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Complements on Growth Envelopes of Spaces with Generalized Smoothness in the Sub-Critical Case

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Abstract. We describe the growth envelope of Besov and Triebel-Lizorkin spaces $B^{\sigma}_{pq}(\mathbb{R}^n)$ and $F^{\sigma}_{pq}(\mathbb{R}^n)$ with generalized smoothness, i.e. instead of the usual scalar regularity index $\sigma \in \mathbb{R}$ we consider now the more general case of a sequence $\sigma = \{\sigma_j\}_{j\in\mathbb{N}_0}$. We take under consideration the range of the parameters σ, p, q which, in analogy to the classical terminology, we call sub-critical.

Keywords: *Besov spaces, generalized smoothness, growth-envelope functions*

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1. Introduction

Let A^s_{pq} be either B^s_{pq} (Besov spaces) or F^s_{pq} (Triebel-Lizorkin spaces) defined in the whole Euclidean space $\mathbb{R}^n$. Suppose also that the parameters s, p, q are chosen so that $A^s_{pq} \subset L_1^{\mathrm{loc}}$ but $A^s_{pq} \not\hookrightarrow L_\infty$. Of interest in this case is the singularity behaviour of $f \in A^s_{pq}$, usually expressed in terms of the distribution function m_f and the non-increasing rearrangement f^*.

We shall be more precise in the sequel: here we remark that answers of final character on this subject can be found in the works of D. Haroske [12, 13] and H. Triebel [20], where the notion of the *growth envelope* $\mathfrak{E}_{\mathrm{G}} A^s_{pq}$ of the spaces A^s_{pq} appears as a useful refinement of the above mentioned tools and as a compact and elegant description of the singularity behaviour of elements in the considered spaces.

Quite recently A. Caetano and S. D. Moura have taken into consideration in [5] the same type of problem sketched above, now for the wider class of spaces $A^{(s,\Psi)}_{pq}$ (where again A stands either for B or F). These spaces, in rough terms, can be considered as a perturbed version of the classical spaces A^s_{pq}, where the usual regularity index s is replaced by a couple (s, Ψ) in which Ψ plays the role of a finer tuning smoothness parameter. Apart from their own interest, these spaces with perturbed smoothness arise naturally in the theory of function spaces defined on some fractal-type sets (see [2, 3, 8, 9, 17, 18]).

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The present paper can be considered a twin paper to [5] in the following sense: instead of considering only perturbed versions to the usual Besov and Triebel-Lizorkin spaces, we take into consideration function spaces (of Besov and Triebel-Lizorkin type) with generalized smoothness A^{σ}_{pq}, where now σ is a sequence. Spaces of this type have been studied by many mathematicians. We refer to [10] for results, references and comments. An application of these spaces (or, from another point of view, a natural motivation for their definition) can be found in [4].

Then we restrict our attention (as it is done in [5]) to a non-limiting case which, in analogy to the classical and the perturbed situation, we call *sub-critical case*. Mutatis mutandis, all the techniques used in [5] (in particular, the powerful tool of interpolation with a function parameter and the atomic representation of the involved spaces) can be applied also in this very general case.

We end up with the growth envelope of spaces with generalized smoothness $\mathfrak{E}_{\mathrm{G}} A^{\sigma}_{pq}$ and this result generalizes in a unified way both the classical and the perturbed sub-critical cases.

2. General notation

In this paper we shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n \quad (n \in \mathbb{N})$ denotes the Euclidean n-space and $\mathbb{R} = \mathbb{R}^1$. We use the equivalence $\sim$ in

$$a_k \sim b_k \qquad \text{or} \qquad \varphi(r) \sim \psi(r)$$

always to mean that there are two numbers $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 a_k \leq b_k \leq c_2 a_k \qquad \text{or} \qquad c_1\varphi(r) \leq \psi(r) \leq c_2\varphi(r)$$

for all admitted values of the discrete variable k or the continuous variable r, respectively. Here a_k, b_k are positive numbers and φ, ψ are positive functions. The word "positive" is always used to mean "strictly positive", both for functions and for real numbers.

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X into Y is continuous. If not otherwise indicated, log is always taken with respect to base 2. We consider here only function spaces defined on $\mathbb{R}^n$ and therefore in most cases we shall omit "$\mathbb{R}^n$" from the notation.

3. The class $\mathbb{B}$, admissible sequences and related indices

As briefly mentioned in the introduction, we shall take into consideration function spaces of Besov and Triebel-Lizorkin type with generalized smoothness, i.e. spaces where the usual regularity index is replaced by a more general sequence. In this section we explain the class of sequences we shall be interested in and some related basic results.

In the sequel we shall be concerned also with the technique of interpolation with a function parameter. Here we collect necessary definitions and results on this topic following [6, 16].

Definition 3.1. A positive and continuous function $g:\ (0,\infty)\to\mathbb{R}$ belongs to the *class* $\mathbb{B}$ if

$$g(t)^{-1}=g(t^{-1}) \tag{3.1}$$
$$\overline{g}(t):=\sup_{s>0}\frac{g(ts)}{g(s)}<\infty$$

for all $t>o$. If $g\in\mathbb{B}$, then the *upper* and *lower Boyd indices* α_g and β_g of g are well defined by

$$\alpha_g=\lim_{t\to\infty}\frac{\log\overline{g}(t)}{\log t}\qquad\text{and}\qquad\beta_g=\lim_{t\to 0}\frac{\log\overline{g}(t)}{\log t},$$

respectively.

Remark 3.2. The class $\mathbb{B}$ has been defined in analogy to the class $\mathcal{B}$ considered in [16]. The latter class differs from the former by condition (3.1), which was not required for $\mathcal{B}$. Our additional requirement (3.1) is simply convenient and does not play any crucial role.

Notice that for any $g\in\mathbb{B}$ one has

$$\overline{g}(\tau^{-1})^{-1}g(s)\le g(\tau s)\le\overline{g}(\tau)g(s)\qquad(s,\tau>0). \tag{3.2}$$

The class of sequences we shall consider has been introduced in [10]. Its definition reads as follows.

