Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 22 (2003), No. 2, 433–453

Delayed Loss of Stability in Systems with Degenerate Linear Parts

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Abstract. We study singularly perturbed scalar and planar differential equations with linear parts independent of time. The associated autonomous equations undergo a bifurcation of equilibria in the scalar case and the Hopf bifurcation in the case of planar systems at a bifurcation point where the zero equilibrium loses stability. We suggest natural sufficient conditions for the phenomenon of delayed loss of stability for the singularly perturbed equations and estimate the asymptotic delay. Bifurcation points, stability of the zero equilibrium, and the asymptotic delay are determined by superlinear terms in the expansions of the right-hand sides of the associated and singularly perturbed equations.

Keywords: *Singular perturbation, delayed loss of stability, periodic solution, Hopf bifurcation*

AMS subject classification: 34D15, 37G15

1. Introduction

We consider singularly perturbed equations

$$\varepsilon \frac{dx}{d\tau} = g(\tau, x) \tag{1.1}$$

and study the asymptotic behavior of solutions of the Cauchy problem for small values of the parameter $\varepsilon > 0$. In this problem, the so-called *associated* to (1.1) equation

$$\frac{dx}{dt} = g(\tau, x) \tag{1.2}$$

where τ is considered as a parameter plays an important role. If $x_*(\tau)$ is an isolated continuous branch of asymptotically stable equilibria of the autonomous equation (1.2), then the behavior of solutions of equation (1.1) (where $x \in \mathbb{R}^N$) is determined by Tikhonov's theorem (see, e.g., [14, 17, 18]). Each solution x of equation (1.1) with an initial value $x(\tau_0)$ which belongs to the basin of attraction of the equilibrium $x_*(\tau_0)$ of equation (1.2) is attracted to the curve $x_* = x_*(\tau)$ and stays in a δ-neighborhood of x_* for all $\tau > \tau_0 + \delta$ where $\delta = \delta(\varepsilon)$ vanishes as $\varepsilon \to 0$.

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ISSN 0232-2064 / \$ 2.50

If two smooth branches of equilibria of equation (1.2) intersect for some $\tau = \tau_*$ and the equilibria $x_*(\tau)$ of one branch are asymptotically stable for $\tau < \tau_*$, then generically this branch consists of unstable equilibria and the other branch $x^*(\tau)$ consists of stable equilibria for $\tau > \tau_*$ (both for transcritical and pitchfork equilibria branching). According to Tikhonov's theorem, each solution x of equation (1.1) starting at some moment $\tau_0 < \tau_*$ in the basin of attraction of the asymptotically stable equilibrium $x_*(\tau_0)$ of equation (1.2) stays near the branch $x_*(\tau)$ for $\tau_0 + \delta < \tau < \tau_*$. Then the following scenarios may be realized:

- either the solution $x(\tau)$ stays near the other branch $x^*(\tau)$ of equilibria (which are stable for $\tau > \tau_*$) for all $\tau > \tau_*$
- or it stays near the same branch $x_*(\tau)$ as before for some time interval $\tau_* < \tau < \tau_1$ and then jumps to the branch $x^*(\tau)$ of stable equilibria.

These two scenarios are referred to as *immediate* and *delayed loss of stability*. The length of the interval $\tau_* < \tau < \tau_1$ in the second scenario is called the *asymptotic delay*.

Immediate loss of stability was studied in [8]. The phenomenon of delayed loss of stability was first noticed for bifurcations of small cycles (Hopf bifurcations); these results were obtained in [16] and extended in [12, 13]. For other types of bifurcations the phenomenon was studied in [1, 3, 10, 15] (see further references therein). In what follows we assume that $x_*(\tau) \equiv 0$. Generically, if stability of the zero equilibrium of equation (1.2) is determined by the linear approximation $x' = g_x(\tau, 0)x$, then this approximation determines also the asymptotic delay in the case of delayed loss of stability for equation (1.1).

In this paper we study equations with linear parts independent of τ. For such equations stability of the zero equilibrium and the bifurcation point τ_* are determined by main superlinear terms in the expansion of $g(\tau, x)$ at the point $x = 0$. We suggest sufficient conditions for delayed loss of stability and estimate the asymptotic delay $\tau_1 - \tau_*$ for scalar equations in the case of a bifurcation of equilibria and for planar equations in the case of the Hopf bifurcation.

The paper is organized as follows. In Section 2 we consider scalar equations. This section explains the results of the main Section 3 and the method of the proofs. In the last subsection of Section 2 a periodic problem for equation (1.1) in the case of delayed loss of stability is considered. Section 3 contains two theorems on planar Hopf bifurcations. We start with an example where the results of Section 2 are directly applied. Theorem 3 is formulated for equations with positively homogeneous main superlinear terms. These terms are polynomial for equations where the function $g(\tau, x)$ admits the Taylor expansion at the point $x = 0$; such equations are considered in Theorem 4.

The proofs are based on the method of differential inequalities and upper and lower solutions close to that of [10, 11]. We do not assume that $g(\tau, x)$ is smooth. Proper modifications of the method can be used for the study of delayed Hopf bifurcations in $\mathbb{R}^N$; this will be the object of another paper.

2. Scalar equations

Consider the scalar equation (1.1). Everywhere it is supposed that the function $g(\tau, x)$ is continuous, $g(\tau, 0) \equiv 0$, and the zero solution $x \equiv 0$ of equation (1.1) does not cross any other its solution. We consider only solutions lying in the upper half-plane $x \geq 0$.

In the main part of this section we assume

$$g(\tau, x) = a(\tau)\phi(x) + o(\phi(x)) \qquad (x \to +0) \tag{2.1}$$

where a and ϕ are continuous,

$$\left.\begin{aligned} &\phi(0) = 0 \\ &\phi(x) > 0 \quad (0 < x \leq \rho_0) \end{aligned}\right\} \tag{2.2}$$

and

$$\Phi(x) := \int_{\rho_0}^{x} \frac{d\xi}{\phi(\xi)} \to -\infty \qquad (x \to +0). \tag{2.3}$$

Here (2.1) is equivalent to

$$\lim_{x\to+0} \sup_{\tau} \left| \frac{g(\tau, x)}{\phi(x)} - a(\tau) \right| = 0.$$

For example, if g is differentiable, then (2.1) holds for $\phi(x) = x$, $a(\tau) = g_x(\tau, 0)$, and $\Phi(x) = \ln x - \ln \rho_0$. If g is sufficiently smooth and its derivatives up to order $n - 1 > 0$ with respect to x at the point $x = 0$ are zero for all τ, then (2.1) is valid for $\phi(x) = x^n$ and $\Phi(x) = -\frac{x^{1-n} - \rho_0^{1-n}}{n-1}$.

Note that due to (2.1), for any solution x of equation (1.1) lying in a sufficiently narrow strip $0 < x < \rho$ for $\tau_1 < \tau < \tau_2$, the derivative

$$\frac{d}{d\tau}\Phi(x(\tau)) = \frac{1}{\phi(x(\tau))}\frac{dx(\tau)}{d\tau} = \frac{g(\tau, x(\tau))}{\varepsilon\phi(x(\tau))}$$

satisfies

$$\varepsilon\left|\frac{d}{d\tau}\Phi(x(\tau))\right| \leq c = c(\rho, \tau_1, \tau_2) < \infty,$$

therefore

$$\left|\Phi(x(\tau)) - \Phi(x(\tau_0))\right| \leq \frac{(\tau_2 - \tau_1)c}{\varepsilon} < \infty \qquad (\tau_1 < \tau,\ \tau_0 < \tau_2).$$

Thus, relation (2.3) implies that each solution of (1.1) starting in the upper half-plane $x > 0$ stays in this half-plane for all τ.

Consider the associated to (1.1) equation (1.2). If (2.1) holds, then the sign of $a(\tau)$ determines stability of the zero equilibrium of equation (1.2) (we consider only the upper half-plane $x \geq 0$). Indeed, the equilibrium $x = 0$ is asymptotically stable if $a(\tau) < 0$ and unstable if $a(\tau) > 0$. If the function a changes the sign at some point

τ_*, then (2.1) implies that for each sufficiently small $x > 0$ the equation $g(\tau, x) = 0$ has a solution $\tau = \tau(x)$ which goes to τ_* as $x \to 0$. It means that equation (1.2) undergoes a bifurcation of equilibria at the point τ_*. Generically, the pairs $\{\tau(x), x\}$ form a continuous curve and $\tau(x) - \tau_*$ has the same sign for all $x > 0$.

