Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 2, 399–410

Fixed Points for Multi-Valued Mixed Increasing Operators in Ordered Banach Spaces with Applications to Integral Inclusions

Nan-jing Huang and Ya-ping Fang

Abstract. Some new fixed point and coupled fixed point theorems for multi-valued mixed increasing operators in ordered Banach spaces are presented in this paper. As applications, we prove the existences of solutions for a class of integral inclusions.

Keywords: *Ordered Banach spaces, multi-valued operators, mixed increasing operators, fixed points, coupled fixed points, integral inclusions*

AMS subject classification: 47H10, 47H07, 47H04, 54H25

1. Introduction and Preliminaries

Fixed point theorems for single-valued increasing operators and coupled fixed point theorems for single-valued mixed increasing operators in ordered Banach spaces are widely investigated and have found various applications to nonlinear integral equations and differential equations. For details, we can refer to [1, 3 - 12, 14, 16, 17, 20] and the references therein. However, only few results on fixed points of multi-valued increasing operators in ordered Banach spaces are obtained. In 1984, Nishnianidze [16] introduced monotone multi-valued operators and proved some fixed point theorems for such operators. Recently, Huy and Khash [11] gave some new fixed point theorems for multi-valued increasing operators in ordered Banach spaces by means of the concept of Nishnianidze [16].

Motivated and inspired by [11], in this paper, we introduce a class of multi-valued mixed increasing operators in Banach spaces and prove some new fixed point and coupled fixed point theorems. As applications, we discuss the existences of solutions for a class of integral inclusions.

Both authors: Sichuan Univ., Dept. Math., Chengdu, Sichuan 610064, P.R. China
nanjinghuang@hotmail.com
This work was supported by the National Natural Science Foundation of China (10171070) and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry.

ISSN 0232-2064 / $ 2.50

Let X be a real Banach space and K be a non-empty pointed closed convex cone in X. Recall that $K \subset X$ is called a pointed closed convex cone if K is closed and the following conditions hold:

(i) $K + K \subset K$

(ii) $tK \subset K$ for all $t \geq 0$

(iii) $K \cap (-K) = \{0\}$.

A partial ordering $\leq$ can be induced by K by

$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$

We always say that $(X, \leq)$ is an ordered Banach space induced by K. A pointed convex cone K is said to be normal if there exists some constant $N > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Definition 1.1. Let $(X, \leq)$ be an ordered Banach space induced by K and let A, B be two non-empty subsets of X. Then we denote

(i) $A \leq_1 B$ if for each $a \in A$ there exists $b \in B$ such that $a \leq b$

(ii) $A \leq_2 B$ if for each $b \in B$ there exists $a \in A$ such that $a \leq b$

(iii) $A \preceq B$ if $A \leq_1 B$ and $A \leq_2 B$

(iv) $A \ll B$ if $a \leq b$ for any $a \in A$ and $b \in B$.

Definition 1.2. Let $(X, \leq)$ be an ordered Banach space induced by K and let $L : X \to X$ be a mapping. We say that L is *positive* if $Lx \in K$ whenever $x \in K$.

Definition 1.3. Let M be a non-empty subset of an ordered Banach space $(X, \leq)$ and let $f : M \times M \to 2^X$ be a multi-valued operator. We say that f is *mixed increasing* if, for any $x_1, x_2, y_1, y_2 \in M$, $x_1 \leq x_2$ and $y_2 \leq y_1$ imply $f(x_1, y_1) \preceq f(x_2, y_2)$.

Definition 1.4. Let M be a non-empty subset of an ordered Banach space $(X, \leq)$ and let $f : M \times M \to 2^X$ be a multi-valued operator. We say that $(x^*, y^*) \in M \times M$ is a *coupled fixed point* of f if $x^* \in f(x^*, y^*)$ and $y^* \in f(y^*, x^*)$. We say that $x^* \in M$ is a *fixed point* of f if $x^* \in f(x^*, x^*)$.

2. Main Results

In this section, we prove some new fixed point and coupled fixed point theorems for multi-valued mixed increasing operators in ordered Banach spaces.

Theorem 2.1. *Let $(X, \leq)$ be an ordered Banach space induced by a pointed closed convex cone K and M be a non-empty closed subset of X. Suppose that $f : M \times M \to 2^M$ is a multi-valued operator satisfying the following conditions:*

(i) *$f(x, y)$ is closed for any $x, y \in M$.*

(ii) *There exist $x_0, y_0 \in M$ such that $\{x_0\} \leq_1 f(x_0, y_0)$ and $f(y_0, x_0) \leq_2 \{y_0\}$.*

(iii) *For any* $u_1, u_2, v_1, v_2 \in M$, $u_1 \le u_2$ *and* $v_2 \le v_1$ *imply*

$$f(u_1, v_1) \subset f(u_2, v_2) - K \cap B\big(r\|u_1 - u_2\| + s\|v_1 - v_2\|\big)$$
$$f(v_1, u_1) \subset f(v_2, u_2) + K \cap B\big(r\|v_1 - v_2\| + s\|u_1 - u_2\|\big)$$

where r, s *are two non-negative constants with* $r + s < 1$ *and* $B(l)$ *denotes the closed ball with radius* l *and center at origin.*

Then f *admits a coupled fixed point* $(x^*, y^*) \in M \times M$.