Definition 3.3. A sequence $\sigma=\{\sigma_j\}_{j\in\mathbb{N}_0}$ is said to be *admissible* if $\sigma_j>0$ for $j\in\mathbb{N}_0$ and there exist two constants $d_0>0$ and $d_1>0$ such that

$$d_0\sigma_j\le\sigma_{j+1}\le d_1\sigma_j\qquad(j\in\mathbb{N}_0). \tag{3.3}$$

Of course, a sequence $\{\sigma_j\}_{j\in\mathbb{N}_0}$ is admissible if and only if $\frac{\sigma_{j+1}}{\sigma_j}$ is bounded away from 0 and infinity uniformly in j. We insist on this observation and define, in analogy to the continuous case, the lower and upper Boyd indices of a given admissible sequence as follows.

Definition 3.4. Let $\sigma=\{\sigma_j\}_{j\in\mathbb{N}_0}$ be an admissible sequence. Define

$$\underline{\sigma}_j=\inf_{k\ge 0}\frac{\sigma_{j+k}}{\sigma_k}\qquad\text{and}\qquad\overline{\sigma}_j=\sup_{k\ge 0}\frac{\sigma_{j+k}}{\sigma_k},\qquad(j\in\mathbb{N}_0).$$

Then we let

$$\alpha_\sigma=\lim_{j\to\infty}\frac{\log\overline{\sigma}_j}{j}\qquad\text{and}\qquad\beta_\sigma=\lim_{j\to\infty}\frac{\log\underline{\sigma}_j}{j} \tag{3.4}$$

be the *upper* and *lower Boyd index* of the given sequence σ, respectively.

Remark 3.5. The above definition is well posed: the sequence $\{\log\overline{\sigma}_j\}_{j\in\mathbb{N}}$ is subadditive and hence the left-hand side limit in (3.4) exists and is finite (since σ is an admissible sequence). The corresponding assertions for the lower counterpart β_σ can be read off observing that $\log\underline{\sigma}_j=-\log(\overline{\sigma^{-1}})_j$. Notice that if σ is an admissible sequence, then $\underline{\sigma}_j \sigma_k \leq \sigma_{j+k} \leq \sigma_k \overline{\sigma}_j$ for any $j, k \in \mathbb{N}_0$. In particular, $\underline{\sigma}_1$ and $\overline{\sigma}_1$ are the best possible constants d_0 and d_1 in (3.3), respectively.

The Boyd index α_σ of an admissible sequence σ (and its lower counterpart β_σ) describes the asymptotic behaviour of the $\overline{\sigma}_j$'s and provides more information than simply $\overline{\sigma}_1$ and, what is more, is stable under the equivalence of sequences: if $\sigma \sim \tau$, then $\alpha_\sigma = \alpha_\tau$ as one readily verifies.

Observe also that given $\varepsilon > 0$, there are two constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$ such that

$$c_1 2^{(\beta_\sigma - \varepsilon)j} \leq \underline{\sigma}_j \leq \overline{\sigma}_j \leq c_2 2^{(\alpha_\sigma + \varepsilon)j} \qquad (j \in \mathbb{N}_0). \tag{3.5}$$

In particular, for each $\varepsilon > 0$,

$$c_1 2^{(\beta_\sigma - \varepsilon)j} \leq \sigma_j \leq c_2 2^{(\alpha_\sigma + \varepsilon)j} \qquad (j \in \mathbb{N}_0) \tag{3.6}$$

for some constants $c_1 = c_1(\varepsilon) > 0$ and $c_2 = c_2(\varepsilon) > 0$.

For our later purposes, given an admissible sequence σ it is useful to construct a function Σ in $\mathbb{B}$ which interpolates σ. The exact definition is of no interest: it suffices to consider any positive and continuous function $\Sigma\colon (0,\infty) \to \mathbb{R}$ in the class $\mathbb{B}$ with $\Sigma(2^j) \sim \sigma_j$ for $j \in \mathbb{N}_0$. An example can be given by the construction

$$\Sigma(t) = \begin{cases} \sigma_0^{-1}\{(2^{-j}t - 1)(\sigma_{j+1} - \sigma_j) + \sigma_j\} & \text{if } t \in [2^j, 2^{j+1}), j \in \mathbb{N}_0 \\ \Sigma(t^{-1})^{-1} & \text{otherwise.} \end{cases} \tag{3.7}$$

Somehow unexpectedly, it turns out that the lower and upper Boyd indices of any such interpolating function do coincide with the corresponding indices of the starting sequence. In the following proposition we state the rigorous assertions.

Proposition 3.6.

(i) *Let $g \in \mathbb{B}$. Then the interpolated sequence $\gamma = \{g(2^j)\}_{j \in \mathbb{N}_0}$ is admissible.*

(ii) *Let σ be an admissible sequence. Let $\Sigma\colon (0,\infty) \to \mathbb{R}$ be a function in $\mathbb{B}$ such that $\Sigma(2^j) \sim \sigma_j$ $(j \in \mathbb{N}_0)$ (for instance, the function defined in (3.7)). Then*

$$\beta_\sigma = \beta_\Sigma \leq \alpha_\Sigma = \alpha_\sigma.$$

Proof. The proof of statement (i) follows immediately from the estimation

$$\overline{g}(2^{-1})^{-1} \leq \frac{g(2^{j+1})}{g(2^j)} \leq \overline{g}(2)$$

which is readily true by virtue of (3.2).

The proof of statement (ii) is not difficult but admittedly tedious: Fixing $\varepsilon > 0$, (3.5) and (3.6) hold true for some constants $c_1 > 0$ and $c_2 > 0$ (depending on ε). Afterwards one has to estimate from above the quotients

$$\frac{\Sigma(2^{j+k})}{\Sigma(2^k)} \qquad (j, k \in \mathbb{Z}).$$

In order to do this one has to take into consideration separately the following six cases:

1) $j > 0,\ k > 0$
2) $j > 0,\ k \le 0,\ j+k > 0$
3) $j > 0,\ k \le 0,\ j+k \le 0$
4) $j \le 0,\ k < 0$
5) $j \le 0,\ k \ge 0,\ j+k > 0$
6) $j \le 0,\ k \ge 0,\ j+k \le 0$.