Suppose that $a(\tau_*) = 0$ and

$$a(\tau_*) \begin{cases} < 0 & \text{for } \tau < \tau_* \\ > 0 & \text{for } \tau > \tau_*. \end{cases} \tag{2.4}$$

Let $\tau_0 < \tau_* < \tau_1$, consider all solutions x^ε of equation (1.1) with initial values $x^\varepsilon(\tau_0) = x_0$ in the interval $0 < x_0 < \rho$. If for some ρ these solutions satisfy pointwise

$$\lim_{\varepsilon \to 0} x^\varepsilon(\tau) = 0 \qquad (\tau_0 < \tau < \tau_1),$$

then the trivial solution $x \equiv 0$ of equation 1 is said to exhibit delayed loss of stability. The value $\tau_1 - \tau_*$ is called the asymptotic delay if for every fixed $x_0 > 0$ and $\delta > 0$ the relation $x^\varepsilon(\tau_0) = x_0$ implies for all sufficiently small ε

$$\sup \left\{ x^\varepsilon(\tau) : \tau_1 - \delta < \tau < \tau_1 + \delta \right\} \geq \rho. \tag{2.5}$$

We consider equations where $g(\tau, x) = o(x)$, for example, $g(\tau, x) = a(\tau)x^\alpha + o(x^\alpha)$ with $\alpha > 1$. The results and the proofs also hold in the well-known case $g(\tau, x) = a(\tau)x + o(x)$.

2.1 Estimates of asymptotic delay. All the functions used below are supposed to be continuous. The functions ϕ and Φ satisfy (2.2) and (2.3), respectively.

Proposition 1. *Let*

$$g(\tau, x) \leq a_+(\tau)\phi(x) \qquad (0 < x < \rho_0) \tag{2.6}$$

and

$$J_+(\tau, \tau_0) := \int_{\tau_0}^{\tau} a_+(s)\, ds < 0 \qquad (\tau_0 < \tau < \tau_1). \tag{2.7}$$

Then all solutions x^ε of equation (1.1) *with initial values $x^\varepsilon(\tau_0) = x_0$ from the interval $0 < x_0 < \rho_0$ satisfy for each small $\delta > 0$*

$$\lim_{\varepsilon \to 0} \max \left\{ x^\varepsilon(\tau) : \tau_0 + \delta \leq \tau \leq \tau_1 - \delta \right\} = 0. \tag{2.8}$$

Proposition 2. *Let*

$$g(\tau, x) \geq a_-(\tau)\phi(x) \qquad (0 < x < \rho_0) \tag{2.9}$$

and

$$J_-(\tau_1, \tau_0) := \int_{\tau_0}^{\tau_1} a_-(s)\, ds > 0 \tag{2.10}$$

for some $\tau_1 > \tau_0$. Let $0 < x_0 < \rho_0$. Then for every sufficiently small ε any solution x^ε of equation (1.1) *with initial value $x^\varepsilon(\tau_0) = x_0$ satisfies $x^\varepsilon(\tau) \geq \rho_0$ at some moment τ from the interval $\tau_0 < \tau < \tau_1$.*

Proof of Proposition 1. Suppose relations (2.6) - (2.7) hold. Let a solution x of equation (1.1) satisfy

$$0 < x(\tau) < \rho_0 \qquad (\tau_0 \leq \tau < T) \tag{2.11}$$

with some $T \leq \tau_1$. From (2.6) it follows that $\varepsilon \frac{d}{d\tau}\Phi(x(\tau)) \leq a_+(\tau)$ for all $\tau_0 \leq \tau < T$, hence

$$\Phi(x(\tau)) - \Phi(x_0) \leq \frac{J_+(\tau, \tau_0)}{\varepsilon} \tag{2.12}$$

with $x_0 = x(\tau_0)$ and (2.7) implies $\Phi(x(\tau)) \leq \Phi(x_0)$. Due to (2.2), the $\Phi(x)$ increases for $x \in (0, \rho_0)$, therefore $x(\tau) \leq x_0$ on the interval $\tau_0 \leq \tau < T$ and consequently $x(T) \leq x_0 < \rho_0$.

Since (2.11) implies $x(T) < \rho_0$, we conclude that (2.11) holds for all $\tau_0 \leq \tau < \tau_1$. Therefore (2.12) is valid for all $\tau_0 \leq \tau < \tau_1$, too. Due to (2.7), for each small $\delta > 0$ there is a $\mu = \mu(\delta) > 0$ such that $J_+(\tau, \tau_0) \leq -\mu$ for $\tau_0 + \delta \leq \tau \leq \tau_1 - \delta$, hence

$$\Phi(x(\tau)) - \Phi(x_0) \leq -\frac{\mu}{\varepsilon} \qquad (\tau_0 + \delta \leq \tau \leq \tau_1 - \delta).$$

This implies (2.8) and proves Proposition 1 ∎

Proof of Proposition 2. Suppose that relations (2.9) - (2.10) are valid. Let a solution x of equation (1.1) satisfy (2.11) on some interval $[\tau_0, T)$ with $T \leq \tau_1$. Then (2.9) implies $\varepsilon \frac{d}{d\tau}\Phi(x(\tau)) \geq a_-(\tau)$ on this interval and therefore

$$\Phi(x(\tau)) - \Phi(x_0) \geq \frac{J_-(\tau, \tau_0)}{\varepsilon} \qquad (\tau_0 \leq \tau < T).$$

Since Φ increases, from (2.11) it follows that $\Phi(x(\tau)) \leq \Phi(\rho_0)$, hence

$$\varepsilon\big(\Phi(\rho_0) - \Phi(x_0)\big) \geq J_-(\tau, \tau_0) \qquad (\tau_0 \leq \tau < T).$$

If ε is sufficiently small, then (2.10) implies the opposite estimate for $\tau = \tau_1$

$$\varepsilon\big(\Phi(\rho_0) - \Phi(x_0)\big) < J_-(\tau_1, \tau_0).$$

Therefore (2.11) cannot hold for all $\tau_0 \leq \tau < \tau_1$. Proposition 2 is proved ∎

Suppose that relation (2.1) is valid. Let

$$\left.\begin{array}{l} a(\tau_0) < 0 \\ g(\tau_0, x) < 0 \quad (0 < x \leq \rho_1) \end{array}\right\} \tag{2.13}$$

(here the first estimate implies the second one for small ρ_1). Then the interval $0 < x \leq \rho_1$ belongs to the basin of attraction of the stable zero equilibrium of autonomous equation (1.2) for each τ close to τ_0. The estimate $a(\tau_0) < 0$ implies for small $\tau - \tau_0 > 0$

$$J(\tau, \tau_0) := \int_{\tau_0}^{\tau} a(s)\, ds < 0.$$

The next simple theorem follows from Propositions 1 and 2.

Theorem 1. *Let relations* (2.13) *be valid. Let*

$$\left.\begin{array}{l} J(\tau_1,\tau_0)=0 \quad \textit{for some } \tau_1>\tau_0 \\ J(\tau,\tau_0)<0 \quad \textit{for all } \tau_0<\tau<\tau_1 \end{array}\right\}. \tag{2.14}$$

Then relation (2.8) *with any small* $\delta>0$ *holds for all solutions* x^ε *of equation* (1.1) *with initial values* $x^\varepsilon(\tau_0)=x_0$ *from the interval* $0<x_0\le\rho_1$. *If*

$$\left.\begin{array}{l} a(\tau_1)>0 \\ g(\tau_1,x)>0 \quad (0<x\le\rho_2) \end{array}\right\}, \tag{2.15}$$

then $x^\varepsilon(\tau_0)\ge\delta$ *implies* (2.5) *with* $\rho=\rho_2$ *for all sufficiently small* ε.

Proof. Consider a sufficiently small segment $\tau_0\le\tau\le\tau_0+\Delta$ such that (2.13) holds for each τ from this segment and therefore the stable zero equilibrium of autonomous equation (1.2) attracts all the trajectories starting in the interval $0<x\le\rho_1$. According to Tikhonov's theorem all solutions x^ε of equation (1.1) with initial values $x^\varepsilon(\tau_0)=x_0$ from the interval $0<x_0\le\rho_1$ satisfy

$$\max\big\{x^\varepsilon(\tau):\ \tau_0+\delta\le\tau\le\tau_0+\Delta\big\}\to 0 \qquad (\varepsilon\to 0) \tag{2.16}$$

for each positive $\delta<\Delta$, i.e. they jump to an arbitrarily small neighborhood of zero for arbitrarily short time δ if ε is sufficiently small and stay in those neighborhood until the moment $\tau=\tau_0+\Delta$.