Proof. From condition (ii), there exist $x_1 \in f(x_0, y_0)$ and $y_1 \in f(y_0, x_0)$ such that $x_0 \le x_1$ and $y_1 \le y_0$. By condition (iii), we can choose $x_2 \in f(x_1, y_1)$ and $y_2 \in f(y_1, x_1)$ such that

$$x_1 \le x_2, \qquad \|x_2 - x_1\| \le r\|x_1 - x_0\| + s\|y_1 - y_0\|$$
$$y_2 \le y_1, \qquad \|y_2 - y_1\| \le r\|y_1 - y_0\| + s\|x_1 - x_0\|.$$

Repeating the arguments above for x_1, x_2, y_1, y_2 in place x_0, x_1, y_0, y_1 and so on, we can construct two sequences

$$\{x_n\}, \qquad x_n \in f(x_{n-1}, y_{n-1})$$
$$\{y_n\}, \qquad y_n \in f(y_{n-1}, x_{n-1})$$

such that

$$\begin{aligned} x_{n-1} \le x_n, &\qquad \|x_n - x_{n-1}\| \le r\|x_{n-1} - x_{n-2}\| + s\|y_{n-1} - y_{n-2}\| \\ y_n \le y_{n-1}, &\qquad \|y_n - y_{n-1}\| \le r\|y_{n-1} - y_{n-2}\| + s\|x_{n-1} - x_{n-2}\|. \end{aligned} \tag{2.1}$$

We claim that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (r+s)^n \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) \\ \|y_{n+1} - y_n\| &\le (r+s)^n \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) \end{aligned} \tag{2.2}$$

for all $n \ge 1$. In fact, for $n = 1$ it follows from (2.1) that

$$\|x_2 - x_1\| \le r\|x_1 - x_0\| + s\|y_1 - y_0\| \le (r+s)\big(\|x_1 - x_0\| + \|y_1 - y_0\|\big)$$
$$\|y_2 - y_1\| \le r\|y_1 - y_0\| + s\|x_1 - x_0\| \le (r+s)\big(\|x_1 - x_0\| + \|y_1 - y_0\|\big).$$

Suppose that (2.2) holds for $n = k$ (≥ 1). For $n = k + 1$ it follows from (2.1) that

$$\begin{aligned} &\|x_{k+2} - x_{k+1}\| \\ &\quad \le r\|x_{k+1} - x_k\| + s\|y_{k+1} - y_k\| \\ &\quad \le r(r+s)^k \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) + s(r+s)^k \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) \\ &\quad = (r+s)^{k+1} \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) \end{aligned}$$

$$\begin{aligned} &\|y_{k+2} - y_{k+1}\| \\ &\quad \le r\|y_{k+1} - y_k\| + s\|x_{k+1} - x_k\| \\ &\quad \le r(r+s)^k \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) + s(r+s)^k \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big) \\ &\quad = (r+s)^{k+1} \big(\|x_1 - x_0\| + \|y_1 - y_0\|\big). \end{aligned}$$

By induction, we can conclude that (2.2) holds for all $n \geq 1$. Since $0 \leq r+s < 1$, from (2.2) we know that $\{x_n\}, \{y_n\}$ are two Cauchy sequences. Let $x_n \to x^*$ and $y_n \to y^*$. Obviously, $(x^*, y^*) \in M \times M$ since M is closed. Further, $x_n \leq x^*$ and $y^* \leq y_n$ for all n since K is closed. Again from condition (iii), we can choose $x^*_{n+1} \in f(x^*, y^*)$ and $y^*_{n+1} \in f(y^*, x^*)$ such that

$$\begin{aligned} \|x^*_{n+1} - x_{n+1}\| &\leq r\|x_n - x^*\| + s\|y_n - y^*\| \\ \|y^*_{n+1} - y_{n+1}\| &\leq r\|y_n - y^*\| + s\|x_n - x^*\|. \end{aligned} \tag{2.3}$$

Since $x_n \to x^*$ and $y_n \to y^*$, it follows from (2.3) that $x^*_n \to x^*$ and $y^*_n \to y^*$. By condition (i), we know that $x^* \in f(x^*, y^*)$ and $y^* \in f(y^*, x^*)$. The proof is complete ∎