By the definition of Σ and the properties of the admissible sequence σ one gets

$$\frac{\Sigma(2^{j+k})}{\Sigma(2^k)} \le c_\varepsilon \max\left\{2^{(\beta_\sigma-\varepsilon)j}, 2^{(\alpha_\sigma+\varepsilon)j}\right\} \qquad (j,k \in \mathbb{Z}) \tag{3.8}$$

for some constant c_ε independent of j and k.

We show in detail how one steps from the discrete estimation (3.8) to the continuous one. By (3.2) there are two constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\Sigma(s) \le \Sigma(\tau s) \le c_2\Sigma(s) \qquad (s > 0)$$

uniformly in $\tau \in [2^{-2}, 2^2]$. For $t, s \in (0,\infty)$, let $j,k \in \mathbb{Z}$ with $2^j \le t \le 2^{j+1}$ and $2^k \le s \le 2^{k+1}$. Then, for appropriately chosen $\tau, \tau' \in [2^{-2}, 2^2]$,

$$\begin{aligned}\frac{\Sigma(ts)}{\Sigma(s)} = \frac{\Sigma(2^{j+k}\tau)}{\Sigma(2^k\tau')} \le c\frac{\Sigma(2^{j+k})}{\Sigma(2^k)} &\le c'_\varepsilon \max\left\{2^{(\beta_\sigma-\varepsilon)j}, 2^{(\alpha_\sigma+\varepsilon)j}\right\} \\ &\le c''_\varepsilon \max\left\{t^{\beta_\sigma-\varepsilon}, t^{\alpha_\sigma+\varepsilon}\right\}.\end{aligned}$$

This shows that

$$\log \overline{\Sigma}(t) \le c_\varepsilon + \max\left\{(\beta_\sigma-\varepsilon)\log t, (\alpha_\sigma+\varepsilon)\log t\right\} \qquad (t > 0).$$

Hence,

$$\frac{\log \overline{\Sigma}(t)}{\log t} \begin{cases} \ge o(1) + \beta_\sigma - \varepsilon & \text{for } t < 1 \\ \le o(1) + \alpha_\sigma + \varepsilon & \text{for } t > 1. \end{cases}$$

This finally proves $\beta_\sigma - \varepsilon \le \beta_\Sigma$ and $\alpha_\sigma + \varepsilon \ge \alpha_\Sigma$. Since one always has $\beta_f \le \alpha_f$ for any $f \in \mathbb{B}$ (see [16: p. 184]) and $\varepsilon > 0$ was arbitrarily chosen, we can infer that

$$\beta_\sigma \le \beta_\Sigma \le \alpha_\Sigma \le \alpha_\sigma.$$

To conclude the proof we have to show the converse inequalities, i.e., we must prove $\beta_\sigma \ge \beta_\Sigma$ and $\alpha_\Sigma \ge \alpha_\sigma$. Fortunately, this turns out to be the easiest part as $\Sigma(2^j) \sim \sigma_j \ \ (j \in \mathbb{N}_0)$ and, by (3.2),

$$\frac{\sigma_{j+k}}{\sigma_k} \sim \frac{\Sigma(2^{j+k})}{\Sigma(2^k)} \le \overline{\Sigma}(2^j) \qquad (j,k \in \mathbb{N}_0).$$

Hence, $\overline{\sigma}_j \le c\overline{\Sigma}(2^j) \ \ (j \in \mathbb{N}_0)$. Therefore $\alpha_\sigma \le \alpha_\Sigma$. Proceeding analogously we can derive the desired estimation also for the lower indices and finally conclude the proof ∎

Example 3.7. We consider some examples of admissible sequences.

(i) For $s \in \mathbb{R}$, $\sigma = \{2^{sj}\}_{j\in\mathbb{N}_0}$ is readily an admissible sequence with $\beta_\sigma = \alpha_\sigma = s$.

(ii) Let $\Phi\colon (0,1] \to \mathbb{R}$ be a slowly varying function (or equivalent to a slowly varying one) in the sense of [1]. Then, for $s \in \mathbb{R}$, $\sigma = \{2^{sj}\Phi(2^j)\}_{j\in\mathbb{N}_0}$ is an admissible sequence. Also here we have $\beta_\sigma = \alpha_\sigma = s$.

(iii) In view of [4: Proposition 1.9.7], the case $\sigma = \{2^{sj}\Psi(2^{-j})\}_{j\in\mathbb{N}_0}$, where now Ψ is an admissible function in the sense of [8], can be regarded as a special case of (ii). We recall that an admissible function Ψ is a positive monotone function defined on $(0,1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$ $(j \in \mathbb{N}_0)$.

(iv) More generally, given any function of the form

$$\Sigma(t) \sim \exp\left\{\int_1^t \xi(s)\,\tfrac{ds}{s}\right\} \tag{3.9}$$

where ξ is a measurable bounded function, the sequence $\sigma = \{\Sigma(2^j)\}_{j\in\mathbb{N}_0}$ is admissible.

One could prove that α_σ is the infimum of the upper bounds of all functions ξ representing Σ as in (3.9) and that β_σ is the supremum of their lower bounds. Conversely, one could even assert that any admissible sequence σ can be represented as $\sigma = \{\Sigma(2^j)\}_{j\in\mathbb{N}_0}$ where Σ has form (3.9). We skip details and we refer to the monograph [1], where in view of our Proposition 3.6 and after an appropriate translation in the language of OR-functions the above assertion can be derived easily from Theorem 2.2.7 on page 74.

4. Function spaces

4.1 Function spaces of generalized smoothness. The definition of Besov and Triebel-Lizorkin spaces in terms of a generalized smoothness has been already considered in some generality: see, for instance, [6, 11, 14 - 16]. We refer to the paper [10] of W. Farkas and H.-G. Leopold which represents a unified and general approach on this topic.

In view of the main Definition 4.4 below, we now collect usual notation and basic concepts. The Schwartz class of all C^∞ functions decreasing rapidly together with all their derivatives is denoted by $\mathcal{S}(\mathbb{R}^n)$ and its dual space of all tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$. If $f \in \mathcal{S}'(\mathbb{R}^n)$, then $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ denote the Fourier and the inverse Fourier transform of f, respectively.