From (2.1) it follows that estimates (2.6) and (2.9) with $a_+(\tau)=a(\tau)+\eta$ and $a_-(\tau)=a(\tau)-\eta$ hold for any $\eta>0$ and any x from a sufficiently small interval $0<x<\rho_0(\eta)$. Moreover, relations (2.14) and $a(\tau_0)<0$ imply for each small $\nu>0$

$$\int_{\tau_0+\delta}^{\tau}(a(s)+\eta)\,ds<0 \qquad (\tau_0+\delta<\tau<\tau_1-\nu)$$

if positive $\eta=\eta(\nu)$ and $\delta=\delta(\nu)$ are sufficiently small. Therefore, by Proposition 1,

$$\max\big\{x^\varepsilon(\tau):\ \tau_0+2\delta\le\tau\le\tau_1-2\nu\big\}\to 0 \qquad (\varepsilon\to 0) \tag{2.17}$$

for all solutions x^ε of equation (1.1) such that $0<x^\varepsilon(\tau_0+\delta)<\rho_0(\eta)$. Combining (2.16) and (2.17), we see that relation (2.8) with any small $\delta>0$ holds for all the solutions x^ε of equation (1.1) with initial values $x^\varepsilon(\tau_0)=x_0$ from the interval $0<x_0\le\rho_1$.

Suppose that relations (2.15) hold. Then

$$g(\tau,x)>0 \qquad (|\tau-\tau_1|<\Delta,\ 0<x\le\rho_2) \tag{2.18}$$

with some $\Delta>0$. Therefore, the zero equilibrium of autonomous equation (1.2) is unstable for each τ close to τ_1. Relations $J(\tau_0,\tau_1)=0$ and $a(\tau_1)>0$ imply that for each small $\nu>0$ there is an $\eta=\eta(\nu)$ such that

$$\int_{\tau_0}^{\tau_1+\nu}(a(s)-\eta)\,ds>0.$$

Take any $\delta>0$. Since (2.9) with $a_-(\tau)=a(\tau)-\eta$ holds for all x from some interval $0<x<\rho_0=\rho_0(\eta)$, it follows from Proposition 2 that each solution x^ε of equation (1.1) with $x^\varepsilon(\tau_0)\ge\delta$ satisfies $x^\varepsilon(\bar\tau)\ge\rho_0$ at some moment $\bar\tau<\tau_1+\nu$ (this moment depends on the solution) if ε is sufficiently small. Also, $\bar\tau>\tau_1-\delta$ for small ε due to (2.8). For $\nu<\delta<\Delta$ estimates $x^\varepsilon(\bar\tau)\ge\rho_0$ $(|\bar\tau-\tau_1|<\delta)$ and (2.18) imply (2.5) with $\rho=\rho_2$ for all sufficiently small ε. Thus, (2.5) with $\rho=\rho_2$ holds for each solution x^ε of equation (1.1) such that $x^\varepsilon(\tau_0)\ge\delta$. This completes the proof ∎

Suppose that all conditions (2.13) - (2.15) of Theorem 1 are satisfied. Then according to (2.5) and (2.8) the behavior of solutions of equation 1 can be described as follows. If x_0 belongs to the basin of attraction of the zero equilibrium of equation (1.2) for $\tau = \tau_0$, then a solution x^ε of equation (1.1) with the initial value $x^\varepsilon(\tau_0) = x_0$ jumps to a small neighborhood of zero, stays there until the moment approximately τ_1, and then jumps out some fixed strip $0 < x < \rho_2$ (the smaller ε, the smaller neighborhood of zero, the shorter times of the jumps).

In the simplest case, the function a has a unique zero τ_* in the interval $\tau_0 < \tau < \tau_1$, i.e. (2.4) holds. Due to (2.8), the zero solution of equation (1.1) exhibits delayed loss of stability, the asymptotic delay equals $\tau_1 - \tau_*$.

2.2 Periodic solutions. Consider equation (1.1) where

$$g(\tau, x) \equiv g(\tau + T, x).$$

We are interested in positive T-periodic solutions x^ε of equation (1.1) (i.e. $x^\varepsilon(\tau) > 0$ for all $\tau \in \mathbb{R}$) that are close to the zero solution on some part of the period and at the same time satisfy $\max x^\varepsilon(\tau) \geq \rho_0$ with some $\rho_0 > 0$ independent of ε.

Let relations (2.1) - (2.3) hold,

$$J(T, 0) = \int_0^T a(s)\, ds > 0 \tag{2.19}$$

and

$$g(\tau, x) < 0 \qquad (x \geq \rho_1 > 0,\ \tau \in \mathbb{R}). \tag{2.20}$$

Assumptions (2.19) - (2.20) are close to that used in [13]. Note that if $a(\tau_0) < 0$ for some τ_0, then relation (2.19) implies (2.14) for some $\tau_1 \in (\tau_0, \tau_0 + T)$.

Theorem 2. *Let relations* (2.19) − (2.20) *hold. Then equation* (1.1) *has at least one positive T-periodic solution x^ε, all such solutions lie in the strip* $0 < x < \rho_1 \quad (\tau \in \mathbb{R})$ *and satisfy* $\max x^\varepsilon(\tau) \geq \rho_0$ *with some* $\rho_0 > 0$ *independent of* ε. *In addition, let for some* τ_0

$$g(\tau_0, x) < 0 \qquad (x > 0), \tag{2.21}$$

let $a(\tau_0) < 0$ *and let* τ_1 *be defined by* (2.14). *Then relation* (2.8) *with any small* $\delta > 0$ *is valid for all positive T-periodic solutions x^ε of equation* (1.1).

Proof. Estimate (2.20) implies $x(T) < x(0)$ for all solutions x of equation (1.1) with initial values $x(0) \geq \rho_1$. On the other hand, due to (2.19) and (2.1) we can choose an $\eta > 0$ such that $J(T, 0) > \eta T$ and a $\rho_0 = \rho_0(\eta)$ such that

$$a(\tau) - \eta < \varepsilon \frac{d}{d\tau} \Phi(x(\tau)) < a(\tau) + \eta \qquad (0 < x(\tau) < \rho_0). \tag{2.22}$$

Then from (2.3) it follows that estimates (2.22) are valid for all $0 \leq \tau \leq T$ if $x(0) > 0$ is sufficiently small. Consequently,

$$0 < \frac{J(T, 0) - \eta T}{\varepsilon} < \Phi(x(T)) - \Phi(x(0))$$

and therefore $x(T) > x(0)$ for all solutions x with sufficiently small initial values $x(0) > 0$. Thus, if the Poincaré map is defined (i.e. the Cauchy problem for equation (1.1) has a unique solution for any initial data), then it maps each segment $\delta \le x(0) \le \rho_1$ with a sufficiently small $\delta > 0$ into itself, which implies the existence of a positive T-periodic solution. If the Poincaré map is not defined, then a positive periodic solution of equation (1.1) can be constructed by the standard approximation procedure as the limit of a sequence $\{x_n\}$, where x_n are periodic solutions of smooth approximations $\varepsilon\frac{dx}{d\tau} = g_n(\tau, x)$ of equation (1.1). This proves the existence of at least one positive T-periodic solution. Moreover, estimate (2.20) implies that all such solutions x satisfy $0 < x(\tau) < \rho_1 \ (\tau \in \mathbb{R})$. From (2.22) and $J(T, 0) - \eta T > 0$ it follows that $\Phi(x(T)) > \Phi(x(0))$ for any solution x lying in the strip $0 < x < \rho_0$ and therefore all the positive T-periodic solutions x satisfy $\max x(\tau) \ge \rho_0$.

If in addition estimate (2.21) holds, then all the conditions of Theorem 1 are satisfied for solutions x from the strip $0 < x < \rho_1$, which implies the last conclusion of Theorem 2 and completes the proof ∎

Condition (2.21) means that the zero equilibrium of the associated to (1.1) equation (1.2) is globally asymptotically for $\tau = \tau_0$.