Theorem 2.2. *Let $(X, \leq)$ be an ordered Banach space induced by a pointed closed convex cone K and M be a non-empty closed subset of X. Suppose that $f : M \times M \to C(M)$ (the family of all non-empty compact subsets of M) is a multi-valued operator satisfying the following conditions:*

- **(i)** *There exist $x_0, y_0 \in M$ such that $\{x_0\} \leq_1 f(x_0, y_0)$ and $f(y_0, x_0) \leq_2 \{y_0\}$.*
- **(ii)** *For any $u_1, u_2, v_1, v_2 \in M$, $u_1 \leq u_2$ and $v_2 \leq v_1$ imply $f(u_1, v_1) \ll f(u_2, v_2)$.*
- **(iii)** *For any fixed $u \in M$, $x \leq y$ implies*

$$\begin{aligned} H\big(f(x,u), f(y,u)\big) &\leq r\|x - y\| \\ H\big(f(u,x), f(u,y)\big) &\leq s\|x - y\| \end{aligned}$$

where r, s are two non-negative constants with $r + s < 1$ and $H(\cdot, \cdot)$ is the Hausdorff metric on $C(M)$.

Then f admits a coupled fixed point $(x^*, y^*) \in M \times M$.

Proof. From condition (i), there exist $x_1 \in f(x_0, y_0)$ and $y_1 \in f(y_0, x_0)$ such that $x_0 \leq x_1$ and $y_1 \leq y_0$. By Nadler [15], we can choose $x'_1 \in f(x_1, y_0)$ and $y'_1 \in f(y_1, x_0)$ such that

$$\begin{aligned} \|x_1 - x'_1\| &\leq H\big(f(x_0, y_0), f(x_1, y_0)\big) \\ \|y_1 - y'_1\| &\leq H\big(f(y_0, x_0), f(y_1, x_0)\big). \end{aligned} \tag{2.4}$$

Since $x'_1 \in f(x_1, y_0)$ and $y'_1 \in f(y_1, x_0)$, again from Nadler [15] we can choose $x_2 \in f(x_1, y_1)$ and $y_2 \in f(y_1, x_1)$ such that

$$\begin{aligned} \|x_2 - x'_1\| &\leq H\big(f(x_1, y_1), f(x_1, y_0)\big) \\ \|y_2 - y'_1\| &\leq H\big(f(y_1, x_1), f(y_1, x_0)\big). \end{aligned} \tag{2.5}$$

It follows from (2.4) - (2.5) and condition (iii) that

$$\begin{aligned} \|x_2 - x_1\| &\leq \|x_2 - x'_1\| + \|x'_1 - x_1\| \\ &\leq H\big(f(x_1, y_1), f(x_1, y_0)\big) + H\big(f(x_0, y_0), f(x_1, y_0)\big) \\ &\leq r\|x_1 - x_0\| + s\|y_1 - y_0\| \end{aligned}$$

$$\begin{aligned} \|y_2 - y_1\| &\leq \|y_2 - y'_1\| + \|y'_1 - y_1\| \\ &\leq H\big(f(y_1, x_1), f(y_1, x_0)\big) + H\big(f(y_0, x_0), f(y_1, x_0)\big) \\ &\leq s\|x_1 - x_0\| + r\|y_1 - y_0\|. \end{aligned}$$

Furthermore, by condition (ii) we know that $x_1 \le x_2$ and $y_2 \le y_1$. Repeating the arguments above for x_1, x_2, y_1, y_2 in place x_0, x_1, y_0, y_1 and so on, we can construct two sequences

$$\{x_n\}, \quad x_n \in f(x_{n-1}, y_{n-1})$$
$$\{y_n\}, \quad y_n \in f(y_{n-1}, x_{n-1})$$

such that

$$\begin{aligned} x_{n-1} \le x_n, &\qquad \|x_n - x_{n-1}\| \le r\|x_{n-1} - x_{n-2}\| + s\|y_{n-1} - y_{n-2}\| \\ y_n \le x_{n-1}, &\qquad \|y_n - y_{n-1}\| \le r\|y_{n-1} - y_{n-2}\| + s\|x_{n-1} - x_{n-2}\|. \end{aligned} \tag{2.6}$$

The rest of proof now follows as in Theorem 2.1 and is therefore omitted ∎

Theorem 2.3. *Let $(X, \le)$ be an ordered Banach space induced by a pointed closed convex normal cone K with normal constant $N > 0$ and M be a non-empty closed subset of X. Suppose that $f : M \times M \to 2^M$ be a multi-valued operator satisfying the following conditions:*