Definition 4.1. By a *resolution of unity* $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ we mean a sequence of compactly supported smooth functions such that

$$\begin{aligned}
&\operatorname{supp}\varphi_0 \subset \left\{\xi \in \mathbb{R}^n : |\xi| \le 2\right\} \\
&\operatorname{supp}\varphi_j \subset \left\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\right\} \quad (j \ge 1) \\
&\sum_{j=0}^\infty \varphi_j(\xi) = 1 \quad (\xi \in \mathbb{R}^n), \ \sup_{\xi\in\mathbb{R}^n} |D^\alpha\varphi_j(\xi)| \le c_\alpha 2^{-j|\alpha|} \quad (\alpha \in \mathbb{N}_0^n).
\end{aligned}$$

Definition 4.2. If $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity and $f \in \mathcal{S}'(\mathbb{R}^n)$, then we set

$$\varphi_j(D)f = \mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$$

for $j \geq 0$.

Definition 4.3. If $\{a_j(x)\}_{j\in\mathbb{N}_0}$ is a sequence of functions defined in $\mathbb{R}^n$, then we put

$$\|a_j|\ell_q(L_p)\| = \big\|\{\|a_j|L_p(\mathbb{R}^n)\|\}_j|\ell_q\big\|$$
$$\|a_j|L_p(\ell_q)\| = \big\|\,\|\{a_j(\cdot)\}_j|\ell_q\||L_p(\mathbb{R}^n)\big\|.$$

Now we are ready for the main definition of this section.

Definition 4.4. Let $\Phi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity and let σ be an admissible sequence.

(i) Let $0 < p, q \leq \infty$. Then

$$B^\sigma_{pq} = \Big\{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|\sigma_j\varphi_j(D)f|\ell_q(L_p)\| < \infty\Big\}.$$

(ii) Let $0 < p, q \leq \infty$ with $p < \infty$. Then

$$F^\sigma_{pq} = \Big\{f \in \mathcal{S}'(\mathbb{R}^n) :\ \|\sigma_j\varphi_j(D)f|L_p(\ell_q)\| < \infty\Big\}.$$

Remark 4.5.

(i) The classical Besov and Triebel-Lizorkin spaces B^s_{pq} and F^s_{pq} $(s \in \mathbb{R})$ are subsumed in the above definitions for $\sigma = \{2^{sj}\}_{j\in\mathbb{N}_0}$ (and, of course, we stick at the usual notation in this case, avoiding the cumbersome expression $B^{\{2^{sj}\}}_{pq}$). As in the classical case, the spaces B^σ_{pq} and F^σ_{pq} are independent, up to equivalent quasi-norms, from the chosen resolution of unity appearing in their definition.

(ii) In [10] the definition of Besov and Triebel-Lizorkin is more general: one can consider more general systems Φ of compactly supported smooth functions φ_j, inducing a fourth parameter $N = \{N_j\}_{j\in\mathbb{N}_0}$, linked to the size of the supports of the φ_j, to appear. We do not go into detail, since the above formulation is sufficient for our future purposes.

As we remarked above, the spaces B^σ_{pq} and F^σ_{pq} coincide with the usual Besov and Triebel-Lizorkin spaces B^s_{pq} and F^s_{pq}, respectively, for $\sigma = \{2^{sj}\}_{j\in\mathbb{N}_0}$. If we let $\sigma = \{2^{sj}\Psi(2^{-j})\}_{j\in\mathbb{N}_0}$, where Ψ is an admissible function, then σ is an admissible sequence (see Example 3.7/(iii)). The corresponding Besov and Triebel-Lizorkin spaces coincide, respectively, with the spaces $B^{(s,\Psi)}_{pq}$ and $F^{(s,\Psi)}_{pq}$ introduced by D. Edmunds and H. Triebel in [8, 9] and also considered by S. D. Moura in [17, 18]. In analogy to our notational agreement confessed above, also in this case we preserve the original notation.

Now we deal with some embedding assertions.

Proposition 4.6. *Let $0 < p, q, q_1, q_2 \leq \infty$ ($p < \infty$ in the F-case) and let σ be an admissible sequence. Then, for $s_1, s_2 \in \mathbb{R}$ with $s_2 < \beta_\sigma \leq \alpha_\sigma < s_1$, $A^{s_1}_{pq_1} \hookrightarrow A^\sigma_{pq} \hookrightarrow A^{s_2}_{pq_2}$.*

Proof. By our assumptions and thanks to (3.6) there exist two constants $c > 0$ and $c' > 0$ such that $c2^{s_2 j} \leq \sigma_j \leq c' 2^{s_1 j}$ $(j \in \mathbb{N}_0)$. Then by standard arguments one concludes the proof ∎

We shall also need the following sharp embedding assertion.

Proposition 4.7. *Let $0 < p_1 < p < p_2 \leq \infty$, $0 < q \leq \infty$ and σ be an admissible sequence. Let σ' and σ'' be two admissible sequences defined by*

$$\sigma'_j = 2^{n(\frac{1}{p_1} - \frac{1}{p})j}\sigma_j \quad \text{and} \quad \sigma''_j = 2^{n(\frac{1}{p_2} - \frac{1}{p})j}\sigma_j \qquad (j \in \mathbb{N}_0).$$

Then

$$B^{\sigma'}_{p_1 u} \hookrightarrow F^{\sigma}_{pq} \hookrightarrow B^{\sigma''}_{p_2 v}$$

if and only if $0 < u \leq p \leq v \leq \infty$.

Proof. We outline the proof, following essentially the arguments used for the proof of [5: Proposition 3.4] taking now advantage of a general lifting operator as it appears in [10: Theorem 3.1.8]. More precisely, for a given admissible sequence τ we consider the operator I_τ defined by the symbol $\mu(t) = \sum_{j=0}^{\infty} \tau_j \varphi_j(t)$, where $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a fixed resolution of unity. This means that I_τ acts on A^{σ}_{pq} as

$$I_\tau f = \mathcal{F}^{-1}(\mu \mathcal{F} f) \qquad (f \in A^{\sigma}_{pq}).$$

Then I_τ maps isomorphically A^{σ}_{pq} onto $A^{\sigma\tau^{-1}}_{pq}$ and $\|I_\tau \cdot \,|\, A^{\sigma\tau^{-1}}_{pq}\|$ is an equivalent quasi-norm in A^{σ}_{pq}. As a consequence we also get that $I_\tau^{-1} = I_{\tau^{-1}}$.