For example, suppose that $g(\tau, x) < 0$ for all $x > 0$ and $0 < \tau < \tau_*$,

$$\begin{aligned} a(0) = a(\tau_*) = 0 \\ a(\tau) < 0 \quad (0 < \tau < \tau_*) \\ a(\tau) > 0 \quad (\tau_* < \tau < T) \end{aligned}$$

and relations (2.19) - (2.20) are valid. Then Theorem 1 and Proposition 2 imply that for each small $\delta > 0$ all T-periodic positive solutions x^ε of equation (1.1) with any ε from a sufficiently small interval $0 < \varepsilon \le \varepsilon_0 = \varepsilon_0(\delta)$ satisfy

$$x^\varepsilon(\tau) \begin{cases} \le \delta & \text{for all } \delta \le \tau \le \tau_1 - \delta \\ \ge \rho_0(\tau) & \text{for all } \tau_1 + \delta \le \tau \le T \end{cases}$$

where $\tau_1 \in (0, T)$ is the root of the equation $J(\tau_1, 0) = 0$; the positive function ρ_0 strictly decreases on the interval $\tau_1 \le \tau < T$ and vanishes at the point $\tau = T$. Thus, periodic solutions stay near the zero solution approximately from the moment $\tau = 0$ until the moment $\tau_1 > \tau_*$, then jump out some strip $0 < x < \rho_2$, and come back to a small neighborhood of zero approximately at the end of the period.

3. Planar Hopf bifurcations

Consider the planar system

$$\left.\begin{aligned} \varepsilon\frac{dx}{d\tau} &= \sigma(\tau)x - w(\tau)y + f_1(\tau, x, y) \\ \varepsilon\frac{dy}{d\tau} &= w(\tau)x + \sigma(\tau)y + f_2(\tau, x, y) \end{aligned}\right\} \tag{3.1}$$

where σ, w and f_1, f_2 are continuous functions and

$$\lim_{|x|+|y|\to 0} \sup_{\tau} \frac{|f_1(\tau,x,y)|}{|x|+|y|} = \lim_{|x|+|y|\to 0} \sup_{\tau} \frac{|f_2(\tau,x,y)|}{|x|+|y|} = 0. \tag{3.2}$$

The associated system has the form

$$\left.\begin{aligned} \frac{dx}{dt} &= \sigma(\tau)x - w(\tau)y + f_1(\tau,x,y) \\ \frac{dy}{dt} &= w(\tau)x + \sigma(\tau)y + f_2(\tau,x,y) \end{aligned}\right\}. \tag{3.3}$$

If $\sigma(\tau_*) = 0$, $w(\tau_*) > 0$, the derivative $\sigma'(\tau_*)$ exists and $\sigma'(\tau_*) \neq 0$, then autonomous system (3.3) satisfies the conditions of Poincaré-Andronov-Hopf's theorem and undergoes the Hopf bifurcation at the point τ_*. It means that for each small $r > 0$ there is a τ_r such that system (3.3) with $\tau = \tau_r$ has a small non-trivial cycle $(x_r(t), y_r(t))$ with a period T_r and

$$\|x_r(t)\|_C + \|y_r(t)\|_C \to 0$$
$$\tau_r \to \tau_* \qquad (r \to 0)$$
$$T_r \to \frac{2\pi}{w(\tau_*)}$$

(further details and related facts can be found, e.g., in [9]). Here r is an auxiliary parameter; usually, it is roughly the amplitude of the cycle. The value τ_* is called a *Hopf bifurcation point* for system (3.3).

Generically, the Hopf bifurcation is either subcritical (all the small cycles of system (3.3) exist for $\tau < \tau_*$) or supercritical (all small cycles exist for $\tau > \tau_*$); from this point of view, the linear system with $f_1 \equiv f_2 \equiv 0$ represents a degenerate case, since all its cycles exist for $\tau = \tau_*$. Which situation takes place is determined by the nonlinear terms f_j. These terms determine also stability of all small cycles.

The zero equilibrium of system (3.3) is asymptotically stable if $\sigma(\tau) < 0$ and unstable if $\sigma(\tau) > 0$. If $\sigma(\tau) < 0$ for $\tau_0 \le \tau < \tau_*$ and $\sigma(\tau) > 0$ for $\tau > \tau_*$, then the zero solution of system (3.1) exhibits delayed loss of stability (see, e.g., [12]). The asymptotic delay $\tau_1 - \tau_*$ is determined from the relation

$$\int_{\tau_0}^{\tau_1} \sigma(s)\, ds = 0.$$

Here we consider systems

$$\left.\begin{aligned} \varepsilon\frac{dx}{d\tau} &= -y + f_1(\tau,x,y) \\ \varepsilon\frac{dy}{d\tau} &= x + f_2(\tau,x,y) \end{aligned}\right\} \tag{3.4}$$

with linear part independent of τ; the continuous non-linearities f_1 and f_2 are supposed to satisfy (3.2). These non-linearities determine Hopf bifurcation points and stability of the zero equilibrium for the associated to (3.4) system

$$\left.\begin{aligned}\frac{dx}{dt} &= -y + f_1(\tau, x, y)\\ \frac{dy}{dt} &= x + f_2(\tau, x, y)\end{aligned}\right\}. \tag{3.5}$$

3.1 Example. Consider system (3.4) with

$$\begin{aligned}f_1(\tau, x, y) &= x(x^2+y^2)(\tau - x^2 - y^2)\\ f_2(\tau, x, y) &= y(x^2+y^2)(\tau - x^2 - y^2).\end{aligned}$$

Passing to the polar coordinates

$$\left.\begin{aligned}x &= r\cos\varphi\\ y &= r\sin\varphi\end{aligned}\right\}$$

in (3.4), we obtain

$$\varepsilon\frac{dr}{dt} = r^3(\tau - r^2) \tag{3.6}$$

and $\varepsilon\frac{d\varphi}{d\tau} \equiv 1$, hence solutions of system (3.4) and equation (3.6) are related by

$$\left.\begin{aligned}x &= r\cos\frac{\tau}{\varepsilon}\\ y &= r\sin\frac{\tau}{\varepsilon}\end{aligned}\right\}.$$

The right-hand side of (3.6) has form (2.1) with $a(\tau) = \tau$ and $\phi(r) = r^3$.

The associated to (3.6) equation

$$\frac{dr}{dt} = r^3(\tau - r^2) \tag{3.7}$$

has the zero equilibrium $r = 0$ for all τ and the equilibrium $r = \sqrt{\tau}$ for $\tau > 0$. For $\tau \le 0$ the zero equilibrium is globally stable. For $\tau > 0$ the zero equilibrium is unstable, the other equilibrium $r = \sqrt{\tau}$ is asymptotically stable and its basin of attraction is the interval $(0, \infty)$.

Theorem 1 implies that the zero solution of equation (3.6) exhibits delayed loss of stability; the asymptotic delay is $-\tau_0$ for any moment $\tau_0 < 0$. The same is true for system (3.4). The associated system (3.5) consists of equations (3.7) and $\frac{d\varphi}{dt} \equiv 1$ (in polar coordinates). This system has a unique globally stable limit cycle $r = \sqrt{\tau}$ for $\tau > 0$, hence it undergoes the Hopf bifurcation at the point $\tau_* = 0$. System (3.4) exhibits delayed Hopf bifurcation. Its solutions starting at a moment $\tau_0 < 0$ jump to a small neighborhood of the origin and stay there until the moment approximately $-\tau_0$. Then they jump to a small neighborhood of the cycle $x^2 + y^2 = \tau$ of system (3.5) and track this cycle while it evolutes in time.

3.2 Positively homogeneous non-linearities. Let functions $F_j = F_j(\tau, x, y)$ be continuous and positively homogeneous of order $\alpha > 1$ for each τ:

$$F_j(\tau, \lambda x, \lambda y) = \lambda^\alpha F_j(\tau, x, y) \qquad (\lambda > 0;\ j = 1, 2).$$

For example, the functions

$$|x|^\beta |y|^\gamma, \quad |x|^\beta |y|^\gamma \operatorname{sign} x, \quad |x|^\beta |y|^\gamma \operatorname{sign} y, \quad |x|^\beta |y|^\gamma \operatorname{sign}(xy)$$

are positively homogeneous of order $\beta + \gamma$. Linear combinations of positively homogeneous functions of order α are positively homogeneous of order α.

