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On the Distributions of Logarithmic Derivative of Differentiable Measures on $\boldsymbol{R}$

By

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In [1], Hora discussed distributions of logarithmic derivative of differentiable probability measures on $\boldsymbol{R}$ and obtained the following theorem with Yamasaki.

Theorem. *Let P be an arbitrary probability distribution with mean 0 which is not Dirac measure δ_0 at 0. Then there exists some differentiable probability measure $d\mu(x)=f(x)dx$ such that $P(E)=\mu(x\,|\,f'(x)/f(x)\in E)$ for all $E\in\mathfrak{B}(\boldsymbol{R})$, where dx is the Lebesgue measure on $\boldsymbol{R}$ and $\mathfrak{B}(\boldsymbol{R})$ is the usual Borel field on $\boldsymbol{R}$.*

In this note, we will give a simple proof of this theorem and add a few comments. First we shall supplement some definitions and a few facts. (See, [1] and [2]).

(a) A probability measure μ is said to be differentiable, if $\mu(E-t)$ is a differentiable function of t for each $E\in\mathfrak{B}(\boldsymbol{R})$.
(b) For the differentiability of μ, it is necessary and sufficient that (1) μ is absolutely continuous with dx and (2) its density $f(x)$ is differentiable almost everywhere on $\boldsymbol{R}$ and $f'(x)\in L^1_{dx}(\boldsymbol{R})$.
(c) If δ_0 would coincide with the distribution μ_f of logarithmic derivative f'/f of μ ($d\mu(x)=f(x)dx$), then it follows that $f'=0$ almost everywhere and that $f\equiv 0$. Thus we must exclude the case $P=\delta_0$ for this problem.
(d) The distribution μ_f has mean 0. Therefore we must consider only probability distributions P with mean 0.

Before beginning the proof of the Theorem, we wish to state some idea which is somewhat formal. For a given P define a function $\omega(t)$ on (0, 1) such that $\omega(t)=\sup\{x\in\boldsymbol{R}\,|\,P((-\infty, x))\leqq t\}$. Then ω is increasing and by the properties of supremum,

(1) $$P((-\infty, \omega(t)))\leqq t \quad \text{for all} \quad t\in(0, 1), \quad \text{and}$$

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$$P((-\infty, x))>t, \qquad \text{if} \quad x>\omega(t). \tag{2}$$

Now let λ be the Lebesgue measure on $[0, 1]$ and define a measure $\omega\lambda$ on $\mathfrak{B}(\boldsymbol{R})$ such that $\omega\lambda(E)=\lambda(t\,|\,\omega(t)\in E)$ for all $E\in\mathfrak{B}(\boldsymbol{R})$. It follows from (2) that $\omega\lambda((-\infty, x))=P((-\infty, x))$ for all $x\in\boldsymbol{R}$. So we have

$$\omega\lambda=P. \tag{3}$$

Consequently,

$$\int_0^1|\omega(t)|\,d\lambda(t)=\int_{-\infty}^{\infty}|x|\,dP(x)<\infty, \qquad \text{and} \tag{4}$$

$$\int_0^1\omega(t)d\lambda(t)=\int_{-\infty}^{\infty}x\,dP(x)=0. \tag{5}$$

Thus the problem is to find f which satisfies $\int_{-\infty}^{\infty}\chi_E(f'(x)/f(x))f(x)dx=\int_0^1\chi_E(\omega(t))dt$, where χ_E is the indicator function of any Borel set E.