Let s_1, s_2, s be such that $s_1 - \frac{n}{p_1} = s - \frac{n}{p} = s_2 - \frac{n}{p_2}$.

In the above commutative diagram the vertical arrows stand for the lift I_τ (or its inverse), where $\tau = \{2^{-sj}\sigma_j\}_{j\in\mathbb{N}_0}$, and the horizontal arrows for the natural injection. The desired result then follows directly from necessary and sufficient conditions for the embeddings corresponding to the lower part of the diagram (cf. [7: pp. 44 - 45] ∎

4.2 Generalized Lorentz spaces. Following [16] we recall the definition of generalized Lorentz spaces, for we shall make use of this type of spaces in the sequel.

First of all we recall that for an a.e. finite measurable function f defined on $\mathbb{R}^n$ its distribution function m_f is defined as

$$m_f(\lambda) = |\{|f(x)| > \lambda\}| \qquad (\lambda \geq 0)$$

where the outer $|\cdot|$ denotes the Lebesgue measure. The non-increasing rearrangement f^* of f is then defined by

$$f^*(t) = \inf\left\{\lambda \geq 0 : m_f(\lambda) \leq t\right\} \qquad (t \geq 0)$$

where we agree on $\inf \emptyset = \infty$.

Definition 4.8. If $\varphi \in \mathcal{B}$ and $0 < q \leq \infty$, then the generalized Lorentz space $L_q(\varphi)$ is the set of all complex measurable functions f on $\mathbb{R}^n$ such that

$$\|f|L_q(\varphi)\| := \begin{cases} \left(\int_0^\infty \big(\varphi(t) f^*(t)\big)^q \, \frac{dt}{t} \right)^{1/q} & \text{if } 0 < q < \infty \\ \sup_{t\in(0,\infty)} \varphi(t) f^*(t) & \text{if } q = \infty \end{cases}$$

is finite.

Remark 4.9. If $\varphi(t) = t^{\frac{1}{p}}(1 + |\log t|)^a$ with $0 < p \leq \infty$ and $a \in \mathbb{R}$, then $L_q(\varphi)$ is the Lorentz-Zygmund space $L_{pq}(\log L)_a$, which in turn is the classical Lorentz space L_{pq} if $a = 0$.

5. Growth envelopes

As we mentioned, the concept of growth envelope has been introduced by D. Haroske in [12] and was considered also by H. Triebel in [20]. Here we quote the basic definitions and results concerning growth envelopes. However, we shall be rather concise and refer mainly to [5, 12, 20] for heuristics, motivations and details on this subject.

The growth envelope function involves the concept of non-increasing rearrangement. So we restrict ourselves to function spaces A^σ_{pq} which are contained in L_1^{loc}.

On the other hand, the cases of interest for studying the growth envelope concern function spaces which are not embedded into L_∞. So, it seems reasonable to take into consideration the spaces A^σ_{pq} with

$$0 < p, q \leq \infty, \qquad n(\tfrac{1}{p} - 1)_+ < \beta_\sigma \leq \alpha_\sigma < \tfrac{n}{p} \tag{5.1}$$

where $a_+ = \max(a, 0)$ for $a \in \mathbb{R}$. As a matter of fact, by virtue of Proposition 4.6 and by the known characterization for $A^s_{pq} \hookrightarrow L_\infty$ and $A^s_{pq} \hookrightarrow L_1^{\mathrm{loc}}$ in terms of s, p, q the above conditions (5.1) guarantee $A^\sigma_{pq} \hookrightarrow L_1^{\mathrm{loc}}$ and $A^\sigma_{pq} \not\hookrightarrow L_\infty$. On the other hand, in a similar way, $\alpha_\sigma < n(\frac{1}{p} - 1)_+$ or $\beta_\sigma > \frac{n}{p}$ lead to $A^\sigma_{pq} \not\hookrightarrow L_1^{\mathrm{loc}}$ and $A^\sigma_{pq} \hookrightarrow L_\infty$, respectively. For the spaces A^s_{pq} and $A^{(s,\Psi)}_{pq}$ conditions (5.1) result in $n(\frac{1}{p} - 1)_+ < s < \frac{n}{p}$ which corresponds to the so-called *sub-critical case*. In analogy, we refer also to (5.1) as to the sub-critical case.

Definition 5.1. Let σ, p, q as in (5.1). We call *growth envelope function of* A^σ_{pq} any positive non-increasing continuous function $\mathcal{E}_{\mathrm{G}} A^\sigma_{pq}$ which is equivalent to

$$\mathcal{E}_{\mathrm{G}}|A^\sigma_{pq}(t) := \sup\left\{ f^*(t) : \|f|A^\sigma_{pq}\| \leq 1 \right\} \qquad (t \in (0,1]) \tag{5.2}$$

in some interval $(0, \varepsilon]$, for some $0 < \varepsilon < 1$.

Let $H(t) := -\log \mathcal{E}_{\mathrm{G}} A^\sigma_{pq}(t)$ $(t \in (0, \varepsilon])$ and let μ_H be the associated Borel measure on $(0, \varepsilon]$. Consider $0 < u \leq \infty$. Then the couple

$$\mathfrak{E}_{\mathrm{G}} A^\sigma_{pq} := ([\mathcal{E}_{\mathrm{G}} A^\sigma_{pq}], u) \tag{5.3}$$

is called *growth envelope for* A^σ_{pq} when

$$\Big(\int_{(0,\varepsilon]}\Big(\frac{f^*(t)}{\mathcal{E}_{\mathrm{G}}A^\sigma_{pq}(t)}\Big)^v\mu_H(dt)\Big)^{\frac{1}{v}}\le c\,\|f|A^\sigma_{pq}\|$$

(modification if $v=\infty$) for some constant $c=c(v)$ and all $f\in A^\sigma_{pq}$ if and only if $u\le v\le\infty$.

Remark 5.2.

(i) The function $\mathcal{E}_{\mathrm{G}}|A^\sigma_{pq}$ (for the prescribed range of parameters) is always non-increasing unbounded and positive on some interval $(0,\varepsilon]$, with $0<\varepsilon\le 1$. This assertion can be easily obtained by Proposition 4.6 and the corresponding assertions for A^s_{pq} (see [20: pp. 189 - 190/12.6]). An equivalent continuous function to $\mathcal{E}_{\mathrm{G}}|A^\sigma_{pq}$ can be then easily exhibited.