Suppose that the non-linearities in (3.4) - (3.5) have the form

$$f_j(\tau, x, y) = F_j(\tau, x, y) + \psi_j(\tau, x, y) \tag{3.8}$$

with

$$\lim_{|x|+|y|\to 0} \sup_\tau \frac{|\psi_j(\tau, x, y)|}{(|x| + |y|)^\alpha} = 0. \tag{3.9}$$

Set

$$a(\tau, \varphi) = F_1(\tau, \cos\varphi, \sin\varphi)\cos\varphi + F_2(\tau, \cos\varphi, \sin\varphi)\sin\varphi \tag{3.10}$$

$$\bar{a}(\tau) = \tfrac{1}{2\pi} \int_0^{2\pi} a(\tau, \varphi)\, d\varphi. \tag{3.11}$$

Theorem 3. *Let*

$$\begin{gathered} \bar{a}(\tau_0) < 0 \\ \bar{J}(\tau, \tau_0) := \int_{\tau_0}^{\tau} \bar{a}(s)\, ds < 0 \quad (\tau_0 < \tau \le \tau_d). \end{gathered} \tag{3.12}$$

Then there exists a ball

$$B(\rho) = \left\{(x_0, y_0) \in \mathbb{R}^2 :\ x_0^2 + y_0^2 < \rho^2\right\}$$

such that all solutions $(x^\varepsilon, y^\varepsilon)$ *of system* (3.4) *with initial values* $(x^\varepsilon(\tau_0), y^\varepsilon(\tau_0)) \in B(\rho)$ *satisfy for each small* $\delta > 0$

$$\lim_{\varepsilon\to 0} \max\left\{|x^\varepsilon(\tau)| + |y^\varepsilon(\tau)| :\ \tau_0 + \delta \le \tau \le \tau_d\right\} = 0. \tag{3.13}$$

Function (3.11) plays an important role in the analysis of Hopf bifurcations for system (3.5) (see [4 - 6]). In particular, every Hopf bifurcation point for this system is a zero of $\bar{a}(\tau)$. Moreover, if $\bar{a}(\tau_*) = 0$ and $\bar{a}(\tau)$ takes values of both sign in each neighborhood of τ_*, then τ_* is a Hopf bifurcation point.

It is easy to check that the sign of $\bar{a}(\tau)$ determines stability of the zero equilibrium of system (3.5) (for example, one can use the argument of Proposition 3 below). The zero equilibrium of system (3.5) is asymptotically stable if $\bar{a}(\tau) < 0$ and unstable if $\bar{a}(\tau) > 0$; Hopf bifurcation points separate the intervals determined by these inequalities.

For simplicity, let the function $\bar{a}$ have a unique zero τ_*. If relations (3.12) hold and $\tau_0 < \tau_* < \tau_d$, then it follows from Theorem 3 that the trivial solution of system (3.4) exhibits delayed loss of stability; the asymptotic delay is estimated from below by the number $\tau_d - \tau_*$. The asymptotic delay equals $\tau_1 - \tau_0$ if $\bar{J}(\tau_1, \tau_0) = 0$.

To prove Theorem 3 we first rewrite system (3.4) in polar coordinates r, φ and make use of representation (3.8). We arrive at

$$\left.\begin{aligned} \varepsilon\frac{dr}{d\tau} &= a(\tau,\varphi)r^\alpha + \zeta(\tau, r, \varphi) \\ \varepsilon\frac{d\varphi}{d\tau} &= 1 + \xi(\tau, r, \varphi) \end{aligned}\right\}$$

where a is function (3.10), $\zeta = o(r^\alpha)$ and $\xi = o(1)$, i.e.

$$\left.\begin{aligned} \sup_{\tau,\varphi} r^{-\alpha}|\zeta(\tau, r, \varphi)| \\ \sup_{\tau,\varphi} |\xi(\tau, r, \varphi)| \end{aligned}\right\} \to 0 \qquad (r \to 0). \tag{3.14}$$

Then we substitute $\varphi = \varepsilon^{-1}\theta$ and change the time $\tau \mapsto \theta$. The resulting system is

$$\left.\begin{aligned} \varepsilon\frac{dr}{d\theta} &= a(\tau, \varepsilon^{-1}\theta)r^\alpha + \zeta_1(\tau, r, \varepsilon^{-1}\theta) \\ \frac{d\tau}{d\theta} &= 1 + \xi_1(\tau, r, \varepsilon^{-1}\theta) \end{aligned}\right\}. \tag{3.15}$$

Relations (3.14) hold for the functions ζ_1 and ξ_1 in place of ζ and ξ, respectively. The time change $\tau \mapsto \theta$ is correct for small r, since $\frac{d\tau}{d\theta}$ is close to 1.

System (3.15) has the form

$$\left.\begin{aligned} \varepsilon\frac{dr}{d\theta} &= g(\tau, r, \varepsilon^{-1}\theta) \\ \frac{d\tau}{d\theta} &= h(\tau, r, \varepsilon^{-1}\theta) \end{aligned}\right\} \tag{3.16}$$

where the functions g and h are continuous, satisfy

$$\begin{aligned} g(\tau, r, \varphi) &\equiv g(\tau, r, \varphi + 2\pi) \\ h(\tau, r, \varphi) &\equiv h(\tau, r, \varphi + 2\pi) \end{aligned} \tag{3.17}$$

and for all τ

$$\bar{h}(\tau) := \tfrac{1}{2\pi}\int_0^{2\pi} h(\tau, 0, \varphi)\, d\varphi > 0. \tag{3.18}$$

Theorem 3 follows from the next Proposition.

Proposition 3. *Let relations* (3.17) – (3.18) *be valid and*

$$g(\tau, r, \varphi) \le a(\tau, \varphi)\phi(r) \qquad (0 < r \le \rho_0) \tag{3.19}$$

where ϕ *satisfies properties* (2.2) – (2.3)*. Let functions* (3.11) *and* (3.18) *satisfy*

$$\begin{aligned} &\bar{a}(\tau_0) < 0 \\ &\int_{\tau_0}^{\tau} \frac{\bar{a}(s)}{\bar{h}(s)}\, ds < 0 \quad (\tau_0 < \tau \le \tau_d). \end{aligned} \tag{3.20}$$

Suppose that for solutions $(r^\varepsilon, \tau^\varepsilon)$ *of system* (3.16) *the estimate* $r^\varepsilon(\theta_0) > 0$ *implies* $r^\varepsilon(\theta) > 0$ *for all* $\theta > \theta_0$*. Then for every sufficiently small* $\rho > 0$ *all solutions* $(r^\varepsilon, \tau^\varepsilon)$ *such that* $0 < r^\varepsilon(\theta_0) = r_0 < \rho$ *and* $\tau^\varepsilon(\theta_0) = \tau_0$ *with any* θ_0 *satisfy*

$$\sup\left\{\tau^\varepsilon(\theta) : \theta \ge \theta_0\right\} > \tau_d \tag{3.21}$$

$$\max\left\{r^\varepsilon(\theta) : \theta_0 \le \theta \le \theta_d\right\} \le \chi(\rho) \tag{3.22}$$

$$\lim_{\varepsilon \to 0} \max\left\{r^\varepsilon(\theta) : \tau^\varepsilon(\theta) \ge \tau_0 + \delta,\ \theta_0 \le \theta \le \theta_d\right\} = 0 \tag{3.23}$$

where θ_d *is defined by*

$$\begin{aligned} &\tau^\varepsilon(\theta_d) = \tau_d \\ &\tau^\varepsilon(\theta) < \tau_d \quad (\theta_0 \le \theta < \theta_d), \end{aligned} \tag{3.24}$$

$\chi(\rho) \to 0$ *as* $\rho \to 0$*, and* $\delta > 0$ *is an arbitrary small number.*

Proof. Let (r, τ) be a solution of system (3.16) on the interval $\theta_0 \le \theta < \Theta$ and let

$$\begin{aligned} &0 < r(\theta) < \rho \le \rho_0 \\ &\tau_0 - 1 \le \tau(\theta) < \tau_d \end{aligned} \tag{3.25}$$

on this interval. Let $\tau(\theta_0) = \tau_0$. Set

$$\begin{aligned} \theta_n &= \theta_0 + 2\pi n\varepsilon \\ r_n &= r(\theta_n) \\ \tau_n &= \tau(\theta_n). \end{aligned}$$

From (3.19) it follows that

$$\frac{\varepsilon}{\phi(r)} \frac{dr}{d\theta} \le a(\tau(\theta), \varepsilon^{-1}\theta) \tag{3.26}$$

and therefore

$$\varepsilon\big(\Phi(r_n) - \Phi(r_{n-1})\big) \le \int_{\theta_{n-1}}^{\theta_n} a(\tau(\theta), \varepsilon^{-1}\theta)\, d\theta.$$

Since $h(\tau, r, \varphi)$ is continuous and periodic in the last variable, the estimate $|h(\tau, r, \varphi)| \le c$ holds for all $\tau_0 - 1 \le \tau \le \tau_d$ and $0 \le r \le \rho_0$, hence

$$|\tau(\theta) - \tau(\theta_n^*)| \le 2\pi c\varepsilon \qquad (\theta_{n-1} \le \theta_n^* \le \theta_n) \tag{3.27}$$

and

$$\int_{\theta_{n-1}}^{\theta_n} \big[a(\tau(\theta),\varepsilon^{-1}\theta) - a(\tau(\theta_n^*),\varepsilon^{-1}\theta)\big]\, d\theta = o(\varepsilon) \qquad (\varepsilon \to 0)$$