In order to find such f, we rewrite the right hand side using integration by substitution with a suitable monotone differentiable function γ on (0, 1). After some calculations (which is omitted here) we reach to a contradiction in the case that γ is strictly increasing. On the other hand if γ is strictly decreasing, then putting $\lim_{t\to 1}\gamma(t)=\alpha$, $\lim_{t\to 0}\gamma(t)=\beta$, we have $\int_0^1\chi_E(\omega(t))dt=-\int_\alpha^\beta\chi_E(\omega(\gamma^{-1}(x))(\gamma^{-1}(x))'dx$. So if we take

$$f'(x)/f(x)=\omega(\gamma^{-1}(x)), \qquad \text{and} \tag{6}$$

$$f(x)=-(\gamma^{-1}(x))'=\frac{-1}{\gamma'(\gamma^{-1}(x))}, \tag{7}$$

then the both sides in the above equality have the same form except the lower limit and upper limit of integration. From (6) and (7) it follows that $\omega(\gamma^{-1}(x))=-\gamma'(\gamma^{-1}(x))f'(x)$ and therefore $\omega(t)=-(f\circ\gamma)'(t)$. Thus for a function defined by $h(t)=\int_0^t\omega(\tau)d\tau$, we have $f(\gamma(t))=-h(t)+\text{const}$ and this constant must be 0, because $f(x)$ must satisfy $\lim_{x\to\pm\infty}f(x)=0$. Further it follows from (7) $\gamma'(t)=h(t)^{-1}$ and $\gamma(t)=\int_{1/2}^t\frac{d\tau}{h(\tau)}+\text{const}$. From now on we shall show that this procedure actually gives the desired function f.

(*Proof* of Theorem)

It is clear that $h(t)\equiv\int_0^t\omega(\tau)d\tau$ is absolutely continuous, and that $h(0)=h(1)=0$. $h(t)$ is negative on (0, 1). In fact suppose that $h(t)$ would be 0 for some $t_0\in(0, 1)$. Then $0=h(1)-h(t_0)=\int_{t_0}^1\omega(\tau)d\tau\geqq\omega(t_0)(1-t_0)$, which shows $\omega(t_0)\leqq 0$. Similarly $0=h(t_0)-h(0)$ shows $\omega(t_0)\geqq 0$, hence $\omega(t_0)=0$. Again from $0=h(1)-$

$h(t_0)=h(t_0)-h(0)$, we have $\omega(\tau)\equiv 0$ on $(0, 1)$, which contradicts to $P\neq\delta_0$. As $\omega(\tau)$ is negative for sufficiently small τ, $h(t)$ is negative on $(0, 1)$. Now we can define a function γ on $(0, 1)$ such that $\gamma(t)=\int_{1/2}^{t}\frac{d\lambda(\tau)}{h(\tau)}$. Then γ is strictly decreasing continuously differentiable function on $(0, 1)$. Put $\lim_{t\to 1}\gamma(t)=\alpha$ $(\geqq -\infty)$ and $\lim_{t\to 0}\gamma(t)=\beta$ $(\leqq\infty)$. Lastly we define a function $f(x)$ on $\boldsymbol{R}$ such that $f(x)=-h(\gamma^{-1}(x))$, if $x\in(\alpha, \beta)$ and $f(x)=0$, otherwise. Since f is absolutely continuous on any closed interval of (α, β) and $\lim_{x\to\alpha}f(x)=\lim_{x\to\beta}f(x)=0$, so it is continuous, differentiable almost everywhere and

$$f'(x)=-\omega(\gamma^{-1}(x))h(\gamma^{-1}(x))=\omega(\gamma^{-1}(x))f(x) \qquad \text{on } (\alpha, \beta). \tag{8}$$

Then

$$\int_{-\infty}^{\infty}f(x)dx=-\int_{\alpha}^{\beta}h(\gamma^{-1}(x))dx=\int_{0}^{1}h(t)\gamma'(t)d\lambda(t)=1, \qquad \text{and} \tag{9}$$

$$\int_{-\infty}^{\infty}|f'(x)|\,dx=\int_{\alpha}^{\beta}|\omega(\gamma^{-1}(x))|\,f(x)dx=\int_{0}^{1}|\omega(t)|\,d\lambda(t)<\infty. \tag{10}$$

Consequently $f(x)$ is an absolutely continuous function on $\boldsymbol{R}$ and a measure defined by $d\mu(x)=f(x)dx$ is differentiable. Now we have $\mu(x\,|\,f'(x)/f(x)\in E)=-\int_{\alpha}^{\beta}\chi_E(\omega(\gamma^{-1}(x))h(\gamma^{-1}(x))dx=\int_{0}^{1}\chi_E(\omega(t))d\lambda(t)=P(E)$ for all Borel sets E.