(i) *$f(x, y)$ is closed for any $x, y \in M$.*

(ii) *There exist $x_0, y_0 \in M$ such that $\{x_0\} \le_1 f(x_0, y_0)$ and $f(y_0, x_0) \le_2 \{y_0\}$.*

(iii) *There exist two positive linear operators $L, S : X \to X$ with $r(S + L) < 1$ such that, for any $u_1, u_2, v_1, v_2 \in M$, $u_1 \le u_2$ and $v_2 \le v_1$ imply:*

(a) *for any $x_1 \in f(u_1, v_1)$, there exists $x_2 \in f(u_2, v_2)$ satisfying $0 \le x_2 - x_1 \le L(u_2 - u_1) + S(v_1 - v_2)$*

(b) *for any $y_1 \in f(v_1, u_1)$, there exists $y_2 \in f(v_2, u_2)$ satisfying $0 \le y_1 - y_2 \le L(v_1 - v_2) + S(u_2 - u_1)$*

where $r(S + L)$ denotes the spectral radius of $S + L$.

Then f admits a coupled fixed point $(x^*, y^*) \in M \times M$.

Proof. From condition (ii), there exist $x_1 \in f(x_0, y_0)$ and $y_1 \in f(y_0, x_0)$ such that $x_0 \le x_1$ and $y_1 \le y_0$. By condition (iii), we can choose $x_2 \in f(x_1, y_1)$ and $y_2 \in f(y_1, x_1)$ such that

$$0 \le x_2 - x_1 \le L(x_1 - x_0) + S(y_0 - y_1)$$
$$0 \le y_1 - y_2 \le L(y_0 - y_1) + S(x_1 - x_0).$$

Repeating the arguments above for x_1, x_2, y_1, y_2 in place x_0, x_1, y_0, y_1 and so on, we can construct two sequences

$$\{x_n\}, \quad x_n \in f(x_{n-1}, y_{n-1})$$
$$\{y_n\}, \quad y_n \in f(y_{n-1}, x_{n-1})$$

such that

$$\begin{aligned} 0 \le x_n - x_{n-1} \le L(x_{n-1} - x_{n-2}) + S(y_{n-2} - y_{n-1}) \\ 0 \le y_{n-1} - y_n \le L(y_{n-2} - y_{n-1}) + S(x_{n-1} - x_{n-2}). \end{aligned} \tag{2.6}$$

We claim that

$$\begin{aligned} 0 \le x_{n+1} - x_n \le (L + S)^n (x_1 - x_0 + y_0 - y_1) \\ 0 \le y_n - y_{n+1} \le (L + S)^n (x_1 - x_0 + y_0 - y_1) \end{aligned} \tag{2.7}$$

for all $n \geq 1$. In fact, for $n = 1$ it follows from (2.6) that

$$0 \leq x_2 - x_1 \leq L(x_1 - x_0) + S(y_0 - y_1) \leq (L+S)(x_1 - x_0 + y_0 - y_1)$$
$$0 \leq y_1 - y_2 \leq L(y_0 - y_1) + S(x_1 - x_0) \leq (L+S)(x_1 - x_0 + y_0 - y_1).$$

Suppose that (2.7) holds for $n = k \quad (k \geq 1)$. For $n = k+1$ it follows from (2.6) that

$$\begin{aligned} 0 &\leq x_{k+2} - x_{k+1} \\ &\leq L(x_{k+1} - x_k) + S(y_k - y_{k+1}) \\ &\leq L\big[(L+S)^k(x_1 - x_0 + y_0 - y_1)\big] + S\big[(L+S)^k(x_1 - x_0 + y_0 - y_1)\big] \\ &= (L+S)^{k+1}(x_1 - x_0 + y_0 - y_1). \end{aligned}$$

By induction, we can conclude that (2.7) holds for all $n \geq 1$. Since K is normal, it follows from (2.7) that

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq N\|(L+S)^n\| \, \|x_1 - x_0 + y_0 - y_1\| \\ \|y_{n+1} - y_n\| &\leq N\|(L+S)^n\| \, \|x_1 - x_0 + y_0 - y_1\|. \end{aligned} \tag{2.8}$$

Since $\lim_{n\to\infty} \|(L+S)^n\| = r(S+L) < 1$, we have

$$\|(L+S)^n\| \leq q^n \tag{2.9}$$

for some $q \in (0,1)$ and n large enough. It follows from (2.8) and (2.9) that

$$\|x_{n+1} - x_n\| \leq Nq^n\|x_1 - x_0 + y_0 - y_1\|$$
$$\|y_{n+1} - y_n\| \leq Nq^n\|x_1 - x_0 + y_0 - y_1\|$$

which implies that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences. Let $x_n \to x^*$ and $y_n \to y^*$. It is easy to see that $x_n \leq x^*$ and $y^* \leq y_n$ for all n since K is closed. Since $x_{n+1} \in f(x_n, y_n)$ and $y_{n+1} \in f(y_n, x_n)$, by condition (iii) we can choose $x^*_{n+1} \in f(x^*, y^*)$ and $y^*_{n+1} \in f(y^*, x^*)$ such that

$$0 \leq x^* - x^*_{n+1} \leq L(x^* - x_n) + S(y_n - y^*)$$
$$0 \leq y^*_{n+1} - y^* \leq L(y_n - y^*) + S(x^* - x_n).$$