(θ_n^* will be fixed later). Here

$$\tfrac{1}{2\pi} \int_{\theta_{n-1}}^{\theta_n} a(\tau(\theta_n^*),\varepsilon^{-1}\theta)\, d\theta = \varepsilon \bar{a}(\tau(\theta_n^*)),$$

therefore

$$\varepsilon\big(\Phi(r_n) - \Phi(r_{n-1})\big) \le 2\pi\varepsilon\bar{a}(\tau(\theta_n^*)) + o(\varepsilon). \tag{3.28}$$

Let us divide the second equation of (3.16) by $\bar{h}(\tau)$ and integrate the result over the segment $\theta_{n-1} \le \theta \le \theta_n$. We obtain

$$H(\tau_n) - H(\tau_{n-1}) = \int_{\theta_{n-1}}^{\theta_n} \frac{h\big(\tau(\theta), r(\theta), \varepsilon^{-1}\theta\big)}{\bar{h}(\tau(\theta))}\, d\theta$$

where $H(\tau) = \int_{\tau_0}^{\tau} \frac{ds}{\bar{h}(s)}$. From estimates (3.27) and $0 < r(\theta) < \rho$ it follows that

$$\int_{\theta_{n-1}}^{\theta_n} \frac{h\big(\tau(\theta), r(\theta), \varepsilon^{-1}\theta\big)}{\bar{h}(\tau(\theta))}\, d\theta - \int_{\theta_{n-1}}^{\theta_n} \frac{h\big(\tau(\theta_n^*), 0, \varepsilon^{-1}\theta\big)}{\bar{h}(\tau(\theta_n^*))}\, d\theta = o(\varepsilon)$$

for $\varepsilon, \rho \to 0$. Here the second integral equals $2\pi\varepsilon$, hence

$$H(\tau_n) - H(\tau_{n-1}) = 2\pi\varepsilon + o(\varepsilon) \qquad (\varepsilon, \rho \to 0). \tag{3.29}$$

Substituting this in (3.28), we arrive at

$$\varepsilon\big(\Phi(r_n) - \Phi(r_{n-1})\big) \le \bar{a}(\tau(\theta_n^*))\big[H(\tau_n) - H(\tau_{n-1})\big] + o(\varepsilon)$$

for $\varepsilon, \rho \to 0$. By the mean value theorem, there is a $\theta_n^* \in [\theta_{n-1}, \theta_n]$ such that

$$\bar{a}(\tau(\theta_n^*))\big[H(\tau_n) - H(\tau_{n-1})\big] = \int_{\tau_{n-1}}^{\tau_n} \bar{a}(\tau)\, dH(\tau) = \int_{\tau_{n-1}}^{\tau_n} \frac{\bar{a}(\tau)}{\bar{h}(\tau)}\, d\tau,$$

hence

$$\varepsilon\big(\Phi(r_n) - \Phi(r_{n-1})\big) \le \int_{\tau_{n-1}}^{\tau_n} \frac{\bar{a}(\tau)}{\bar{h}(\tau)}\, d\tau + o(\varepsilon) \qquad (\varepsilon, \rho \to 0). \tag{3.30}$$

We suppose that ε and ρ are sufficiently small. Then relation (3.29) implies $H(\tau_n) > H(\tau_{n-1})$ and therefore the sequence $\{\tau_n\}$ increases. Let us sum inequalities (3.30) over $1 \le n \le N$, where $2\pi N\varepsilon < \Theta - \theta_0$. We obtain

$$\varepsilon\big(\Phi(r_N) - \Phi(r_0)\big) \le \int_{\tau_0}^{\tau_N} \frac{\bar{a}(\tau)}{\bar{h}(\tau)}\, d\tau + No(\varepsilon).$$

The sum of relations (3.29) over $1 \le n \le N$ is

$$H(\tau_N) - H(\tau_0) = N(2\pi\varepsilon + o(\varepsilon)), \tag{3.31}$$

therefore

$$N = \frac{1}{2\pi\varepsilon + o(\varepsilon)} \int_{\tau_0}^{\tau_N} \frac{d\tau}{\bar{h}(\tau)}$$

and

$$\varepsilon\big(\Phi(r_N) - \Phi(r_0)\big) \le \int_{\tau_0}^{\tau_N} \frac{\bar{a}(\tau)}{\bar{h}(\tau)}\, d\tau + \int_{\tau_0}^{\tau_N} \frac{d\tau}{\bar{h}(\tau)} \cdot o(1) \qquad (\varepsilon, \rho \to 0)$$

(here $o(1)$ is independent of N). Due to (3.20), there is a $\mu > 0$ such that for all $\tau \in [\tau_0, \tau_d]$

$$\int_{\tau_0}^{\tau} \frac{\bar{a}(\tau)}{\bar{h}(\tau)}\, d\tau + \eta \int_{\tau_0}^{\tau} \frac{d\tau}{\bar{h}(\tau)} \le -\mu(\tau - \tau_0)$$

if $|\eta|$ is sufficiently small, hence

$$\Phi(r_N) - \Phi(r_0) \le -\frac{\mu}{\varepsilon}\,(\tau_N - \tau_0) \le 0. \tag{3.32}$$

Set

$$a_0 = \max\big\{|a(\tau,\varphi)| :\ \tau_0 - 1 \le \tau \le \tau_d,\ 0 \le \varphi \le 2\pi\big\}.$$

Estimate (3.26) implies

$$\Phi(r(\theta)) - \Phi(r_N) \le (\theta - \theta_N)\varepsilon^{-1} a_0 \qquad (\theta \ge \theta_N).$$

For a given θ we define N by $\theta_N \le \theta < \theta_{N+1}$. Then $\Phi(r(\theta)) - \Phi(r_N) \le 2\pi a_0$ and due to (3.32)

$$\Phi(r(\theta)) \le \Phi(r_0) + 2\pi a_0 \qquad (\theta_0 \le \theta < \Theta). \tag{3.33}$$

Moreover, since $|\tau(\theta) - \tau_N| \le 2\pi c\,\varepsilon$, estimate (3.32) implies

$$\Phi(r(\theta)) - \Phi(r_0) \le -\frac{\mu}{\varepsilon}\,(\tau(\theta) - \tau_0) + 2\pi(a_0 + \mu c) \qquad (\theta_0 \le \theta < \Theta). \tag{3.34}$$

Relation (3.31) is equivalent to

$$H(\tau_N) - H(\tau_0) = (1 + o(1))(\theta_N - \theta_0).$$

From $0 \le \theta - \theta_N < 2\pi\varepsilon$ it follows that $H(\tau(\theta)) - H(\tau_N) = O(\varepsilon)$ and

$$H(\tau(\theta)) - H(\tau_0) = (1 + o(1))(\theta - \theta_0) + O(\varepsilon).$$

For $|o(1)| < \frac{1}{2}$ and $|O(\varepsilon)| < H(\tau_0) - H(\tau_0 - 1)$ this implies

$$H(\tau(\theta)) - H(\tau_0 - 1) \ge \frac{\theta - \theta_0}{2} \qquad (\theta_0 \le \theta < \Theta). \tag{3.35}$$

Here $\tau_0 - 1 \le \tau(\theta) < \tau_d$ by assumption, hence $H(\tau(\theta)) < H(\tau_d)$ and

$$\Theta \le 2\big(H(\tau_d) - H(\tau_0 - 1)\big) + \theta_0 < \infty. \tag{3.36}$$

To conclude the proof, let us fix any sufficiently small ρ and ε such that (3.25) implies relations (3.33) - (3.35) and take any $r_0 \in (0, \rho)$ such that

$$\Phi(r_0) + 2\pi a_0 < \Phi(\rho) \tag{3.37}$$

(this estimate for all sufficiently small r_0 follows from (2.3)). Consider a solution (r, τ) of system (3.16) with initial data $r(\theta_0) = r_0$ and $\tau(\theta_0) = \tau_0$. Let Θ be the largest number such that (3.25) holds on the interval $\theta_0 \le \theta < \Theta$ for this solution. Then either $\tau(\Theta) = \tau_0 - 1$ or $\tau(\Theta) = \tau_d$ or $r(\Theta) = \rho$. But relation (3.35) implies $H(\tau(\Theta)) > H(\tau_0 - 1)$; relations (3.33) and (3.37) imply $\Phi(r(\Theta)) < \Phi(\rho)$. Therefore $\tau(\Theta) > \tau_0 - 1$ and $r(\Theta) < \rho$. This proves $\tau(\Theta) = \tau_d$, which is equivalent to $\Theta = \theta_d$. Now conclusions (3.22) - (3.23) of Proposition 3 follow from relations (3.33) - (3.34) and $\Phi(r) \to -\infty$ as $r \to 0$. The function χ can be defined by $\chi(\rho) = \Phi^{-1}(\Phi(\rho) + 2\pi a_0)$; estimate (3.34) implies

$$\begin{aligned} &\max\big\{r(\theta) :\, \tau(\theta) \ge \tau_0 + \delta,\ \theta_0 \le \theta \le \theta_d\big\} \\ &\qquad \le \Phi^{-1}\big(-\mu\delta\varepsilon^{-1} + \Phi(r_0) + 2\pi(a_0 + \mu c)\big). \end{aligned}$$