(ii) If $\|\cdot\,|\,A^\sigma_{pq}\|_1$ and $\|\cdot\,|\,A^\sigma_{pq}\|_2$ are two equivalent quasi-norms, then (in obvious notation) $\mathcal{E}_{\mathrm{G}}|_1A^\sigma_{pq}\sim\mathcal{E}_{\mathrm{G}}|_2A^\sigma_{pq}$, so that the definition of the growth envelope is independent of the particular quasi-norm considered in the space taken into consideration.

(iii) The brackets $[\cdot]$ in (5.3) mean that we take the equivalence class of all possible envelope functions $\mathcal{E}_{\mathrm{G}}A^\sigma_{pq}$. In the future we shall be less rigorous and we shall adopt the following sloppy convention: if we write, say, $\mathfrak{E}_{\mathrm{G}}B^\sigma_{pq}=(f(t),q)$ where f is a distinguished function (maybe not continuous or not monotone), then we tacitly assert that f is equivalent to some $\mathcal{E}_{\mathrm{G}}A^\sigma_{pq}$ in some neighbourhood $(0,\varepsilon)$ of zero.

(iv) If H is as in the above definition, the corresponding Borel measure μ_H is defined on $[a,b]\subset(0,\varepsilon]$ as $\mu_H([a,b])=H(b)-H(a)$ and then prolonged in the usual way on the σ-field of all Borel sets. If H is continuously differentiable, then $\frac{d\mu_H}{dt}=H'(t)$. We shall use this below in (6.15).

6. Results

Now we are ready for the main results of this paper.

Proposition 6.1. *Let* $0<p,q\le\infty$ *and* σ *be an admissible sequence with*

$$n\big(\tfrac{1}{p}-1\big)_+<\beta_\sigma\le\alpha_\sigma<\tfrac{n}{p}. \tag{6.1}$$

Let $\Sigma\in\mathbb{B}$ *with* $\Sigma(2^j)\sim\sigma_j$ $(j\in\mathbb{N}_0)$ *(for instance, the function defined in* (3.7)*). Then*

$$\mathcal{E}_{\mathrm{G}}|B^\sigma_{pq}(t)\le c\,t^{-\frac{1}{p}}\Sigma(t^{\frac{1}{n}})\qquad(t\in(0,1]) \tag{6.2}$$

and, for each $v\in(q,\infty]$*, there exists a constant* $c(v)>0$ *such that*

$$\Big(\int_0^1\big(t^{\frac{1}{p}}\Sigma(t^{-\frac{1}{n}})\,f^*(t)\big)^v\frac{dt}{t}\Big)^{\frac{1}{v}}\le c\,\|f|B^\sigma_{pq}\| \tag{6.3}$$

(with modification (6.9) *if* $v=\infty$*) for all* $f\in B^\sigma_{pq}$.

Proof. Thanks to (6.1) it is clearly possible to choose two real numbers s_1 and s_2 such that

$$n\big(\tfrac{1}{p}-1\big)_+ < s_1 < \beta_\sigma \le \alpha_\sigma < s_2 < \tfrac{n}{p}. \tag{6.4}$$

Let us consider the function g defined by

$$g(t) = t^{-\frac{s_1}{s_2-s_1}}\,\Sigma(t^{\frac{1}{s_2-s_1}}) \qquad (t>0).$$

By straightforward calculations, it follows that

$$\overline{g}(t) = \sup_{s>0}\frac{g(ts)}{g(s)} = t^{-\frac{s_1}{s_2-s_1}}\,\overline{\Sigma}(t^{\frac{1}{s_2-s_1}}) \qquad (t>0)$$

and hence, g belongs to the class $\mathbb{B}$. Moreover, taking into account (6.4) and Proposition 3.6,

$$\alpha_g = -\frac{s_1}{s_2-s_1} + \frac{\alpha_\Sigma}{s_2-s_1} = \frac{-s_1+\alpha_\sigma}{s_2-s_1} < 1$$
$$\beta_g = -\frac{s_1}{s_2-s_1} + \frac{\beta_\Sigma}{s_2-s_1} = \frac{-s_1+\beta_\sigma}{s_2-s_1} > 0.$$

Thus, we can apply [16: Theorem 13] complemented by [6: Remark 5.4] to conclude that, for each $v\in(0,\infty]$,

$$B^\sigma_{pv} = (B^{s_1}_{p1}, B^{s_2}_{p1})_{g,v} \tag{6.5}$$

where in the right-hand side is an interpolation space with a function parameter.

On the other hand, if r_1, r_2 are defined by the equations $s_i - \frac{n}{p} = -\frac{n}{r_i}$ $(i=1,2)$, then $r_i\in(1,\infty)$ and $r_i>p$, and

$$B^{s_i}_{p1} \hookrightarrow L_{r_i} \qquad (i=1,2) \tag{6.6}$$

(cf., e.g., [19: 11.4/(iii) and 10.5/(i)]). By virtue of [16: Theorem 3] we get

$$(L_{r_1}, L_{r_2})_{g,v} = L_v(\varphi) \tag{6.7}$$

where

$$\varphi(t) := t^{\frac{1}{r_1}}\,g\big(t^{\frac{1}{r_1}-\frac{1}{r_2}}\big)^{-1} = t^{\frac{1}{p}}\,\Sigma(t^{\frac{1}{n}})^{-1} \qquad (t>0).$$