Since K is normal, it follows that

$$\|x^*_{n+1} - x^*\| \leq \|L\| \, \|x_n - x^*\| + \|S\| \, \|y_n - y^*\|$$
$$\|y^*_{n+1} - y^*\| \leq \|L\| \, \|y_n - y^*\| + \|S\| \, \|x_n - x^*\|.$$

This implies that $x^*_n \to x^*$ and $y^*_n \to y^*$. By condition (i) we know that $x^* \in f(x^*, y^*)$ and $y^* \in f(y^*, x^*)$. The proof is complete ∎

Theorem 2.4. *Let $(X, \leq)$ be an ordered Banach space induced by a pointed closed convex normal cone K with normal constant $N > 0$, and let $x_0, y_0 \in X$ with $x_0 \leq y_0$. Denote*

$$D = [x_0, y_0] = \{x \in X : x_0 \leq x \leq y_0\}$$

and let $f : D \times D \to 2^X$ be a multi-valued mixed increasing operator satisfying the following conditions:

(i) *$f(x, y)$ is closed for any $x, y \in M$.*

(ii) *$\{x_0\} \leq_1 f(x_0, y_0)$ and $f(y_0, x_0) \leq_2 \{y_0\}$.*

(iii) *There exists a positive linear operator $L : X \to X$ with $r(L) < 1$ such that, for any $x, y \in D$, $x \leq y$ implies $0 \leq v - u \leq L(y - x)$ for any $u \in f(x, y)$ and any $v \in f(y, x)$.*

Then there exists $x^* \in D$ such that $\{x^*\} = f(x^*, x^*)$.

Proof. First we show that $f(x, x)$ is single-valued for each $x \in D$. Indeed, since K is normal, it follows from condition (iii) that

$$\|u - v\| \leq N\|L\| \, \|x - x\| = 0 \qquad (u, v \in f(x, x))$$

which implies that $f(x, x)$ is single-valued for every $x \in D$. From condition (ii), there exist $x_1 \in f(x_0, y_0)$ and $y_1 \in f(y_0, x_0)$ such that $x_0 \leq x_1$ and $y_1 \leq y_0$. Further, since $x_0 \leq y_0$, it follows from condition (iii) that $x_0 \leq x_1 \leq y_1 \leq y_0$. Since f is mixed increasing, we can choose $x_2 \in f(x_1, y_1)$ and $y_2 \in f(y_1, x_1)$ such that $x_1 \leq x_2$ and $y_2 \leq y_1$. Again from condition (iii), we know that $x_1 \leq x_2 \leq y_2 \leq y_1$. Repeating the arguments above for x_1, x_2, y_1, y_2 in place x_0, x_1, y_0, y_1 and so on, we can construct two sequences

$$\begin{matrix} \{x_n\}, & x_{n+1} \in f(x_n, y_n), \\ \{y_n\}, & y_{n+1} \in f(y_n, x_n), \end{matrix} \quad x_n \leq x_{n+1} \leq y_{n+1} \leq y_n.$$

From here and condition (iii) we have

$$0 \leq x_{n+1} - x_n \leq y_n - x_n \leq L(y_{n-1} - x_{n-1}) \leq L^n(y_0 - x_0)$$
$$0 \leq y_n - y_{n+1} \leq y_n - x_n \leq L(y_{n-1} - x_{n-1}) \leq L^n(y_0 - x_0).$$

Since K is normal, we now have

$$\left.\begin{array}{r} \|x_{n+1} - x_n\| \\ \|y_{n+1} - y_n\| \\ \|x_n - y_n\| \end{array}\right\} \leq N\|L\|^n \, \|y_0 - x_0\|.$$

Since $r(L) < 1$, it follows that both $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences with the same limit. Let $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$. It is easy to see that $x_n \leq x^* \leq y_n$ for all $n \geq 0$. Since f is mixed increasing, we can choose $x^*_{n+1} \in f(x^*, x^*)$ and $y'_{n+1} \in f(y_n, x_n)$ such that $x_{n+1} \leq x^*_{n+1} \leq y'_{n+1}$, i.e.