Finally, we have proved that $\max \tau(\theta) \ge \tau_d$; since relations (3.20) hold for $\tau_0 \le \tau \le \tau_d + \eta$ with any small $\eta > 0$, estimate (3.21) is true, which completes the proof ∎

Condition (3.19) of Proposition 3 can be replaced by

$$g(\tau, r, \varphi) = a(\tau, \varphi)\phi(r) + o(\phi(r)) \qquad (r \to 0). \tag{3.38}$$

Indeed, (3.38) implies that estimates (3.19) and (3.20) with $a(\tau, \varphi) + \eta$ and $\bar{a}(\tau) + \eta$ in place of $a(\tau, \varphi)$ and $\bar{a}(\tau)$ are valid for any sufficiently small $\eta > 0$ and $\rho_0 = \rho_0(\eta) > 0$. Now to prove Theorem 3 it suffices to note that for system (3.15) relations (3.38) with $\phi(r) = r^\alpha$ and $\bar{h}(\tau) \equiv 1$ hold, estimates (3.20) have the form (3.12) and, due to (3.22), each solution of system (3.15) with a sufficiently small initial value $r^\varepsilon(\theta_0) = r_0$ is close to the zero solution for all $\theta_0 \le \theta \le \theta_d$. Since systems (3.4) and (3.15) are equivalent in some neighborhood of zero, relations (3.13) and (3.23) are equivalent for small ρ.

For example, let

$$F_j(\tau, x, y) = a_1^j x^2 + a_2^j x|x| + a_3^j xy + a_4^j x|y| + a_5^j |x|y + a_6^j |x||y| + a_7^j y|y| + a_8^j y^2$$

where $a_k^j = a_k^j(\tau)$. Then $\alpha = 2$, $\phi(r) = r^2$ and function (3.11) equals

$$\bar{a}(\tau) = \frac{2}{3\pi}\big(2a_2^1(\tau) + a_4^1(\tau) + a_5^2(\tau) + 2a_7^2(\tau)\big).$$

If the functions $f_j(\tau, x, y)$ admit the Taylor expansion (with some finite number of terms) at the point $x = y = 0$ for each τ, then F_j are homogeneous polynomials of order $\alpha \ge 2$ of the variables x and y. Theorem 3 can not be used in the main case $\alpha = 2$, since for quadratic polynomials F_j function (3.11) is identical zero. This case is studied in the next subsection by simple modification of Proposition 3.

Suppose that all quadratic terms in the Taylor expansions of the functions $f_j = f_j(\tau, x, y)$ at the point $x = y = 0$ are zero for all τ, i.e.

$$f_j(\tau, x, y) = a_1^j(\tau)x^3 + a_2^j(\tau)x^2y + a_3^j(\tau)xy^2 + a_4^j(\tau)y^3 + o(|x|^3 + |y|^3). \tag{3.39}$$

Then $\alpha = 3$, $\phi(r) = r^3$ and function (3.11) is defined by

$$\bar{a}(\tau) = \frac{1}{8}\big(3a_1^1(\tau) + a_3^1(\tau) + a_2^2(\tau) + 3a_4^2(\tau)\big). \tag{3.40}$$

3.3 Polynomial non-linearities. Let

$$\begin{aligned} P_j(\tau, \tilde{x}, \tilde{y}) &= p_1^j\tilde{x}^2 + p_2^j\tilde{x}\tilde{y} + p_3^j\tilde{y}^2 \\ Q_j(\tau, \tilde{x}, \tilde{y}) &= q_1^j\tilde{x}^3 + q_2^j\tilde{x}^2\tilde{y} + q_3^j\tilde{x}\tilde{y}^2 + q_4^j\tilde{y}^3 \end{aligned}$$

where $p_k^j = p_k^j(\tau)$ and $q_k^j = q_k^j(\tau)$. Consider the system

$$\left.\begin{aligned} \frac{d\tilde{x}}{dt} &= -\tilde{y} + P_1(\tau, \tilde{x}, \tilde{y}) + Q_1(\tau, \tilde{x}, \tilde{y}) + \psi_1(\tau, \tilde{x}, \tilde{y}) \\ \frac{d\tilde{y}}{dt} &= \tilde{x} + P_2(\tau, \tilde{x}, \tilde{y}) + Q_2(\tau, \tilde{x}, \tilde{y}) + \psi_2(\tau, \tilde{x}, \tilde{y}) \end{aligned}\right\} \tag{3.41}$$

where the terms ψ_j satisfy (3.9) with $\alpha = 3$. The standard change of variables

$$\begin{aligned} x &= \tilde{x} + b_1^1(\tau)\tilde{x}^2 + b_2^1(\tau)\tilde{x}\tilde{y} + b_3^1(\tau)\tilde{y}^2 \\ y &= \tilde{y} + b_1^2(\tau)\tilde{x}^2 + b_2^2(\tau)\tilde{x}\tilde{y} + b_3^2(\tau)\tilde{y}^2 \end{aligned} \tag{3.42}$$

transforms this system to system (3.5) with non-linearities (3.39) that do not contain quadratic terms. The coefficients a_k^j and b_k^j in (3.39) and (3.42) are determined by straightforward calculations. Each coefficient b_k^j is a linear combination of the coefficients p_ℓ^m of the polynomials P_1 and P_2; every coefficient a_k^j is the sum of the corresponding coefficient q_k^j of Q_j with a quadratic form of the coefficients p_ℓ^m of P_1 and P_2 (see, e.g., [2, 7]). Using the expressions for a_k^j, one arrives at the following formula for the function $\bar{a} = \bar{a}(\tau)$ given by (3.40):

$$\begin{aligned} \bar{a} = \frac{1}{8}\Big(&2(p_3^1p_3^2 - p_1^1p_1^2) + p_2^1(p_1^1 + p_3^1) \\ &- p_2^2(p_1^2 + p_3^2) + 3q_1^1 + q_3^1 + q_2^2 + 3q_4^2\Big). \end{aligned} \tag{3.43}$$

Since systems (3.5) and (3.41) related by equalities (3.42) are equivalent in some neighborhood of the origin, their Hopf bifurcation points coincide; hence, each Hopf bifurcation point τ_* is a zero of the function (3.43). The sign of (3.43) at any other point τ determines stability of the zero equilibrium for system (3.41).

Now consider the system

$$\left.\begin{aligned}\varepsilon\frac{d\tilde{x}}{d\tau} &= -\tilde{y} + P_1(\tau,\tilde{x},\tilde{y}) + Q_1(\tau,\tilde{x},\tilde{y}) + \psi_1(\tau,\tilde{x},\tilde{y})\\ \varepsilon\frac{d\tilde{y}}{d\tau} &= \tilde{x} + P_2(\tau,\tilde{x},\tilde{y}) + Q_2(\tau,\tilde{x},\tilde{y}) + \psi_2(\tau,\tilde{x},\tilde{y})\end{aligned}\right\}. \tag{3.44}$$

Suppose that all coefficients $p_\ell^m(\tau)$ of the polynomials P_1 and P_2 are continuously differentiable. Let system (3.5) with non-linearities f_j of form (3.39) be obtained from the associated to (3.44) system (3.41) by transformation (3.42). Then formulas (3.42) transform system (3.44) to

$$\left.\begin{aligned}\varepsilon\frac{dx}{d\tau} &= -y + f_1(\tau,x,y) + \varepsilon\gamma_1(\tau,x,y)\\ \varepsilon\frac{dy}{d\tau} &= x + f_2(\tau,x,y) + \varepsilon\gamma_2(\tau,x,y)\end{aligned}\right\} \tag{3.45}$$

where $\gamma_j = O(x^2+y^2)$. This system does not have form (3.4) (due to the presence of the terms $\varepsilon\gamma_j$). Nevertheless, the following analog of Theorem 3 holds.

