligned}$$

So $\hat{\mu}(m) = im\hat{g}(m)$. It follows that $\hat{g}(m) = i\hat{\tilde{f}}(m)$ for all $m \neq 0$. But we know that $g \in BV[-\pi,\pi]$ and our claim is proven.

For the converse it suffices to show that if $f \in \Lambda(1,1)$, then $\|\tilde{\sigma}_n(f) - \tilde{f}\|_1 = O(\frac{1}{n})$. Fix a value of n. From the formula for $\widetilde{K}_n$, where K_n is the Fejér kernel, we have

$$\begin{aligned} \tilde{\sigma}_n(f)(x) - \tilde{f}(x) &= \frac{1}{n\pi} \int_0^{\pi} \big(f(x+t) - f(x-t)\big) \frac{\sin nt}{(2\sin\frac{t}{2})^2}\, dt \qquad (10) \\ &= \int_0^{\frac{1}{n}} + \int_{\frac{1}{n}}^{\pi} \\ &= P(x) + Q(x), \end{aligned}$$

say. Suppose that $\|\Delta_t f\|_1 \leq M|t|$ for some M. We have

$$\|P\|_1 \leq \tfrac{1}{n\pi} \int_0^{\frac{1}{n}} 2Mtnt\big(\tfrac{2t}{\pi}\big)^{-2}\, dt = O\big(\tfrac{1}{n}\big).$$

Next suppose that $f = g$ a.e. where $g \in BV[-\pi,\pi]$. Bearing in mind the definition of the conjugate function, it is clear that the left-hand side of (10) has the same value for all x, whether we use f or g. We assume therefore that on the right-hand side $f \in BV[-\pi,\pi]$. To show that $\|Q\|_1 = O(\frac{1}{n})$, we use an approximation process. We replace f by $\sigma_m(f)$ and Q by Q_m above. We know that $\lim_{m\to\infty}\|f-\sigma_m(f)\|_1 = 0$, and a standard application of Fubini's theorem shows that $\lim_{m\to\infty}\|Q-Q_m\|_1 = 0$. Noting that

$$\left|\int_t^{\pi} \frac{\sin nu}{(2\sin\frac{u}{2})^2}\,du\right| \le \frac{2}{n(2\sin\frac{t}{2})^2} \le \frac{A}{nt^2}$$

let $R_n(t)$ denote the integral on the left. So $R_n(\pi)=0$. Let $\|f\|_*$ denote the BV-norm of f and apply integration by parts to the formula for Q_m:

$$\begin{aligned} Q_m(x) = \tfrac{1}{n\pi}\Big(& -R_n\big(\tfrac{1}{n}\big)\big[\sigma_m\big(x+\tfrac{1}{n}\big)-\sigma_m\big(x-\tfrac{1}{n}\big)\big] \\ & - \int_{\frac{1}{n}}^{\pi} R_n(t)\big[\sigma_m'(x+t)+\sigma_m'(x-t)\big]\,dt\Big) \end{aligned}$$

whence

$$\begin{aligned} |Q_m(x)| \le \tfrac{1}{n\pi}\Big(& \big|R_n\big(\tfrac{1}{n}\big)\big|\,\big[\big|\sigma_m\big(x+\tfrac{1}{n}\big)-\sigma_m\big(x-\tfrac{1}{n}\big)\big|\big] \\ & + \int_{\frac{1}{n}}^{\pi} |R_n(t)|\,\big[|\sigma_m'(x+t)|+|\sigma_m'(x-t)|\big]\,dt\Big). \end{aligned}$$

Integrating with respect to x we get

$$\|Q_m\|_1 \le \tfrac{1}{n\pi}\left(An\big(\tfrac{2M}{n}\big) + 2A\|f\|_*\int_{\frac{1}{n}}^{\pi}\frac{dt}{nt^2}\right) = O\big(\tfrac{1}{n}\big)$$

uniformly in m. We have used the fact that

$$\big\|\sigma_m(f)\big(\cdot+\tfrac{1}{n}\big)-\sigma_m(f)\big(\cdot-\tfrac{1}{n}\big)\big\|_1 \le 2M\tfrac{1}{n}$$

which follows readily from the definition of σ_m, and also that $\|\sigma_m(f)\|_* \to \|f\|_*$ as $m\to\infty$ [11: Chapter 4]. Letting $m\to\infty$, it follows that $\|Q\|_1 = O(\frac{1}{n})$.

Combining our estimates, we get $\|P+Q\|_1 = O(\frac{1}{n})$, and therefore $\|\tilde{\sigma}_n(f)-\tilde{f}\|_1 = O(\frac{1}{n})$. This completes the proof ∎

Remark. If only the weaker condition $\|\sigma_{2n}(f)-\sigma_n(f)\|_1 = O(\frac{1}{n})$ holds and f is analytic, then Theorem 5 with $s=1$ says that $f\in A^1_{1\infty}$.

If we recall that, for an analytic function f, $f\in\Lambda(1,1)$ precisely when $f'\in H^1$, the following corollary represents a partial addition to the results of Theorems 4′ and 9.

Corollary 11. *Suppose that f is analytic. Then $\|\sigma_n(f)-f\|_1 = O(\frac{1}{n})$ if and only if $f'\in H^1$.*

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