In view of the interpolation property, (6.5) - (6.7) imply $B^\sigma_{pv}\hookrightarrow L_v(\varphi)$, that is, there exists a constant $c>0$ such that

$$\bigg(\int_0^\infty \big(t^{\frac{1}{p}}\,\Sigma(t^{\frac{1}{n}})^{-1}\,f^*(t)\big)^v\frac{dt}{t}\bigg)^{\frac{1}{v}} \le c\,\|f|B^\sigma_{pv}\|$$

for all $f\in B^\sigma_{pv}$. If $v\ge q$, then $B^\sigma_{pq}\hookrightarrow B^\sigma_{pv}$, and hence

$$\bigg(\int_0^\infty \big(t^{\frac{1}{p}}\,\Sigma(t^{-\frac{1}{n}})\,f^*(t)\big)^v\frac{dt}{t}\bigg)^{\frac{1}{v}} \le c\,\|f|B^\sigma_{pq}\| \tag{6.8}$$

for some constant c and for all $f\in B^\sigma_{pq}$. This proves (6.3). As for (6.2), this follows easily by considering just the particular case of $v=\infty$ in (6.8), that is,

$$\sup_{t\in(0,1]} t^{\frac{1}{p}}\,\Sigma(t^{-\frac{1}{n}})\,f^*(t) \le c\,\|f|B^\sigma_{pq}\| \tag{6.9}$$

for all $f\in B^\sigma_{pq}$ ∎

Proposition 6.2. *Let $0 < p, q \leq \infty$ and let σ be an admissible sequence with property* (6.1)*. Consider an interpolating function Σ (say, as in* (3.7)*). Then there exist $\varepsilon \in (0,1)$ and a constant $c > 0$ such that*

$$\mathcal{E}_{\mathrm{G}}|B^{\sigma}_{pq}(t) \geq c\, t^{-\frac{1}{p}}\, \Sigma(t^{\frac{1}{n}}) \qquad (t \in (0, \varepsilon]), \tag{6.10}$$

and for each $v \in (0, q)$ there is no $c(v) > 0$ such that

$$\left(\int_0^1 \left(t^{\frac{1}{p}}\, \Sigma(t^{-\frac{1}{n}})\, f^*(t) \right)^v \frac{dt}{t} \right)^{\frac{1}{v}} \leq c\, \|f|B^{\sigma}_{pq}\|$$

for all $f \in B^{\sigma}_{pq}$.

Proof. We follow closely the arguments of [20: 15.2] with appropriate modifications (see also the proof of [5: Proposition 4.2]).

For each $j \in \mathbb{N}$, let A_j be given by

$$A_j(x) = \sigma_j^{-1} 2^{j\frac{n}{p}}\, \Phi(2^j x) \qquad (x \in \mathbb{R}^n)$$

where Φ is defined by

$$\Phi(x) = \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Since $\beta_\sigma > n(\frac{1}{p} - 1)_+$, A_j is (up to a constant factor) an atom for the space B^{σ}_{pq}. It can be easily seen that

$$A_j^*(d 2^{-jn}) \sim \sigma_j^{-1} 2^{j\frac{n}{p}} \qquad (j \in \mathbb{N})$$

for some $d > 0$ depending only on the function Φ. By (5.2) and the atomic decomposition theorem (cf. [10: Theorem 4.4.3]) we infer that

$$\mathcal{E}_{\mathrm{G}}|B^{\sigma}_{pq}(d 2^{-jn}) \geq c_1 A_j^*(d 2^{-jn}) \geq c_2\, \sigma_j^{-1} 2^{j\frac{n}{p}} \qquad (j \in \mathbb{N})$$

which implies (6.10).

Assume now that for some $v \in (0, q)$ there exists a constant $c(v) > 0$ such that

$$\left(\int_0^1 \left(t^{\frac{1}{p}}\, \Sigma(t^{-\frac{1}{n}})\, f^*(t) \right)^v \frac{dt}{t} \right)^{\frac{1}{v}} \leq c\, \|f|B^{\sigma}_{pq}\| \tag{6.11}$$

for all $f \in B^{\sigma}_{pq}$. For each $J \in \mathbb{N}$, set

$$f_J(x) = \sum_{j=1}^{J} \sigma_j^{-1} 2^{j\frac{n}{p}}\, \Phi(2^j x - x^0) \qquad (x \in \mathbb{R}^n)$$

with Φ as above and $x^0 \in \mathbb{Z}^n$ chosen in such a way that the supports of the functions $\Phi(2^j \cdot - x^0)$ $(j \in \mathbb{N})$ are disjoint (it is enough to take x^0 with $|x^0| > 3$). For $k \in \{1, \ldots, J\}$ one has, for some $a, b > 0$ depending only on Φ,

$$m_{f_J}(b\, \sigma_k^{-1} 2^{k\frac{n}{p}}) \geq 2a 2^{-kn}.$$

Therefore,

$$f_J^*(a2^{-kn}) \geq b\,\sigma_k^{-1} 2^{k\frac{n}{p}} \qquad (k = 1, \dots, J). \tag{6.12}$$

We remark that, by the atomic decomposition theorem, f_J belongs to B^σ_{pq} and, moreover,

$$\|f_J|B^\sigma_{pq}\| \leq c\Big(\sum_{j=1}^{J} 1\Big)^{\frac{1}{q}} = c\,J^{\frac{1}{q}} \qquad (J \in \mathbb{N}). \tag{6.13}$$

Let $k_0 \in \mathbb{N}$ be such that $a2^{-k_0 n} \leq \varepsilon$. Inserting (6.13) and (6.12) into (6.11) and using the monotonicity of f_J^*, we obtain, for any $J \geq k_0$,

$$\begin{aligned}
J^{\frac{1}{q}} &\geq c_1 \bigg(\sum_{k=k_0}^{\infty} \int_{a2^{-(k+1)n}}^{a2^{-kn}} \big(t^{\frac{1}{p}}\, \Sigma(t^{-\frac{1}{n}})\, f_J^*(t)\big)^v \,\frac{dt}{t}\bigg)^{\frac{1}{v}}\\
&\geq c_2 \bigg(\sum_{k=k_0}^{J} f_J^*(a2^{-kn})^v\, \Sigma(2^k)^v \int_{a2^{-(k+1)n}}^{a2^{-kn}} t^{\frac{v}{p}-1}\,dt\bigg)^{\frac{1}{v}}\\
&\geq c_3\,(J - k_0 + 1)^{\frac{1}{v}}
\end{aligned}$$

where the constants involved are independent of J. But this is clearly impossible as $v \in (0, q)$ ∎

Theorem 6.3. *Let $0 < p < \infty$, $0 < q \leq \infty$ and σ be an admissible sequence with $n(\frac{1}{p} - 1)_+ < \beta_\sigma \leq \alpha_\sigma < \frac{n}{p}$. Consider a corresponding interpolating function Σ (say, as in* (3.7)*). Then:*

(i) $\mathfrak{E}_{\mathrm{G}} B^\sigma_{pq} = (t^{-\frac{1}{p}}\, \Sigma(t^{\frac{1}{n}}), q)$.