Theorem 4. *Let estimates* (3.12) *be valid for function* (3.43). *Then there is a* $\rho > 0$ *such that for any small* $\delta > 0$ *the relation*

$$\lim_{\varepsilon\to 0}\max\left\{|\tilde{x}^\varepsilon(\tau)| + |\tilde{y}^\varepsilon(\tau)| : \ \tau_0+\delta \le \tau \le \tau_d\right\} = 0$$

holds for all solutions $(\tilde{x}^\varepsilon, \tilde{y}^\varepsilon)$ *of system* (3.44) *satisfying* $\big(\tilde{x}^\varepsilon(\tau_0), \tilde{y}^\varepsilon(\tau_0)\big) \in B(\rho)$.

Equivalently, it means that solutions of system (3.45) satisfy (3.13).

To prove this fact, we rewrite system (3.45) in polar coordinates (r,φ), substitute $\varphi = \varepsilon^{-1}\theta$, and change the time $\tau \mapsto \theta$. As a result, we obtain the system

$$\left.\begin{aligned}\varepsilon\frac{dr}{d\theta} &= g(\tau,r,\varepsilon^{-1}\theta,\varepsilon)\\ \frac{d\tau}{d\theta} &= h(\tau,r,\varepsilon^{-1}\theta,\varepsilon)\end{aligned}\right\} \tag{3.46}$$

equivalent to (3.44) and (3.45) in some neighborhood of the origin, where

$$\begin{aligned}g(\tau,r,\varphi,\varepsilon) &= a(\tau,\varphi)r^3 + o(r^3) + \varepsilon O(r^2)\\ h(\tau,r,\varphi,\varepsilon) &= 1 + O(r)\end{aligned} \tag{3.47}$$

and integral (3.11) of the function $a(\tau,\varphi)$ is defined by (3.43).

Theorem 4 follows from the next proposition.

Proposition 4. *Let relations* (3.47) *hold, functions* (3.47) *be continuous and* 2π-*periodic in* φ. *Let estimates* (3.12) *be valid for function* (3.11). *Suppose that for solutions* $(r^\varepsilon,\tau^\varepsilon)$ *of system* (3.46) *the estimate* $r^\varepsilon(\theta_0) > 0$ *implies* $r^\varepsilon(\theta) > 0$ *for all* $\theta > \theta_0$. *Then for every sufficiently small* $\rho > 0$ *and* $\delta > 0$ *all solutions* $(r^\varepsilon,\tau^\varepsilon)$ *of system* (3.46) *such* *that* $0 < r^\varepsilon(\theta_0) = r_0 < \rho$ *and* $\tau^\varepsilon(\theta_0) = \tau_0$ *with any* θ_0 *satisfy relations* (3.21) – (3.23) *where* θ_d *is defined by* (3.24) *and* $\chi(\rho) \to 0$ *as* $\rho \to 0$.

Proof. We need to modify only slightly the proof of Proposition 3. Let us fix any α from the interval $\frac{1}{2} < \alpha < 1$ and any sufficiently small $\eta > 0$ such that relations (3.12) hold for $\bar{a}(\tau)+\eta$ in place of $\bar{a}(\tau)$. Let $\tau(\theta_0) = \tau_0$, estimates (3.25) be valid for a solution (r,τ) of system (3.46) on some interval $\theta_0 \le \theta < \Theta$ and let additionally $r(\theta) > \varepsilon^\alpha$ on this interval. Set

$$\phi(r) = r^3, \quad \Phi(r) = -\frac{1}{2r^2}, \quad \bar{h}(\tau) \equiv 1, \quad H(\tau) = \tau.$$

If ρ is sufficiently small, then (3.47) implies

$$\left| g(\tau, r, \varphi, \varepsilon) - a(\tau,\varphi) r^3 \right| \le \eta r^3 \qquad (\varepsilon^\alpha < r \le \rho) \tag{3.48}$$

and therefore the solution (r,τ) satisfies for all $\theta_0 \le \theta < \Theta$

$$\frac{\varepsilon}{r^3}\frac{dr}{d\theta} \le a(\tau(\theta), \varepsilon^{-1}\theta) + \eta.$$

This estimate is the analog of (3.26). From this point, we can apply exactly the same argument as in the proof of Proposition 3; the only difference is that $a(\tau,\varphi)$ and $\bar{a}(\tau)$ should be everywhere replaced by $a(\tau,\varphi)+\eta$ and $\bar{a}(\tau)+\eta$, respectively. As a result, we prove relations (3.33) - (3.34), but now they are valid on the maximal interval $\theta_0 \le \theta < \Theta$ where either $\tau(\Theta) = \tau_d$ or $r(\Theta) = \varepsilon^\alpha$ (this alternative is due to the additional requirement $r(\theta) > \varepsilon^\alpha$ to the solution).

Fix a ν such that $\frac{1}{2} < \alpha - 2\nu < \alpha < 1$. Estimate (3.48) implies

$$\frac{\varepsilon}{r^3}\frac{dr}{d\theta} \ge -(a_0 + \eta) \qquad (\theta_0 \le \theta < \Theta)$$

and therefore

$$\frac{\varepsilon}{2}\left(\frac{1}{r^2(\theta_0)} - \frac{1}{r^2(\Theta)}\right) \ge -(a_0+\eta)(\Theta - \theta_0)$$

where $a_0 = \max|a(\tau,\varphi)|$ and $\Theta - \theta_0 \le 2(\tau_d - \tau_0 + 1) =: c_1$ according to (3.36). If $r(\theta_0) > \varepsilon^{\alpha-\nu}$, then

$$r^2(\Theta) \ge \frac{\varepsilon^{2\alpha-2\nu}}{1 + 2c_1(a_0+\eta)\varepsilon^{2\alpha-2\nu-1}}.$$

Since $2\alpha - 2\nu > 1$, for all sufficiently small ε this implies $r(\Theta) > \varepsilon^\alpha$ and from the alternative above it follows that $\tau(\Theta) = \tau_d$, i.e. $\Theta = \theta_d$. Thus, for solutions (r,τ) of system (3.46) with initial values $r(\theta_0) > \varepsilon^{\alpha-\nu}$ estimates (3.33) - (3.34) hold for all $\theta_0 \le \theta \le \theta_d$, consequently for such solutions relations (3.22) - (3.23) are valid, i.e. the conclusion of the proposition is proved for them.

If $r(\theta_0) \le \varepsilon^{\alpha-\nu}$, then relations (3.22) - (3.23) follow from rough estimates of g. Consider any solution (r,τ) of system (3.46) with initial data $r(\theta_0) \le \varepsilon^{\alpha-\nu}$ and $\tau(\theta_0) =$

τ_0 on the maximal interval $\theta_0 \le \theta < \Theta_1$ such that $0 < r(\theta) < \varepsilon^{\alpha-2\nu}$ for all θ and $\Theta_1 \le \theta_d$. From (3.47) it follows

$$g(\tau, r, \varphi, \varepsilon) = O(\varepsilon^{3\alpha-6\nu}) \qquad (0 < r \le \varepsilon^{\alpha-2\nu}).$$

This relation and the estimates $\Theta_1 - \theta_0 \le \theta_d - \theta_0 \le c_1$ imply $r' = O(\varepsilon^{3\alpha-6\nu-1})$ on the interval $\theta_0 \le \theta < \Theta_1$ and $r(\Theta_1) - r(\theta_0) = O(\varepsilon^{3\alpha-6\nu-1})$. Since $2\alpha - 4\nu > 1$, we conclude that $r(\Theta_1) < \varepsilon^{\alpha-2\nu}$ for small ε, which implies according to the definition of Θ_1 that $\Theta_1 = \theta_d$ and

$$\max\{r(\theta) : \theta_0 \le \theta \le \theta_d\} < \varepsilon^{\alpha-2\nu} < \sqrt{\varepsilon}.$$

This proves (3.22) - (3.23) for solutions with $r(\theta_0) \le \varepsilon^{\alpha-\nu}$ ∎

Acknowledgements. The research was done while D. Rachinskii was visiting the *Weierstraß-Institut für Angewandte Analysis und Stochastik* in Berlin and the *Technische Universität Münich* supported by the Research Fellowship from the *Alexander von Humboldt-Stiftung*. His work was partially supported by the *Russian Science Support Foundation*, the *Russian Foundation for Basic Research* (Grants No. 01-01-00146, 03-01-00258), and the *Grants of the President of Russia* (Grants No. MD-87.2003.01, NS-1532.2003.1). The support is gratefully acknowledged.

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Received 11.07.2002