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Remark 1. *$f_k(x)\equiv f(x+k)$ (k: an arbitrary constant) also satisfies $\mu_{f_k}=P$, because the translation of f does not change the distribution of logarithmic derivative.*

Remark 2. *If P is a symmetric distribution i.e., $P(E)=P(-E)$ for all $E\in\mathfrak{B}(\boldsymbol{R})$, then f is an even function and $f(0)>0$.*

Proof. Take any $t\in(0, 1/2)$. Then $P((-\infty, \omega(t+1/2)+\varepsilon))>t+1/2$ and $P((-\varepsilon-\omega(1/2-t), \infty))=P((-\infty, \omega(1/2-t)+\varepsilon))>1/2-t$. It follows that $\omega(t+1/2)+\varepsilon>-\varepsilon-\omega(1/2-t)$ for all $\varepsilon>0$ and hence $\omega(t+1/2)+\omega(1/2-t)\geqq 0$. Since $0=\int_{0}^{1}\omega(t)d\lambda(t)=\int_{0}^{1/2}\{\omega(t+1/2)+\omega(1/2-t)\}\,d\lambda(t)$, so $\omega(t+1/2)+\omega(1/2-t)=0$ for almost all $t\in(0, 1/2)$. Consequently it follows from (5) $h(t+1/2)=h(1/2-t)$ and from this $\gamma(t+1/2)=-\gamma(1/2-t)$ for all $t\in(0, 1/2)$. Thus we have $f(0)=-h(\gamma^{-1}(0))=-h(1/2)>0$, and $f(\gamma(t+1/2))=-h(t+1/2)=-h(1/2-t)=f(\gamma(1/2-t))=f(-\gamma(t+1/2))$.

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Conversely, it is evident that if f is an even function then μ_f is symmetric.

Example 1. $P=U_{-a,a}$ $(a>0)$: Uniform distribution on $[-a, a]$.

By simple computations, we have $\omega(t)=a(2t-1)$, $h(t)=at(t-1)$ and $\gamma(t)=a^{-1}\log(t^{-1}(1-t))$. Therefore $\gamma^{-1}(x)=\{1+\exp(ax)\}^{-1}$ and $f(x)=a\exp(ax)\{1+\exp(ax)\}^{-2}$.

Example 2. $P=N(0, 1)$: Normal distribution with mean 0 and variance 1.

Put $G(x)=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-x^2/2)dx$. Then it is easy to see that $\omega(t)=G^{-1}(t)$, $h(t)=-(2\pi)^{-1/2}\exp(-G^{-1}(t)^2/2)$ and $\gamma(t)=-G^{-1}(t)$. Thus we have $\gamma^{-1}(x)=G(-x)$ and $f(x)=(2\pi)^{-1/2}\exp(-x^2/2)$.

Remark 3. *As we have seen in Remark* 1, *a function f which satisfies $\mu_f=P$ for a given P is not unique. By the way we can take f as an even function, if P is symmetric. However such an even function is not uniquely determined as it will be seen in the following example.*

Example 3. Put $g(x)=1/2|x|\exp(-|x|)$. Then $d\mu(x)=g(x)dx$ is a differentiable measure and after some calculations we have,

$$\mu(x\,|\,g'(x)/g(x)\in E)=1/2\int_E\left\{\frac{1}{(1+|x|)^3}\exp\left(-\frac{1}{1+|x|}\right)+\chi_{[-1,1]}(x)\frac{1}{(1-|x|)^3}\exp\left(-\frac{1}{1-|x|}\right)\right\}dx\,.$$

Thus for the measure P defined by the right hand side in the above equality, g and f obtained in the proof of Theorem are even solutions of $\mu_f=P$. However they does not coincide with each other, because $g(0)=0$ and $f(0)>0$.

References

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