(ii) $\mathfrak{E}_{\mathrm{G}} F^\sigma_{pq} = (t^{-\frac{1}{p}}\, \Sigma(t^{\frac{1}{n}}), p)$.

Proof. *Step* 1: Consider the function

$$g(t) = t^{\frac{1}{p}}\, \Sigma(t^{\frac{1}{n}})^{-1} \qquad (t \geq 0).$$

By virtue of Proposition 3.6/(ii) it can be easily seen that the above function g belongs to the class $\mathcal{B}$ and, moreover, for its lower Boyd index we have

$$\beta_g = \frac{1}{p} - \frac{\beta_\Sigma}{n} = \frac{1}{p} - \frac{\beta_\sigma}{n} > 0.$$

Using a result due to Merucci, namely [16: Proposition 4], we can guarantee the existence of an increasing function $h \in \mathcal{B}$ with $h \sim g$ which is a C^1-diffeomorphism of $(0, \infty)$ and satisfies

$$0 < \inf_{t>0} t\,\frac{h'(t)}{h(t)} \leq \sup_{t>0} t\,\frac{h'(t)}{h(t)} < \infty.$$

In particular,

$$\frac{h'(t)}{h(t)} \sim \frac{1}{t} \qquad (t \geq 0) \tag{6.14}$$

and $h(t)^{-1} \sim t^{-\frac{1}{p}}\,\Sigma(t^{\frac{1}{n}})$ in $(0,\varepsilon]$, for $\varepsilon \in (0,1)$ according to Proposition 6.2. By (6.2) and (6.10) we conclude that $h(t)^{-1}$ is a continuous representative in the class $[\mathcal{E}_{\mathrm{G}}|B^\sigma_{pq}]$. We set $H(t) = \log h(t)$ in $(0,\varepsilon]$ and consider μ_H – the Borel measure associated to H. Since h is continuously differentiable in $(0,\varepsilon]$, we have

$$\mu_H(dt) = \frac{h'(t)}{h(t)}\,dt. \tag{6.15}$$

Taking into account (6.14) and (6.15), Propositions 6.1 and 6.2 assert that, for a given $v \in (0,\infty]$, there exists a constant $c(v) > 0$ such that

$$\left(\int_{(0,\varepsilon]} \big(h(t)\,f^*(t)\big)^v \mu_H(dt)\right)^{\frac{1}{v}} \le c\,\|f|B^\sigma_{pq}\| \quad (f \in B^\sigma_{pq}) \quad \Longleftrightarrow \quad v \ge q.$$

This completes the proof of assertion (i).

Step 2: We choose p_1, p_2 such that $0 < p_1 < p < p_2 \le \infty$ and consider the sequences σ', σ'' defined by

$$\left.\begin{aligned} \sigma'_j &= 2^{n(\frac{1}{p_1}-\frac{1}{p})j}\sigma_j \\ \sigma''_j &= 2^{n(\frac{1}{p_2}-\frac{1}{p})j}\sigma_j \end{aligned}\right. \qquad (j \in \mathbb{N}_0).$$

Then, by Proposition 4.7,

$$B^{\sigma'}_{p_1p} \hookrightarrow F^\sigma_{pq} \hookrightarrow B^{\sigma''}_{p_2p}. \tag{6.16}$$

Due to part (i) proved above,

$$\begin{aligned} \mathcal{E}_{\mathrm{G}}|B^{\sigma'}_{p_1p}(t) &\sim t^{-\frac{1}{p_1}}\,\Sigma'(t^{\frac{1}{n}}) \sim t^{-\frac{1}{p}}\,\Sigma(t^{\frac{1}{n}}) \\ \mathcal{E}_{\mathrm{G}}|B^{\sigma''}_{p_2p}(t) &\sim t^{-\frac{1}{p_2}}\,\Sigma''(t^{\frac{1}{n}}) \sim t^{-\frac{1}{p}}\,\Sigma(t^{\frac{1}{n}}) \end{aligned}$$

in some interval $(0,\varepsilon]$, for some $\varepsilon \in (0,1)$. Then assertion (ii) follows from (6.16), thanks to Proposition 2.4/(iv) and [12: Proposition 3.5] ∎

Remark 6.4. The use of the interpolating function Σ in the above theorem is somehow immaterial. In fact, it was useful for technical reasons, in particular in connection to the interpolation with a function parameter in the proof of Proposition 6.1. Indeed, of interest is the singular behaviour of the growth envelope function in a neighbourhood of the origin and this can simply be described by $\mathcal{E}_{\mathrm{G}} A^\sigma_{pq}(2^{-jn}) \sim 2^{j\frac{n}{p}}\,\sigma_j^{-1}$ for any large enough $j \in \mathbb{N}_0$.

Remark 6.5. We compare our results with the known ones. For $A^{(s,\Psi)}_{pq}$, with Ψ an admissible function, which corresponds to the choice $\sigma = \{2^{sj}\Psi(2^{-j})\}_{j\in\mathbb{N}_0}$, then $\Sigma(t) \sim t^s\,\Psi(t)^{-1}$ $(t \in (0,1])$. According to the above theorem, for $0 < p,q \le \infty$ and $n(\frac{1}{p}-1)_+ < s < \frac{n}{p}$, we have

$$\begin{aligned} \mathfrak{E}_{\mathrm{G}} B^{(s,\Psi)}_{pq} &= \big(t^{-\frac{1}{r}}\,\Psi(t^{\frac{1}{n}})^{-1}, q\big) \\ \mathfrak{E}_{\mathrm{G}} F^{(s,\Psi)}_{pq} &= \big(t^{-\frac{1}{r}}\,\Psi(t^{\frac{1}{n}})^{-1}, p\big) \end{aligned}$$

where $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$. As for an admissible function Ψ, $\Psi(t^{\frac{1}{n}}) \sim \Psi(t)$ $(t \in (0,1])$ (cf. [5: Lemma 2.3]), we recover the results of A. Caetano and S. D. Moura in [5]. In particular, we also extended the results of D. Haroske and H. Triebel regarding the spaces A^s_{pq} in the sub-critical case (cf. [12, 13, 20]).

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