$$0 \leq x^*_{n+1} - x_{n+1} \leq y'_{n+1} - x_{n+1}.$$

This and condition (iii) imply

$$\|x^*_{n+1} - x_{n+1}\| \leq N\|y'_{n+1} - x_{n+1}\| \leq N\|L\| \, \|y_n - x_n\|.$$

Since $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x^*$, then $x^*_n \to x^*$. It follows from condition (i) that $\{x^*\} = f(x^*, x^*)$. The proof is complete ∎

3. Applications

Let $(E,\leq)$ be a real ordered separable Banach space induced by a pointed closed convex normal cone P with normal constant $N>0$, let $C([0,1],E)=\{u:[0,1]\to E|\text{ u is continuous}\}$ and $P_c=\{u\in C([0,1],E):u(t)\geq 0\ (t\in[0,1])\}$. For each $u\in C([0,1],E)$ we define $\|u\|_c=\max_{t\in[0,1]}\|u(t)\|$. Then $C([0,1],E)$ is a real Banach space with norm $\|\cdot\|_c$ and P_c is a pointed closed convex normal cone with normal constant N. In this section, we also denote by $\leq$ the order induced by P_c.

Let (Ω,Σ) be a measurable space and X be a non-empty subset of E. We will use the notations

$$\begin{aligned} P_f(X)&=\big\{A\subseteq X:\ A\text{ non-empty, closed}\big\}\\ P_{kc}(X)&=\big\{A\subseteq X:\ A\text{ non-empty, compact, convex}\big\}. \end{aligned}$$

A multi-valued mapping $F:\Omega\to P_f(X)$ is said to be measurable if, for every $x\in X$,

$$\omega\to d(x,F(\omega))=\inf_{z\in F(\omega)}\|x-z\|$$

is measurable.

In the following, we always suppose that $x\in C([0,1],E)$, $k:[0,1]\times[0,1]\to(-\infty,+\infty)$ is a non-negative continuous function, and $f:[0,1]\times E\times E\to 2^E$ is a multi-valued operator.

Theorem 3.1. *Assume that the following conditions hold:*

(C1) $f:[0,1]\times E\times E\to 2^E$ *is a multi-valued operator such that*

(a) $f(\cdot,\cdot,\cdot)$ *has values in* $P_{kc}(E)$

(b) *for each* $u,v\in C(I,E)$, $t\mapsto f(t,u(t),v(t))$ *is measurable*

(c) *for each* $t\in I$ *and* $u,v\in C(I,E)$, $\sup_{x\in f(\cdot,u(\cdot),v(\cdot))}\|x\|\in L^1_+$.

(C2) *There exist* $u_0,v_0\in C([0,1],E)$ *such that*

$$\begin{aligned} \{u_0(t)\}&\leq_1 x(t)+\int_0^t k(t,s)f\big(s,u_0(s),v_0(s)\big)ds\\ \{v_0(t)\}&\geq_2 x(t)+\int_0^t k(t,s)f\big(s,v_0(s),u_0(s)\big)ds \end{aligned}$$

(C3) *There exist two non-negative constants* L',S' *such that, for any* $u_1,u_2,v_1,v_2\in C([0,1],E)$, $u_1\leq u_2$ *and* $v_2\leq v_1$ *imply*

(a) *for any* $x_1(t)\in\int_0^t k(t,s)f\big(s,u_1(s),v_1(s)\big)ds$, *there exists* $x_2(t)\in\int_0^t k(t,s)\times f\big(s,u_2(s),v_2(s)\big)ds$ *such that*

$$\begin{aligned} 0&\leq x_2(t)-x_1(t)\\ &\leq\int_0^t L'k(t,s)\big(u_2(s)-u_1(s)\big)ds+\int_0^t S'k(t,s)\big(v_1(s)-v_2(s)\big)ds \end{aligned}$$

(b) *for any* $y_1(t) \in \int_0^t k(t,s)f\big(s, v_1(s), u_1(s)\big)ds$, *there exists* $y_2(t) \in \int_0^t k(t,s) \times f\big(s, v_2(s), u_2(s)\big)ds$ *such that*

$$\begin{aligned} 0 &\le y_1(t) - y_2(t) \\ &\le \int_0^t L'k(t,s)\big(v_1(s) - v_2(s)\big)ds + \int_0^t S'k(t,s)\big(u_2(s) - u_1(s)\big)ds. \end{aligned}$$

(C4) *There exists a constant* $K \ge 0$ *such that* $K(L'+S') < 1$ *and* $\int_0^t k(t,s)\,ds \le K$. *Then there exist* $u^*, v^* \in C(I,E)$ *such that*

$$\begin{aligned} u^*(t) &\in x(t) + \int_0^t k(t,s)f\big(s, u^*(s), v^*(s)\big)ds \\ v^*(t) &\in x(t) + \int_0^t k(t,s)f\big(s, v^*(s), u^*(s)\big)ds. \end{aligned}$$

Proof. Define $F : C([0,1],E) \times C([0,1],E) \to 2^{C([0,1],E)}$ as

$$F(u,v)(t) = x(t) + \int_0^t k(t,s)f(s,u(s),v(s))\,ds \qquad (u,v \in C([0,1],E). \tag{3.1}$$

From condition (C1) we know that F has non-empty values. Because of the Rädstrom embedding theorem (see Klein and Thompson [13]), it is easy to see that

$$\int_0^t k(t,s)f(s,u(s),v(s))\,ds \in P_{kc}(E) \qquad (t \in [0,1]).$$

So a straightforward application of the Arzela and Ascoli theorem tells us that F has values in $P_{kc}(C[[0,1],E])$. It follows from condition (C2) and (3.1) that $\{u_0\} \le_1 F(u_0,v_0)$ and $F(v_0,u_0) \le_2 \{v_0\}$. We now define $L, S : C([0,1],E) \to C([0,1],E)$ by

$$\begin{aligned} Lu(t) &= \int_0^t L'k(t,s)u(s)\,ds \\ Su(t) &= \int_0^t S'k(t,s)u(s)\,ds. \end{aligned}$$

From here and conditions (C3) - (C4), it is easy to see that condition (iii) of Theorem 2.3 holds for F. Thus, by Theorem 2.3, there exist $u^*, v^* \in C([0,1],E)$ such that

$$\begin{aligned} u^*(t) &\in x(t) + \int_0^t k(t,s)f(s,u^*(s),v^*(s))\,ds \\ v^*(t) &\in x(t) + \int_0^t k(t,s)f(s,v^*(s),u^*(s))\,ds. \end{aligned}$$

The proof is complete ∎

Remark 3.1. If $\dim E < \infty$, then condition (C4) of Theorem 3.1 can be relaxed by requiring only $KS' < 1$. In fact, L in the proof of Theorem 3.1 is a compact Volterra operator, and so the operator $S+T$ has the same spectrum as S by [2: Theorem 2.3]; in particular, $r(S+T) = r(S)$. Using this fact and $KS' < 1$, we know that $r(S+T) < 1$.

Remark 3.2. If $f : [0,1] \times E \times E \to P_{kc}(E)$ is a multi-valued operator such that, for all $u, v \in C([0,1], E)$, $t \mapsto f(t, u(t), v(t))$ is integrably bounded (see, for example, [12] or [19]), then condition (C1) of Theorem 3.1 holds. If $f : [0,1] \times E \times E \to E$ is a single-valued operator satisfying the Carathéodory condition, then condition (C1) of Theorem 3.1 can be satisfied.

Theorem 3.2. *Let $u_0, v_0 \in C([0,1], E)$ with $u_0 \le v_0$, let $D = [u_0, v_0] = \{u \in C([0.1], E) : u_0 \le u \le v_0\}$ and let $f : [0,1] \times E \times E \to 2^E$ be a mixed increasing operator satisfying the following conditions:*

(C1) (a) *$f(\cdot,\cdot,\cdot)$ has values in $P_{kc}(E)$*

(b) *for each $u, v \in D$, $t \mapsto f(t, u(t), v(t))$ is measurable*

(c) *for each $t \in [0,1]$ and $u, v \in D$, $\sup_{x \in f(\cdot, u(\cdot), v(\cdot))} \|x\| \in L^1_+$.*

(C2) *u_0 and v_0 are such that*

$$\{u_0(t)\} \le_1 x(t) + \int_0^t k(t,s) f(s, u_0(s), v_0(s))\, ds$$
$$\{v_0(t)\} \ge_2 x(t) + \int_0^t k(t,s) f(s, v_0(s), u_0(s))\, ds.$$

(C3) *There exists a non-negative constant L' such that for any $\mu, \nu \in D$, $\mu \le \nu$ implies*

$$0 \le v(t) - u(t) \le \int_0^t L' k(t,s)(\nu(s) - \mu(s))\, ds$$

for any $v(t) \in \int_0^t k(t,s) f(s, \nu(s), \mu(s))\, ds$ and $u(t) \in \int_0^t f(s, \mu(s), \nu(s))\, ds$.

(C4) *There exists a constant $K \ge 0$ such that $KL' < 1$ and $\int_0^t k(t,s)\, ds \le K$.*

Then there exists $u^* \in D$ such that

$$\{u^*(t)\} = x(t) + \int_0^t k(t,s) f(s, u^*(s), u^*(s))\, ds.$$

Proof. By using Theorem 2.4 and the similar arguments in Theorem 3.1, the conclusion can be proved but we omit the details ∎

Example 3.1. Let $u_0, v_0 \in C([0,1], E)$ with $u_0 \le v_0$. Let

$$D = [u_0, v_0] = \big\{u \in C([0,1], E) : u_0 \le u \le v_0\big\}$$

and let $f : [0,1] \times E \times E \to E$ be a single-valued mixed increasing operator satisfying the following conditions:

(C1) For each $u, v \in D$, $t \mapsto f(t, u(t), v(t))$ is measurable.

(C2) u_0 and v_0 are such that

$$u_0(t) \leq x(t) + \int_0^t k(t,s) f(s, u_0(s), v_0(s))\, ds$$
$$v_0(t) \geq x(t) + \int_0^t k(t,s) f(s, v_0(s), u_0(s))\, ds.$$

(C3) There exists a non-negative constant L' such that, for any $\mu, \nu \in D$, $\mu \leq \nu$ implies

$$0 \leq f(t, \nu(t), \mu(t)) - f(t, \mu(t), \nu(t)) \leq L'(\nu(t) - \mu(t)).$$

(C4) There exists a constant $K \geq 0$ such that $KL' < 1$ and $\int_0^t k(t,s)ds \leq K$.

Then by using Theorem 3.2, there exists $u^* \in D$ such that

$$u^*(t) = x(t) + \int_0^t k(t,s) f(s, u^*(s), u^*(s))\, ds.$$

However, the standard technique used in [18] is invalid since f is not continuous.

Acknowledgment. The authors would like to express their deep gratitude to the referees for their helpful comments and suggestions.

References

[1] Amann, H.: *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces.* SIAM Rev. 18 (1976), 620 – 709.

[2] Biberdorf, E. A. and M. Väth: *On the spectrum of orthomorphisms and Barbashin operators.* Z. Anal. Anw. 18 (1999), 859 – 873.

[3] Chang, S. S., Cho, Y. J. and N. J. Huang: *Coupled fixed point theorems with applications.* J. Korean Math. Soc. 33 (1996), 575 – 585.

[4] Du, Y. H.: *Fixed points of increasing operators in ordered Banach spaces with applications.* Appl. Anal. 38 (1990), 1 – 20.

[5] Guo, D. J.: *Fixed points of mixed monotone operators with applications.* Appl. Anal. 31 (1988), 215 – 224.

[6] Guo, D. J.: *Existence and uniqueness of positive fixed point of mixed monotone operators with applications.* Appl. Anal. 46 (1992), 91 – 100.

[7] Guo, D. J.: *Existence and uniqueness of positive fixed points for noncompact decreasing operators.* Indian J. Pure Appl. Math. 31 (2000), 551 – 562.

[8] Guo, D. J. and V. Lakshmikantham: *Coupled fixed points of nonlinear operators with applications.* Nonlin. Anal. – TMA 11 (1987), 623 – 632.

[9] Heikkilä, S. and V. Lakshmikantham: *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations.* New York: Dekker 1994.

[10] Huang, N. J., Tang, Y. Y. and Y. P. Liu: *Some new existence theorems for nonlinear inclusion with an application.* Nonlin. Funct. Anal. Appl. 6 (2001), 341 – 350.

[11] Huy, N. B. and N. H. Khanh: *Fixed point for multivalued increasing operators.* J. Math. Anal. Appl. 250 (2000), 368 – 371.

[12] Kandilakis, D. A. and N. S. Papageorgiou: *On the properties of the Aumann integral with applications to differential inclusions and control systems.* Czech. Math. J. 39 (1989), 1 – 15.

[13] Klein, E. and A. Thompson: *Theory of Correspondences.* New York: Wiley 1984.

[14] Krasnoselskii, M. A. and P. P. Zabriko: *Geometrical Methods of Nonlinear Analysis.* Berlin: Springer-Verlag 1984.

[15] Nadler, S. B.: *Multivalued contraction mappings.* Pacific J. Math. 30 (1969), 475 – 488.

[16] Nishnianidze, Z. G.: *Fixed points of monotone multivalued operators.* Soobshch. Akad. Nauk Gruzin. SSR 114 (1984), 489 – 491.

[17] Sun, J. X. and Z. Q. Zhao: *Fixed point theorems of increasing operators and applications to nonlinear integro-differential equations with discontinuous terms.* J. Math. Anal. Appl. 175 (1993), 33 – 45.

[18] Sun, J. X.: *The solutions of ordinary differential equations in Banach spaces.* Acta Math. Sinica 33 (1990), 374 – 380.

[19] Tolstonogov, A.: *Differential Inclusions in a Banach Space.* Dordrecht: Kluwer Acad. Publ. 2000.

[20] Zhang, Z. T.: *New fixed point theorems of mixed monotone operators and applications.* J. Math. Anal. Appl. 204 (1996), 307 – 319.

Received 27.06.2002; in revised form 10.12.2002